Dynamic Bifurcations and Melting Boundary Convection

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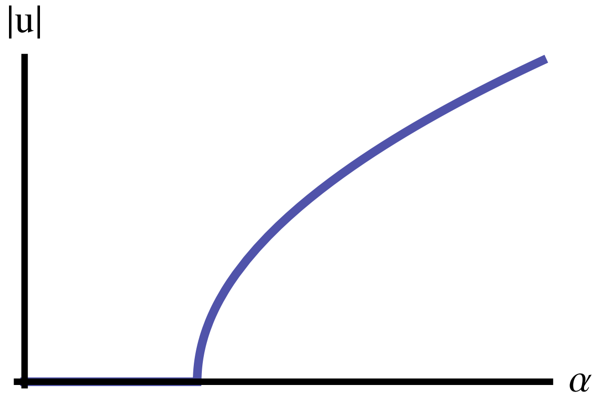
Mathematical Motivation

Consider a general stability problem

$$\frac{du}{dt} = F(u;\alpha)$$



This is the same at the "coupled dynamical system"

$$\frac{du}{dt} = F(u;\alpha) \qquad \frac{d\alpha}{dt} = 0$$

Mathematical Motivation

What if we make it a bit more interesting?

$$\frac{du}{dt} = F(u;\alpha) \qquad \frac{d\alpha}{dt} = \epsilon\, G(\alpha)$$

$\epsilon \longrightarrow 0$ Recovers the previous case

Now it is a (one-way coupled) multi-timescale problem where

$$\alpha = \alpha(\epsilon t)$$

Mathematical Motivation

What if we make it even more interesting?

$$\frac{du}{dt} = F(u;\alpha) \quad \frac{d\alpha}{dt} = \epsilon\, G(\alpha;u)$$

$\epsilon \longrightarrow 0$ Recovers the original case

Now it is a (two-way coupled) multi-timescale problem where the bifurcation "parameter" is simply a slowly varying dynamical variable of the system

Mathematical Motivation

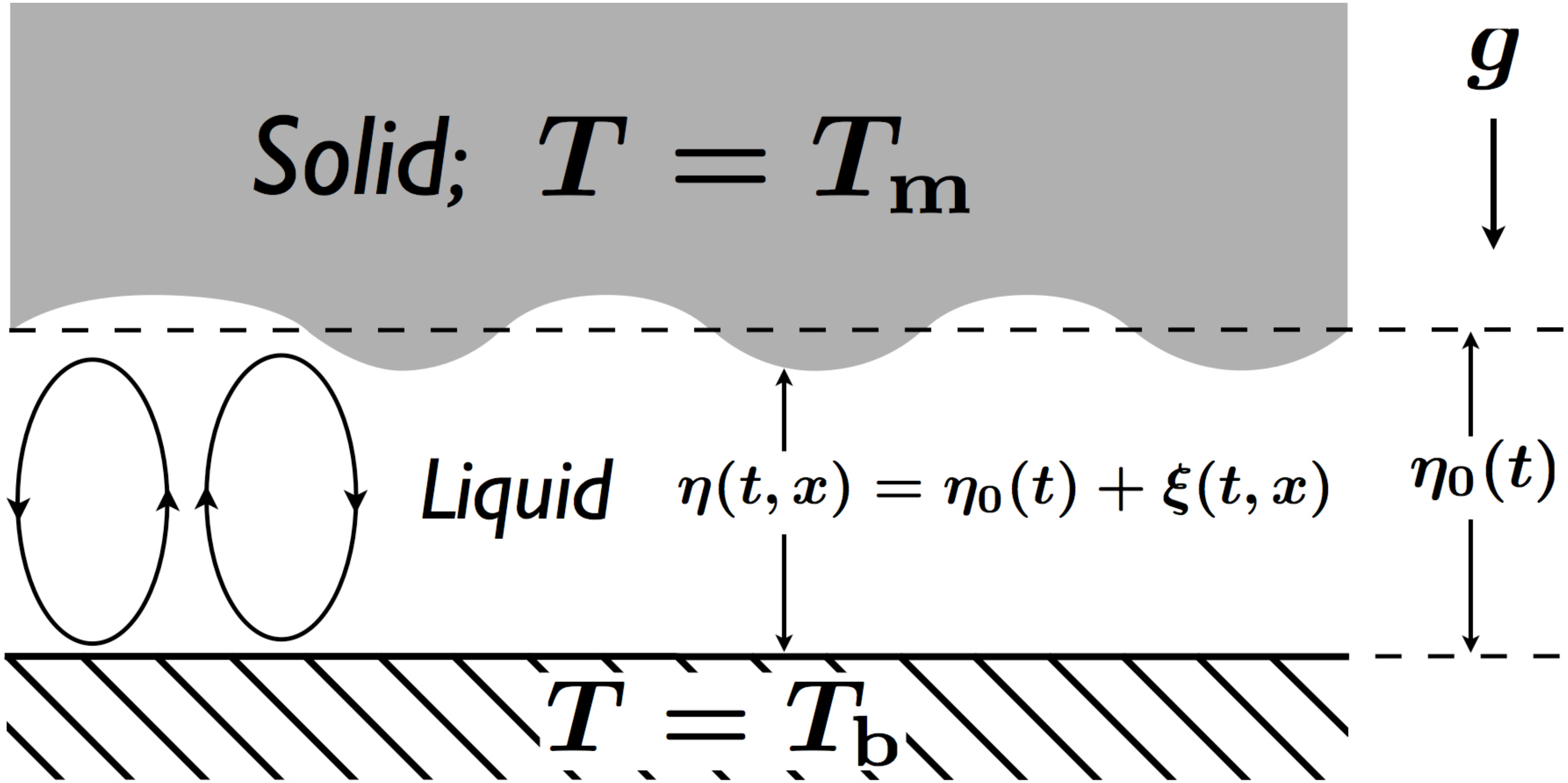
$$\frac{du}{dt} = F(u;\alpha) \quad \frac{d\alpha}{dt} = \epsilon\, G(\alpha;u)$$

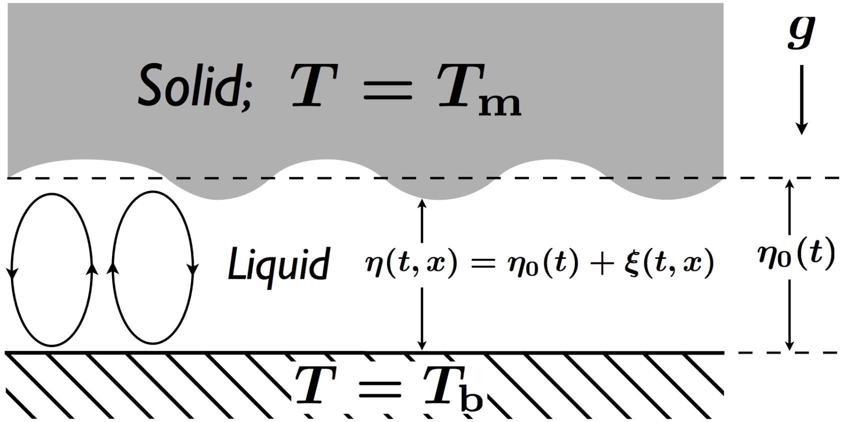
Rescale time

$$t \to \epsilon^{-1/2} t, \quad \frac{d}{dt} \to \epsilon^{1/2}\frac{d}{dt}$$

$$\epsilon^{1/2}\frac{du}{dt} = F(u;\alpha) \quad \frac{d\alpha}{dt} = \epsilon^{1/2} G(\alpha;u)$$

A Model Problem





$$\mathrm{Ra} = \frac{g\alpha(T_{\mathrm{b}} - T_{\mathrm{m}})H_0^3}{\nu\kappa}$$

$$\mathrm{Pr} = \frac{\nu}{\kappa}$$

$$S = \frac{L_{\mathrm{fusion}}}{c_{\mathrm{p}}(T_{\mathrm{b}} - T_{\mathrm{m}})}$$

$$\frac{1}{\mathrm{Pr}}(\partial_t u + u\cdot\nabla u) + \nabla P = \mathrm{Ra}\,T\hat{z} + \Delta u$$

$$\nabla\cdot u = 0$$

$$\partial_t T + u\cdot\nabla T = \Delta T$$

at $z = \eta(x,y)$, $u = 0 \quad T = 0$

at $z = 0$, $u = 0 \quad T = 1$

The Stefan Condition

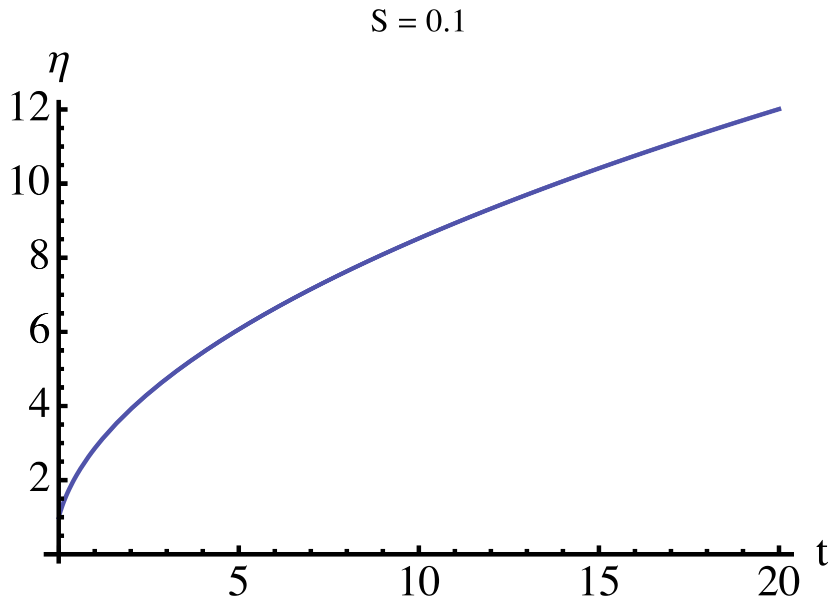
$$S\partial_t\eta + \hat{n}\cdot\nabla T = 0 \quad \text{at } z = \eta(x,y)$$

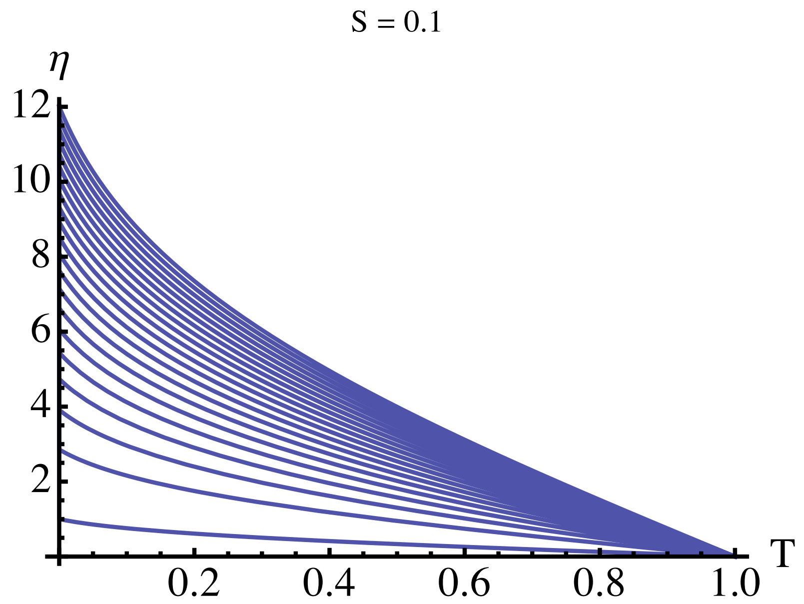
Background State (Similarity Solution)

$$\partial_t T_0 = \partial_z^2 T_0 \quad S\partial_t \eta_0 = -\partial_z T_0|_{z=\eta_0}$$

$$\frac{1}{S} = \sqrt{\pi}\beta \mathrm{Erf}(\beta)\exp(\beta^2) \approx 2\beta^2 \quad \text{if} \quad S \gg 1$$

$$\eta_0 = \sqrt{4\beta^2 t + 1} \qquad T_0 = 1 - \frac{\mathrm{Erf}(\beta z/\eta_0)}{\mathrm{Erf}(\beta)} \approx 1 - \frac{z}{\eta_0}$$





$$\epsilon \;=\; S^{-1/4} \ll 1$$ Slow time evolution

$$\mathrm{Ra} \;=\; \mathrm{Ra_c} \;\approx\; 1708$$

$$\partial_x = ik_c + \epsilon\,\partial_X \quad \partial_t = \epsilon^2 \partial_\tau$$

$$\vec{u} = \epsilon\, A(\tau, X)\vec{U}_c(z)e^{ik_c x} + \mathrm{c.c.}$$

$$T = 1 - z + \epsilon\, A(\tau, X)\Theta_c(z)e^{ik_c x} + \mathrm{c.c.}$$

$$\eta = 1 + \epsilon^2\,\tau + \epsilon^3\,\Xi(\tau, X)e^{ik_c x} + \mathrm{c.c.}$$

Put it all together...

We get something like the Ginzburg-Landau Equation

$$\partial_\tau A = \gamma\tau A + \Xi - |A|^2 A + \partial_X^2 A$$

$$\partial_\tau \Xi = A$$

$$\gamma = \frac{3\int_0^1 W_c \Theta_c \, dz}{|\partial_z \Theta_c|^2_{z=1}} \approx 3.747739$$

These systems have been studied before

Ginzburg-Landau (GL)

$$\partial \tau A = \mu A - |A|^2 A + \partial_X^2 A$$

$$\frac{du}{dt} = F(u;\alpha) \qquad \frac{d\alpha}{dt} = 0$$

Dynamic GL1

$$\partial_\tau A = \gamma \tau A - |A|^2 A + \partial_X^2 A$$

$$\frac{du}{dt} = F(u;\alpha) \qquad \frac{d\alpha}{dt} = \epsilon G(\alpha)$$

Dynamic GL2

$$\partial_\tau A = \gamma \tau A + \Xi - |A|^2 A + \partial_X^2 A$$

$$\partial_\tau \Xi = A$$

$$\frac{du}{dt} = F(u;\alpha) \qquad \frac{d\alpha}{dt} = \epsilon G(\alpha;u)$$

This system has some interesting new dynamics

Purely Real, No Spatial Dependence

$$\partial_\tau A = \mu A - A^3$$

$A \sim e^{\mu\tau}$ for $|A| \ll 1$

$A \to \pm\sqrt{\mu}$ as $\tau \to \infty$

For $\mu < 0$, the solutions decay very rapidly

$$\partial_\tau A = \gamma\tau A - A^3$$

$A \sim e^{\gamma\tau^2/2}$ for $|A| \ll 1$

$A \to \pm\sqrt{\gamma\tau}$ as $\tau \to \infty$

For $\tau < 0$, the solutions also decay very rapidly

Linear, No Spatial Modulation

$$\partial_\tau A = \gamma \tau A + \Xi \qquad \partial_\tau \Xi = A$$

$$A \sim \int_0^\infty \zeta^{\frac{1}{\gamma}} \exp\left[-\frac{\zeta^2}{2\gamma} \pm \zeta\tau\right] \mathrm{d}\zeta \text{ for } |A| \ll 1$$

$$A \to \frac{\Gamma\left(1+\frac{1}{\gamma}\right)}{|\tau|^{\frac{1}{\gamma}+1}} \text{ for } \tau \ll 0$$

$$A \to \sqrt{2\pi\gamma}\,(\gamma\tau)^{\frac{1}{\gamma}} \exp\left[\frac{\gamma\tau^2}{2}\right] \text{ for } \tau \gg 0$$

For $\tau < 0$, the solutions can *grow* algebraically.
There is a morphological instability before there is a convective instability

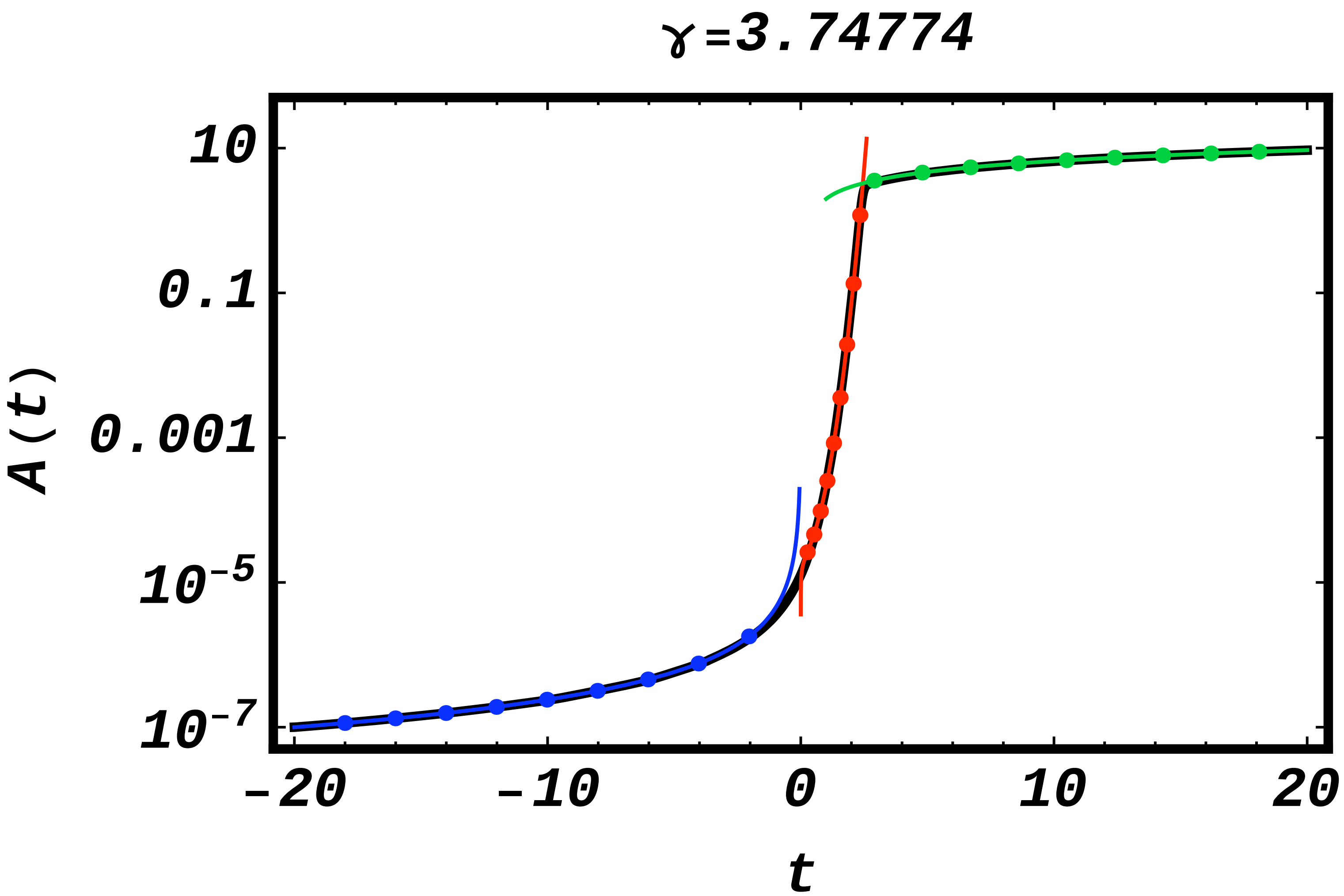
Nonlinear, Spatial Modulation

One of the most surprising new features...

$$\boldsymbol{A \rightarrow \sqrt{(\gamma + 2/3)\,\tau}\, e^{i\psi_0(X)} \text{ as } \tau \rightarrow \infty}$$

The "locked-in" phase function is strongly dependent on the initial conditions

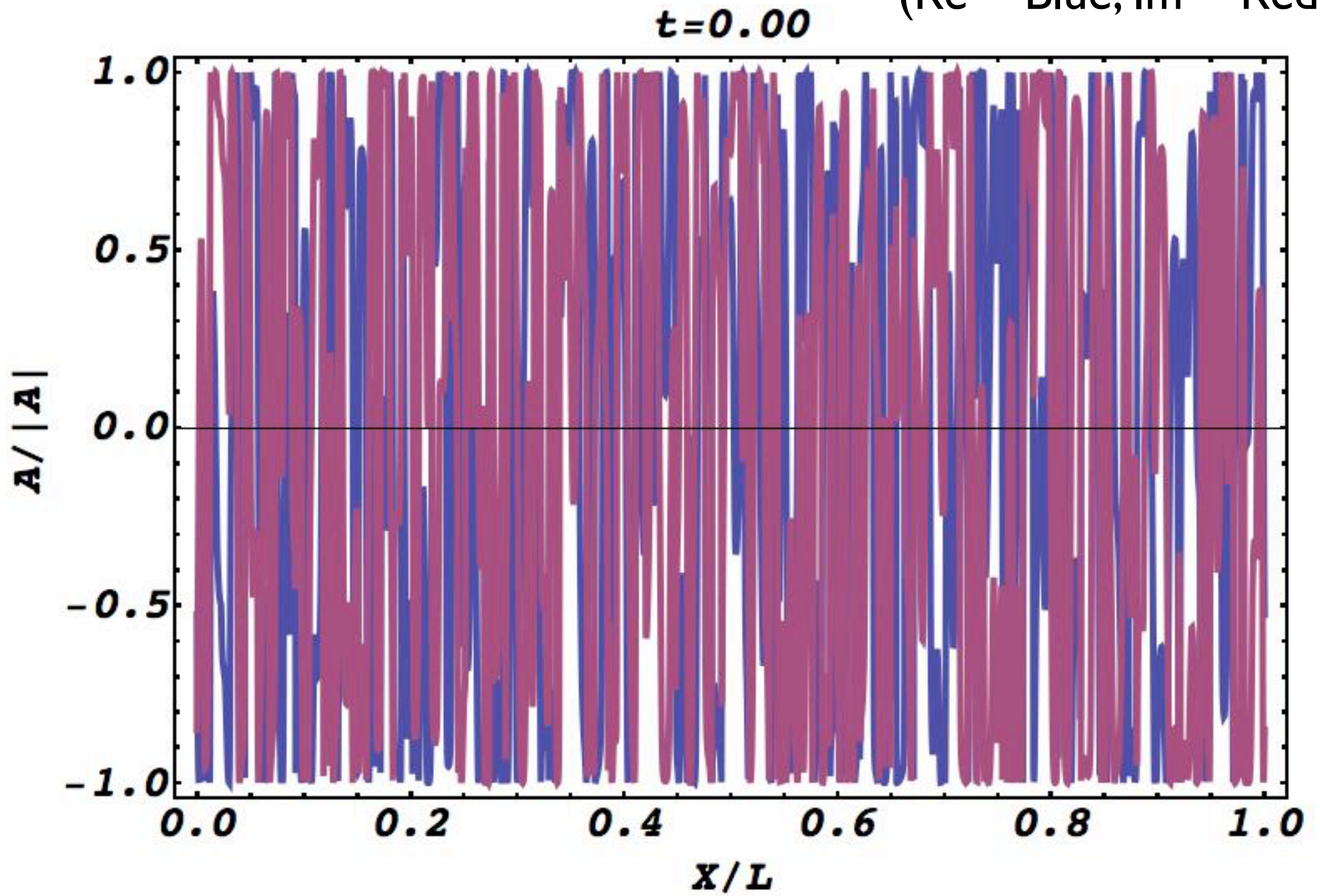
Purely Real, No Spatial Modulation



What about more complex patterns?

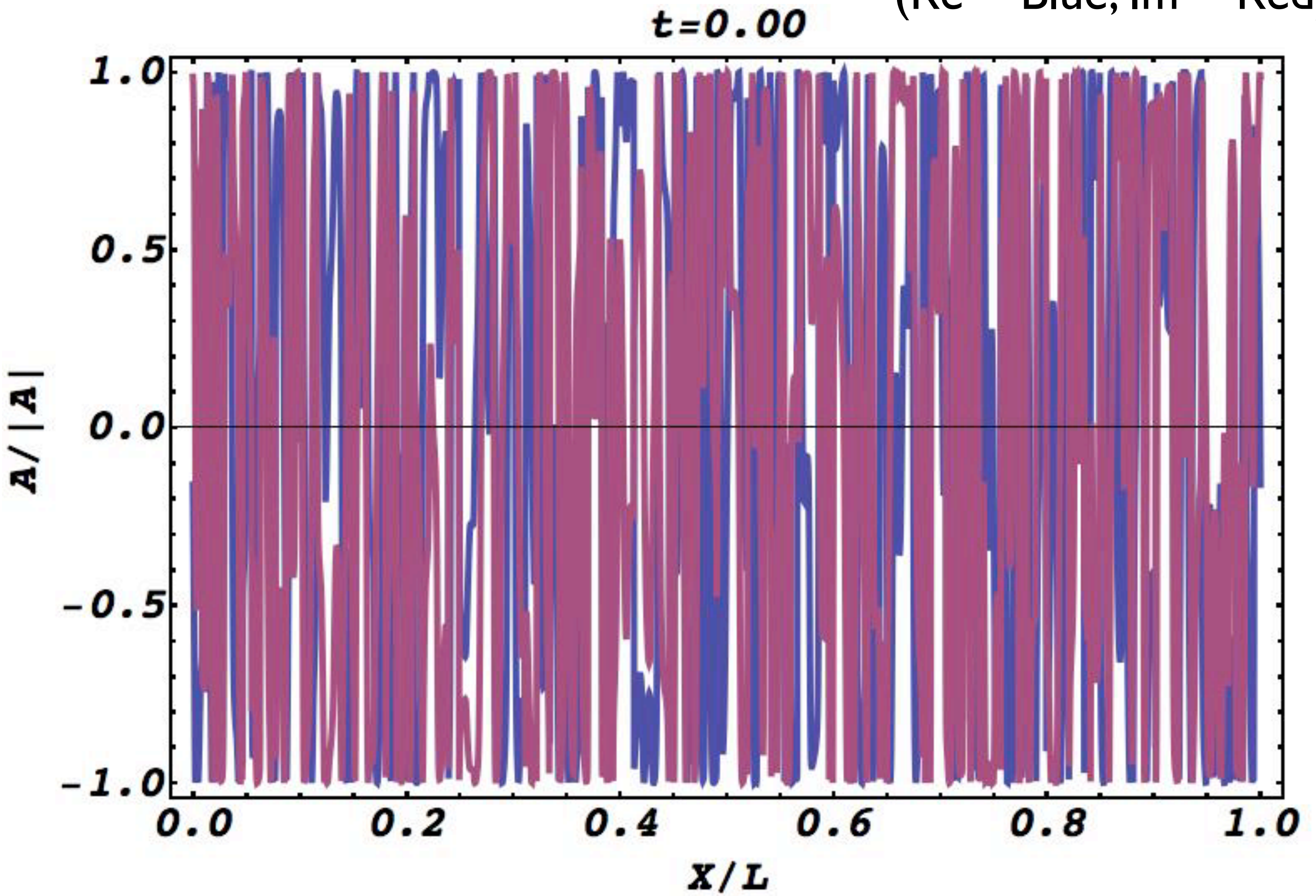
GL

(Re = Blue, Im = Red)



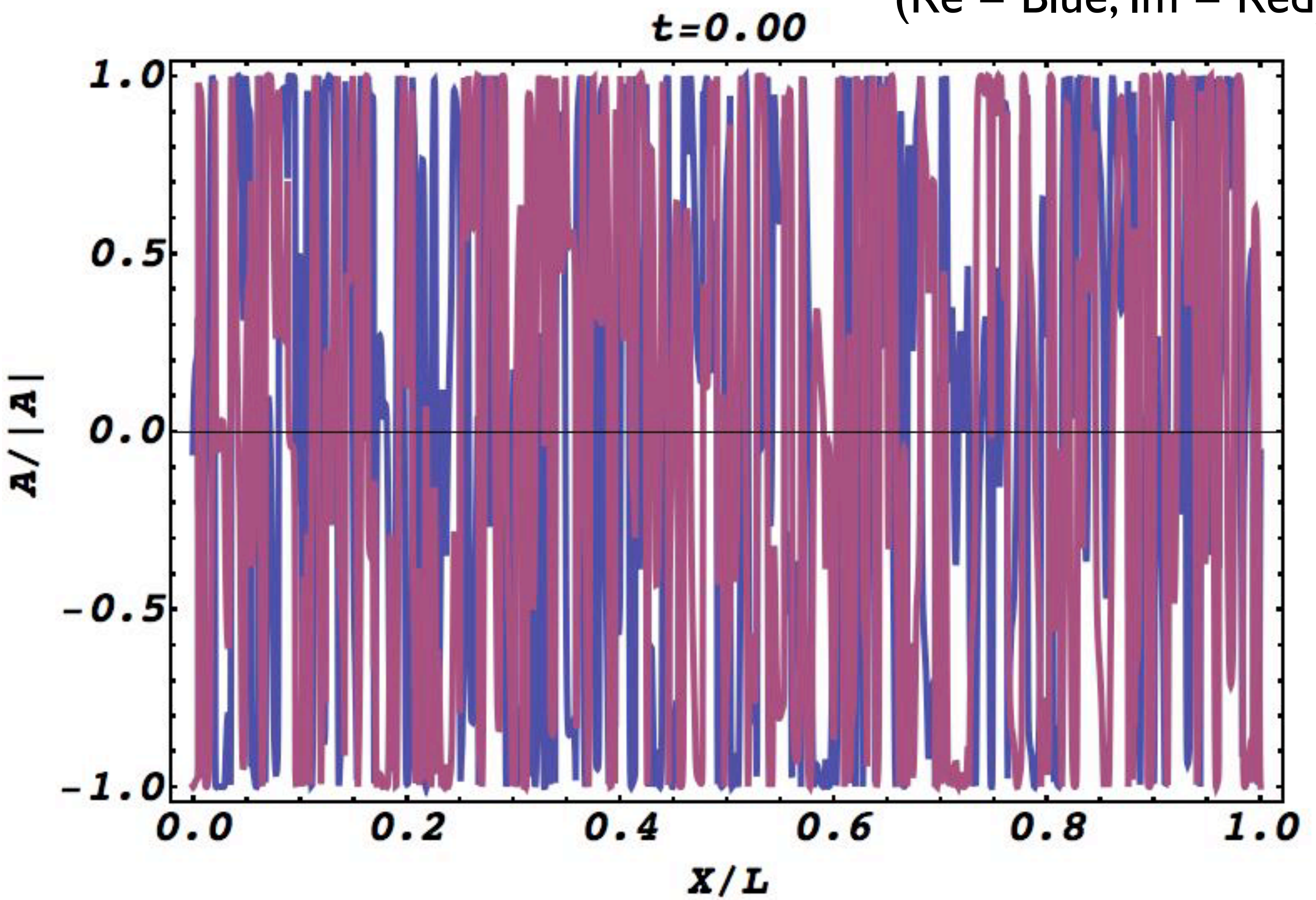
DGL-I

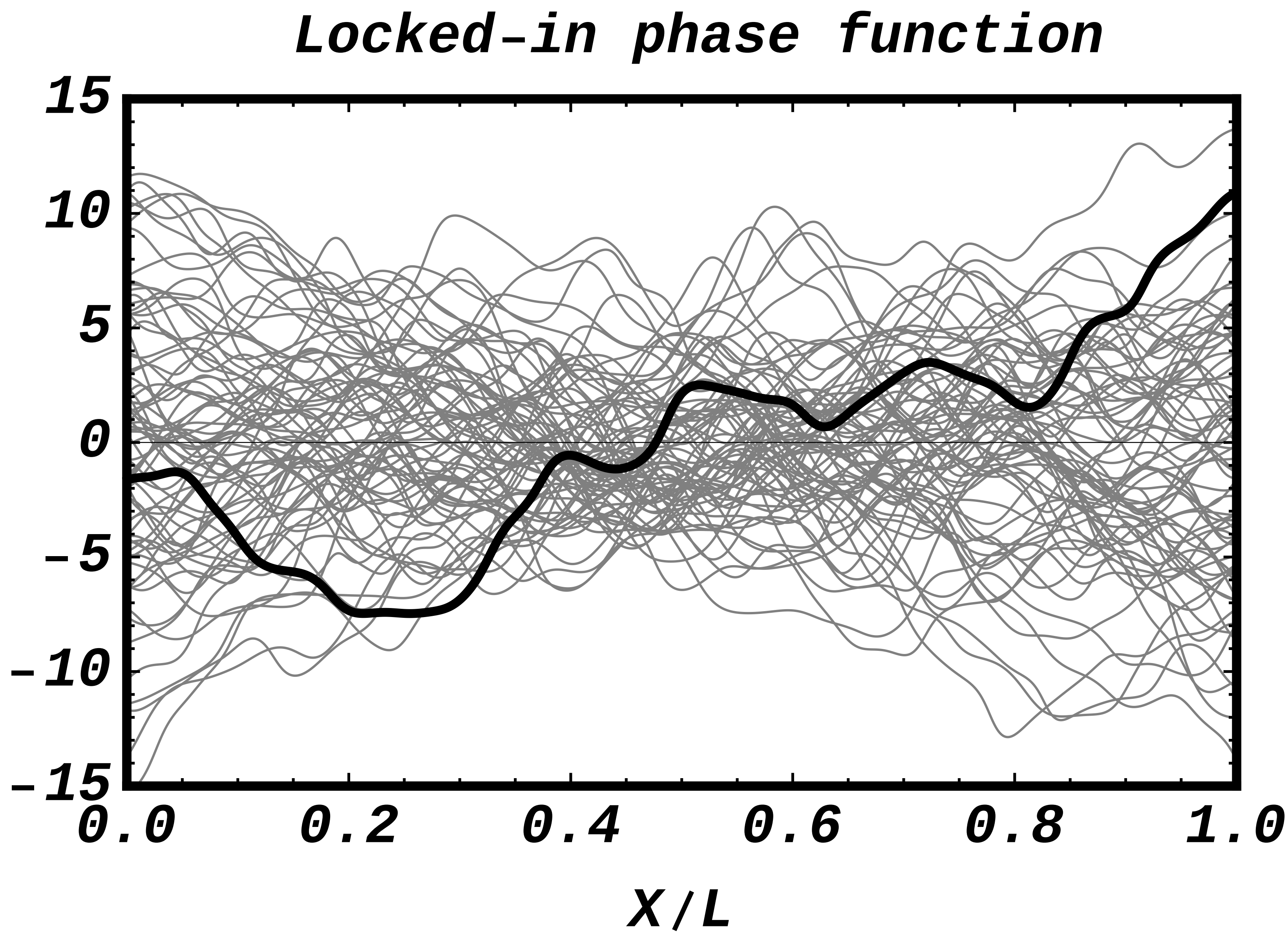
(Re = Blue, Im = Red)



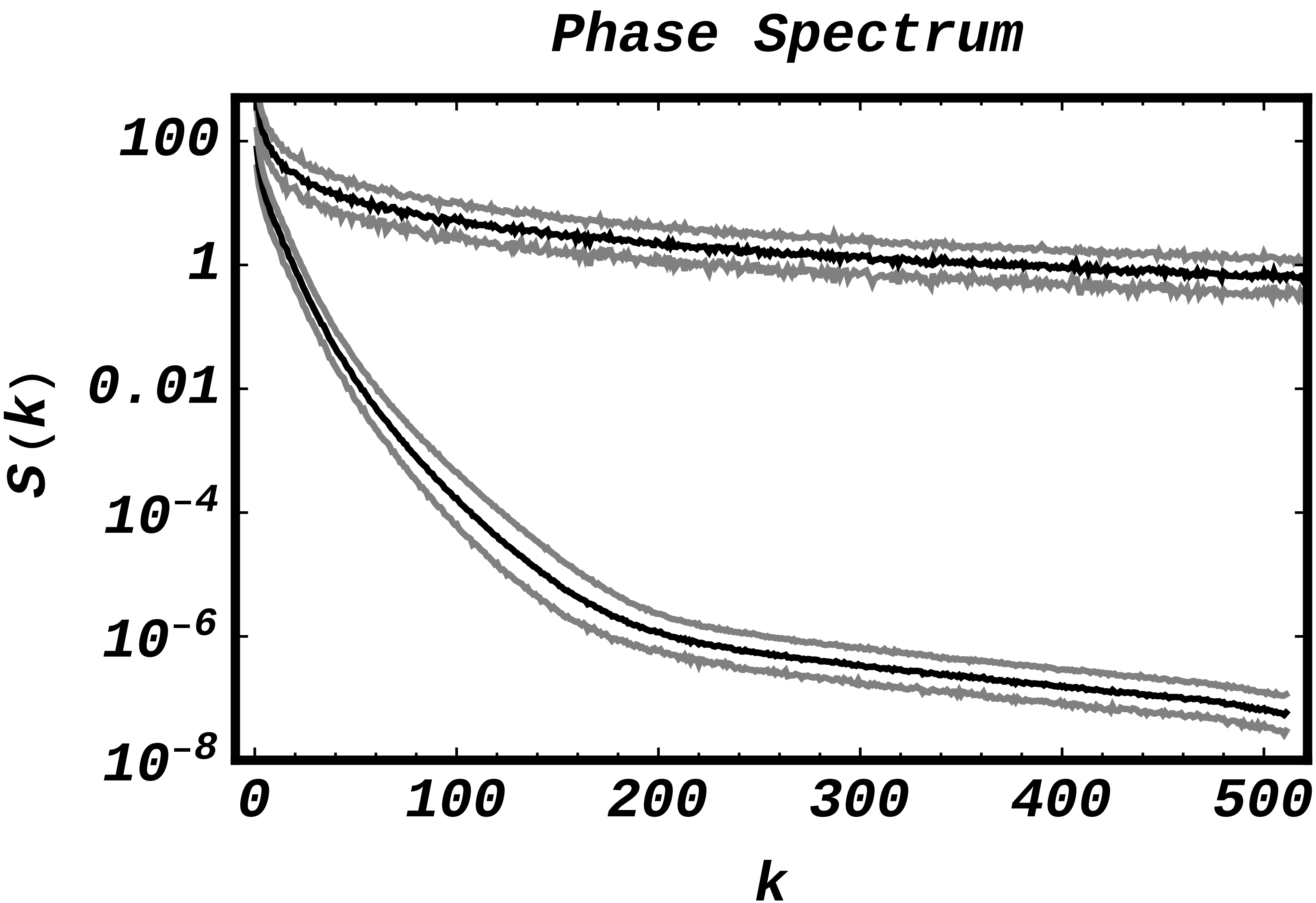
DGL-II

(Re = Blue, Im = Red)



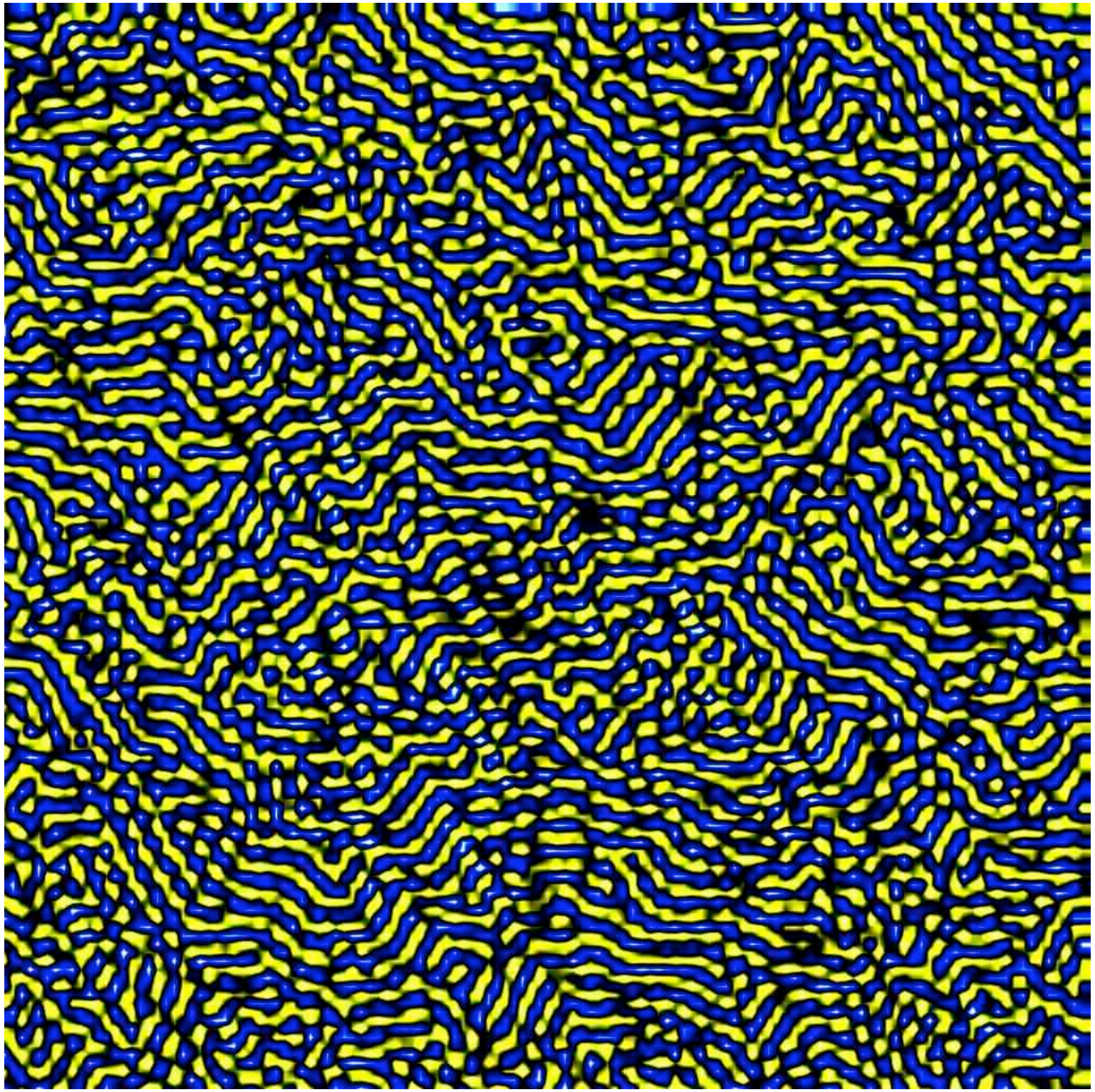


Locked-in phase function
15
10
5
0
−5
−10
−15
0.0
0.2
0.4
0.6
0.8
1.0
ψ0(X)
X/L



Phase Spectrum
100
1
0.01
10^-4
10^-6
10^-8
S(k)
0
100
200
300
400
500
k

Patterns freeze in 2D as well



Some Thoughts...

Even strongly nonlinear systems often pass though a time when the growth rate of the instability is comparable to the rate of evolution of the background.

Conditions in this early stage can have dramatic consequences for the long-time dynamics of the system.

The pattern can even lock in broken-symmetry states that would otherwise be unstable.

We're hoping to set up an experiment with wax (or something) sometime soon.

I'm always looking for other systems to apply these basic methods to….