A finite element approach of nonlinear MHD problems in heterogeneous domains

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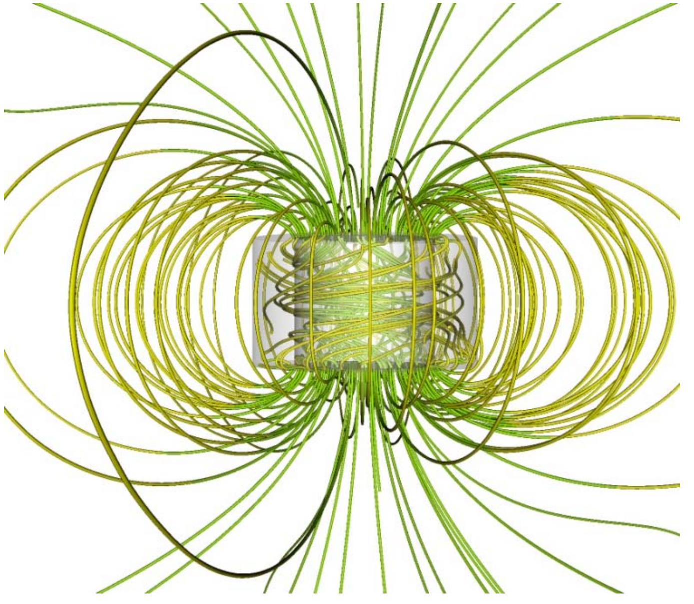


Cargèse sept. 2010

Co-workers
J. L. Guermond (Texas A&M), J. Léorat (Obs. Paris)
F. Luddens (LIMSI, TAMU), A. Ribeiro (LIMSI), A. Giesecke (FZR, Dresden)

Motivation

MHD equations and transmission conditions

SFEMaNS code (Spectral Finite Element code for Maxwell and Navier-Stokes equations)

μ jump: analytical solution of induction in a composite sphere

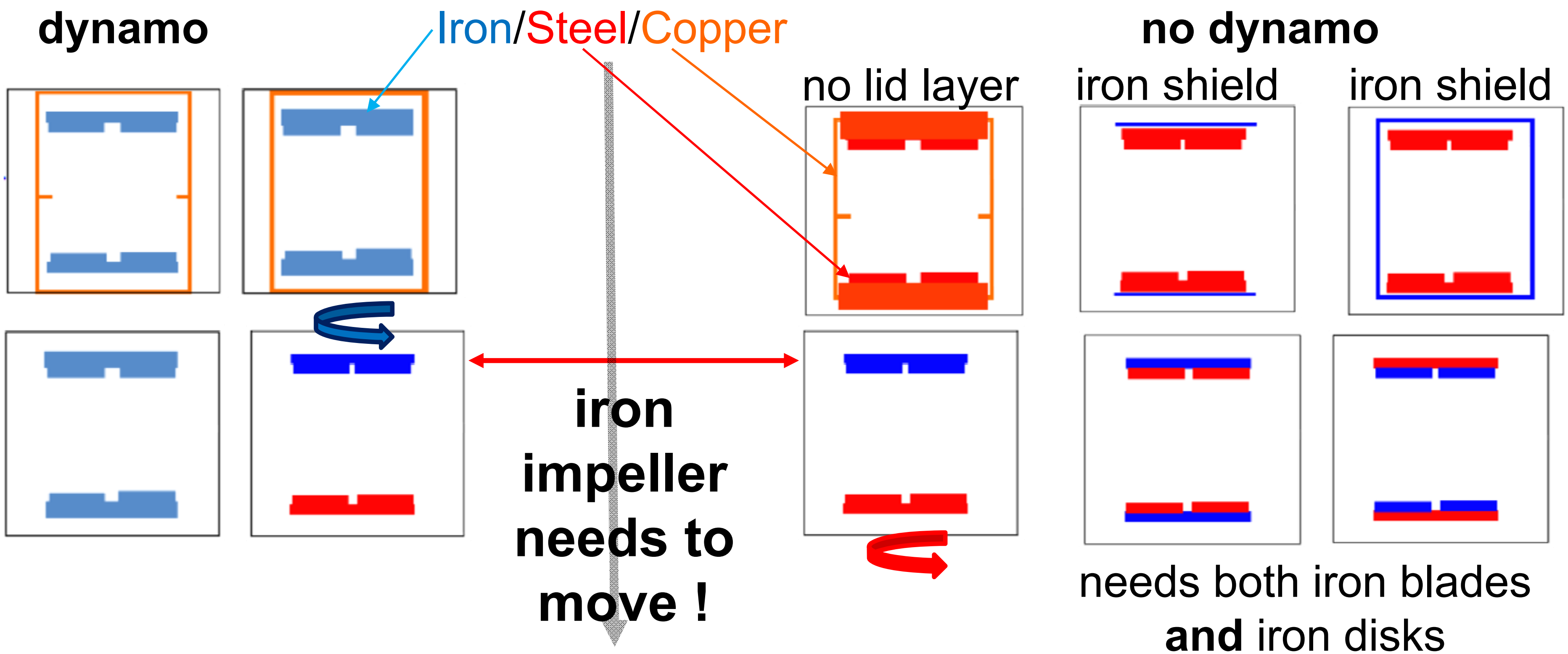
μ jump: induction in different cylindrical geometries

σ or μ jump in a von Kármán geometry: Ohmic decay

σ or μ jump in a von Kármán geometry: kinematic dynamo

Conclusions

Results up to June 2010 of the VKS experiment (from S. Aumaitre)



❖*What is the role of boundary conditions: ferromagnetic (soft iron) impellers against non-ferromagnetic (steel) impellers?*
ferromagnetic impellers lower the threshold but NOT ONLY

❖Need new algorithms dedicated to jump in magnetic permeabilities
❖Start with a kinematic code with axisymmetric velocity and compare TWO different algorithms (SFEMaNS, Orsay – FV/BEM, Dresden)

H-Φ formulation for Maxwell equations

$$\mu^c \partial_t \vec{H}^c = -\nabla \times \vec{E}^c \text{ in } \Omega_c$$

$$\nabla \times \vec{H}^c = \sigma(\vec{E}^c + \vec{U} \times \mu^c \vec{H}^c) \text{ in } \Omega_c$$

$$\vec{E}^c \times \vec{n}^c \Big|_{\Gamma_c} = \vec{a}$$

$$\vec{H}^c \Big|_{t=0} = \vec{H}_0^c$$

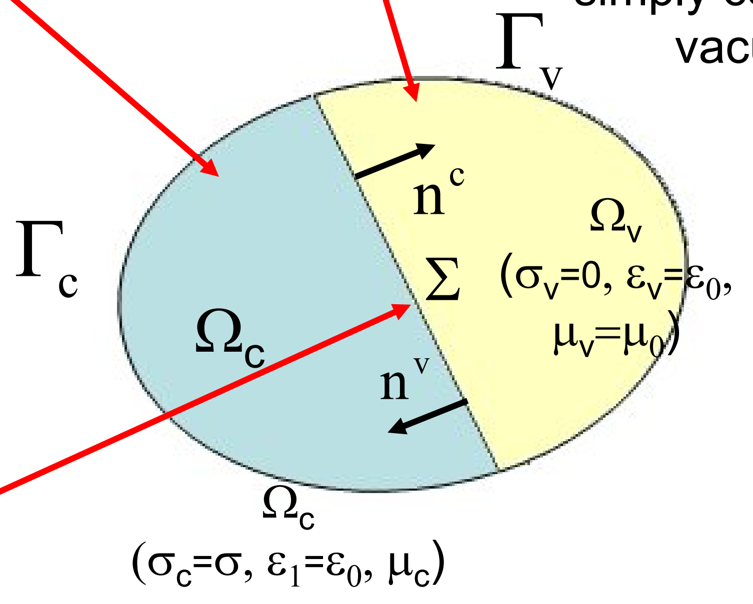
$$\mu^v \partial_t \nabla \phi = -\nabla \times \vec{E}^v \text{ in } \Omega_v$$

$$\nabla . \vec{E}^v = 0 \text{ in } \Omega_v$$

$$\vec{E}^v \times \vec{n}^v \Big|_{\Gamma_v} = \vec{a}$$

$$\vec{H} = \nabla \phi, \phi\Big|_{t=0} = \phi_0 \, (i.c.)$$

simply connected vacuum



$$\vec{H}^c \times \vec{n}^c + \nabla \phi \times \vec{n}^v = \vec{0} \text{ on } \Sigma$$

$$\vec{E}^c \times \vec{n}^c + \vec{E}^v \times \vec{n}^v = \vec{0} \text{ on } \Sigma$$

H-Φ formulation for Maxwell equations

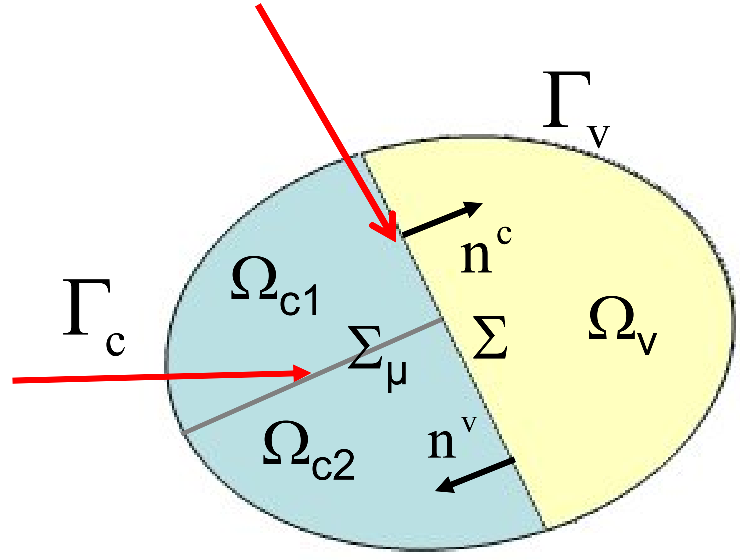
New algorithm to describe jump in magnetic permeabilities (Discontinuous Galerkin Method in weak formulation)

$$\vec{H}^c \times \vec{n}^c + \nabla\phi \times \vec{n}^v = \vec{0} \text{ on } \Sigma$$

$$\mu^c \vec{H}^c . \vec{n}^c + \mu^v \nabla\phi . \vec{n}^v = \vec{0} \text{ on } \Sigma$$

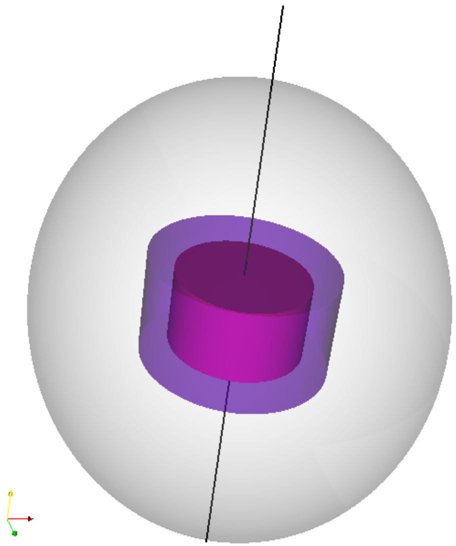
$$\vec{H}^{c1} \times \vec{n}^{c1} + \vec{H}^{c2} \times \vec{n}^{c2} = \vec{0} \text{ on } \Sigma_\mu$$

$$\vec{E}^{c1} \times \vec{n}^{c1} + \vec{E}^{c2} \times \vec{n}^{c2} = \vec{0} \text{ on } \Sigma_\mu$$



SFEMaNS code = Spectral + FEM method

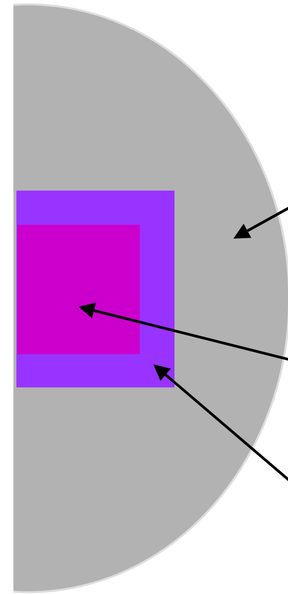
3D initial problem with axisym. interfaces



Cylindrical coordinates (r,θ,z)

Fourier decomposition in the azimuthal direction with azimuthal **m**

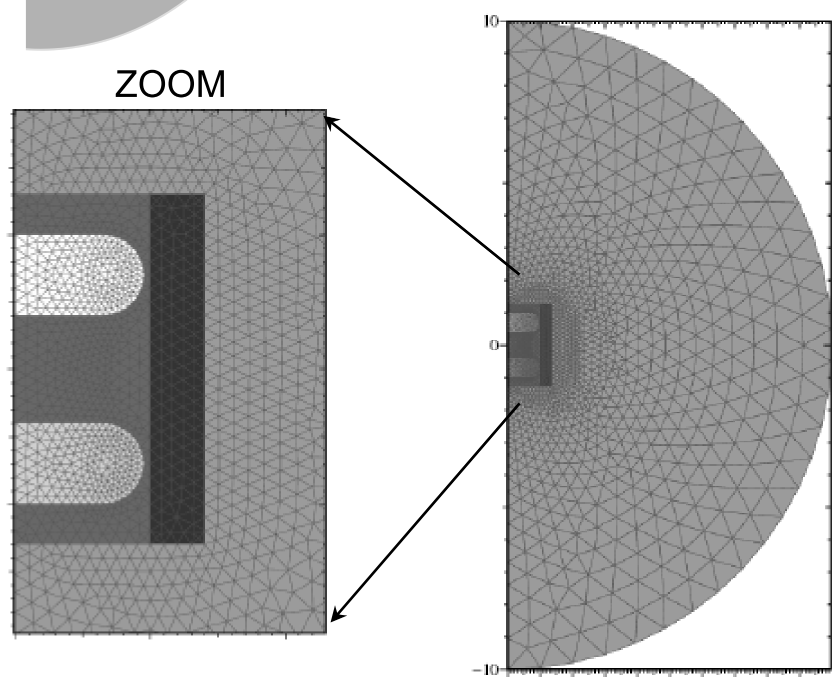
Finite element method in meridian plane



Insulating domain $\sigma = 0, \mu$ $\vec{H} = \nabla \phi$

Conducting domain 1 σ_1, μ_1

Conducting domain 2 σ_2, μ_2



Guermond et al., JCP, 2007, 2009

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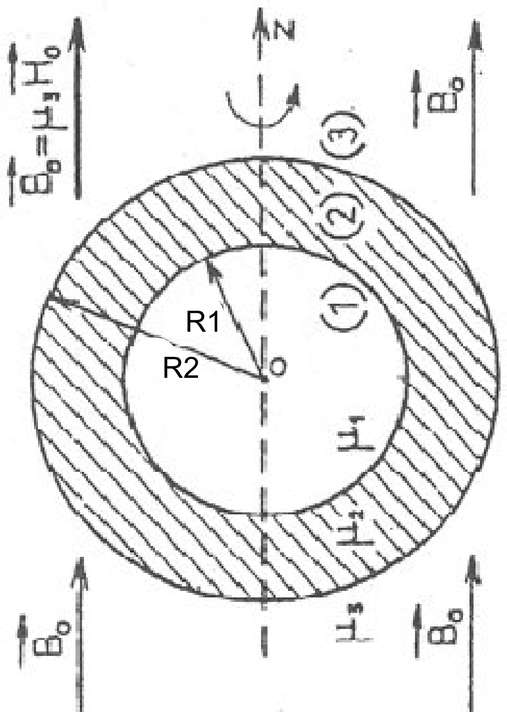
μ jump: analytical solution of induction in a composite sphere



Durand, Magnétostatique, 1968

∇xH=0 and ∇.(μ H)=0 –> potential H=∇ψ

Steady solution

$$\psi(\varrho,\theta,\vartheta) = \begin{cases} -A\rho\cos\vartheta, & \text{for } \varrho \le R_1 \\ -\left(B\rho + C\dfrac{R_1^3}{\rho^2}\right)\cos\vartheta & \text{for } R_1 \le \varrho \le R_2 \\ -\left(D\dfrac{R_1^3}{\rho^2} - H_0\rho\right)\cos\vartheta & \text{for } R_2 \le \varrho, \end{cases}$$

Expressions for A,B,C and D

for $\mu_1 = \mu_0$ and $\mu := \mu_2/\mu_0$

$$A = -\frac{9\mu H_0}{(2\mu+1)(\mu+2) - 2(\mu-1)^2\left(\frac{a}{b}\right)^3}$$

$$D = \frac{(2\mu+1)(\mu-1)\left[\left(\frac{b}{a}\right)^3 - 1\right]H_0}{(2\mu+1)(\mu+2) - 2(\mu-1)^2\left(\frac{a}{b}\right)^3}$$

$$B = \frac{1}{3}\left(2+\frac{1}{\mu}\right)A, \qquad C = \frac{1}{3}\left(1-\frac{1}{\mu}\right)A.$$

The magnetic field is given by:

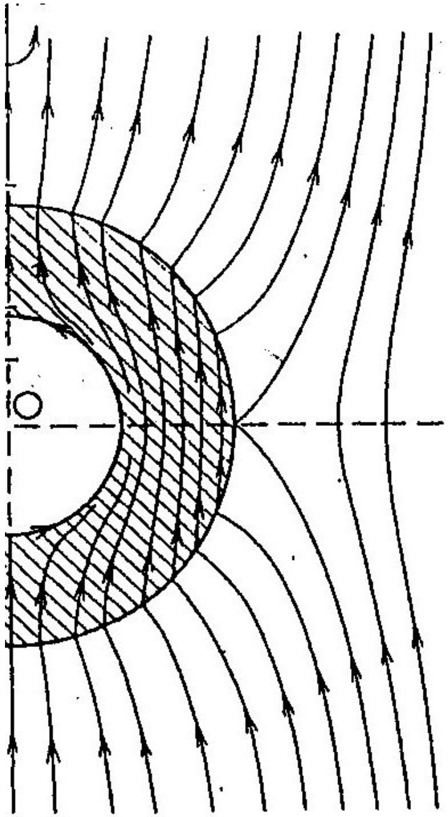
- **for ρ < R_1**

$$H_r = 0$$
$$H_\theta = 0$$
$$H_z = -A$$

- **for R_1< ρ < R_2**

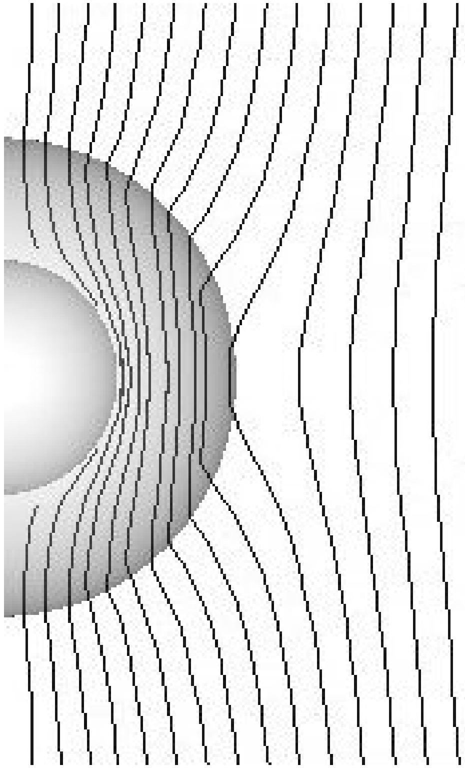
$$H_r = 3CR_1^3\, zr/\rho^5$$
$$H_\theta = 0$$
$$H_z = C\left(R_1/\rho\right)^3\left(3\frac{z^2}{\rho^2} - 1\right) - B$$

Magnetic induction B lines



Analytical magnetic induction lines

($\mu \rightarrow \infty$, Durand, Magnétostatique)



Numerical magnetic induction lines

(µ=100)

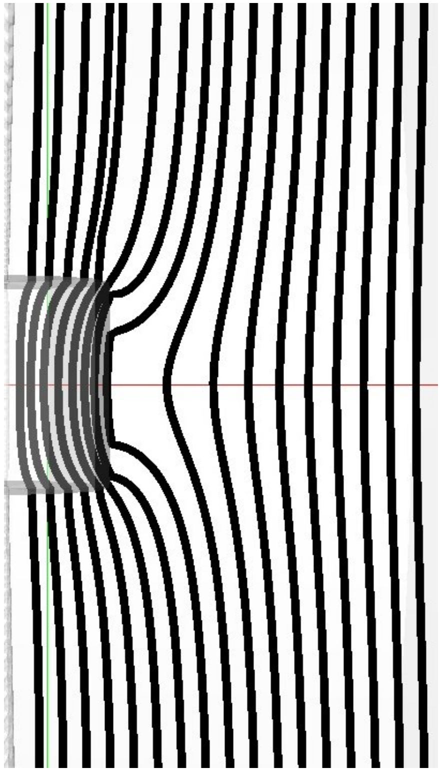
Magnetic induction lines are curved next to the ferromagnetic domain

VALIDATION by convergence towards steady solution

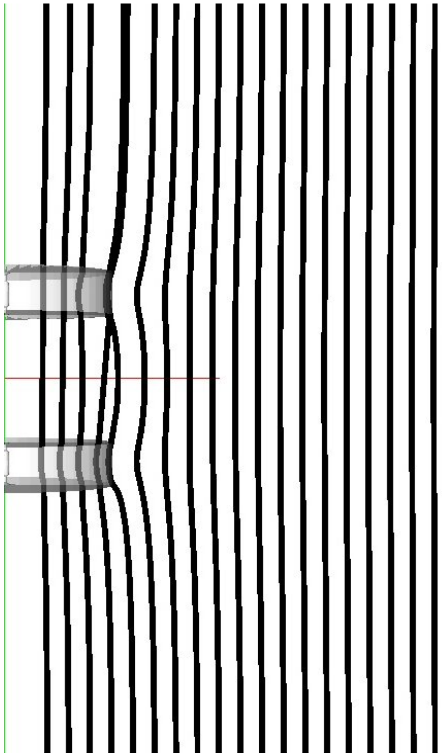
(need a lot of points on the interface HH)

μ jump: induction in different cylindrical geometries

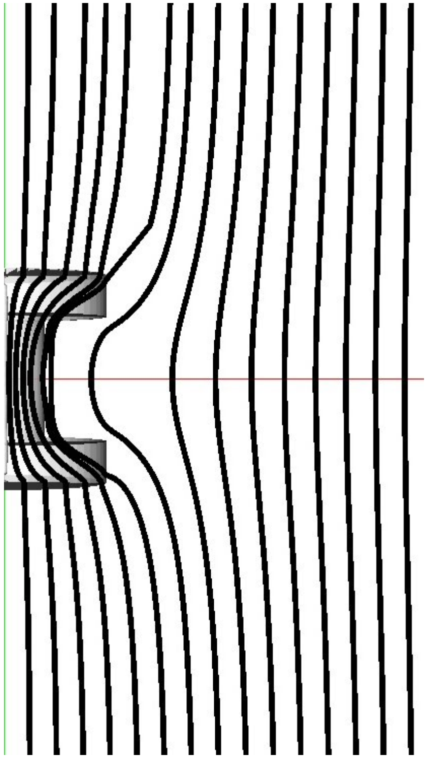
Magnetic induction B lines with μ =100



a) Full Cylinder



b) 2 disks



c) Reel geometry

➡ **The ferromagnetic material concentrates the magnetic field lines**

As μ → ∞ magnetic B lines connect perpendicularly

but different from the Hxn = 0 (pseudo-vacuum or VTF) condition

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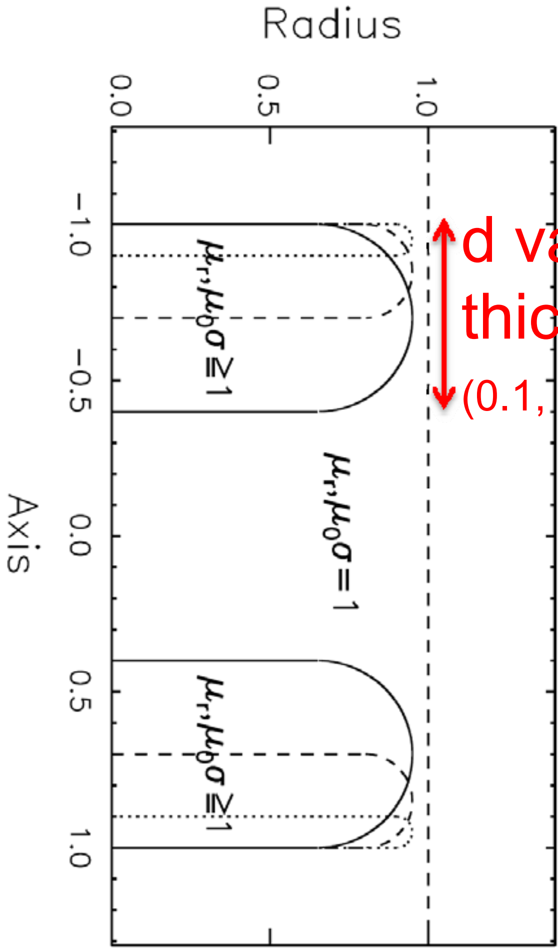


$$\frac{\partial \mu^c \boldsymbol{H}}{\partial t} = -\nabla \times \left(\frac{\nabla \times \boldsymbol{H}}{\sigma} \right)$$

σ jump: continuity of H

(tangential and normal components)

≠

μ jump: continuity of H_{tang} and

discontinuity of H_n

$\Downarrow$

Vacuum around cylinder

$$\mu_r^{\text{eff}} = \frac{1}{V} \int_V \mu_r(\boldsymbol{r}) dV \qquad \sigma^{\text{eff}} = \frac{1}{V} \int_V \sigma(\boldsymbol{r}) dV$$

$$\frac{\mu_r^{\text{eff}}}{\mu} - 1 = (\mu_r - 1) \frac{2\pi R_d^2 d}{V_{\text{cyl}}}$$

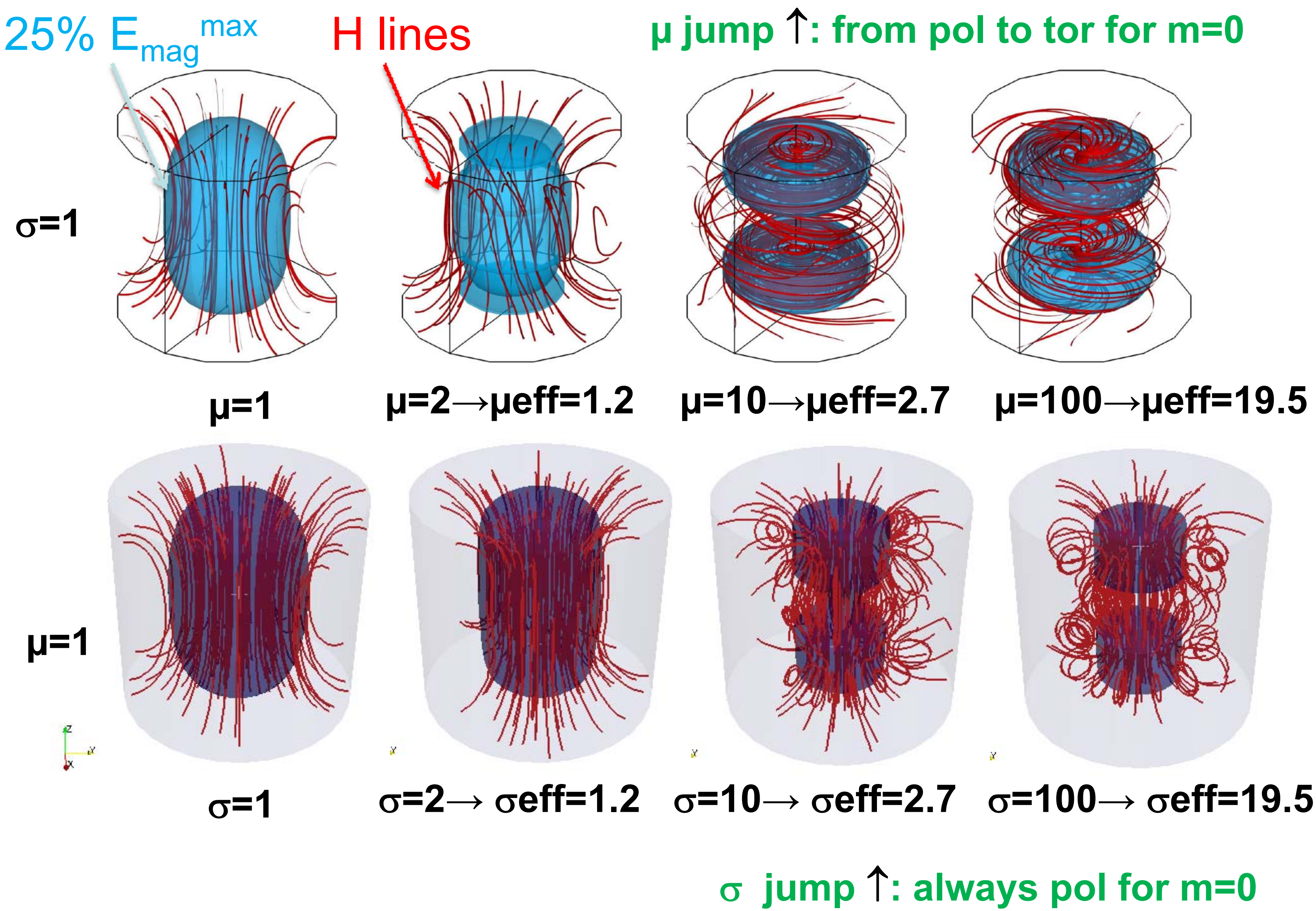
comparison between 2 codes

FV/BEM (Giesecke et al, 08)

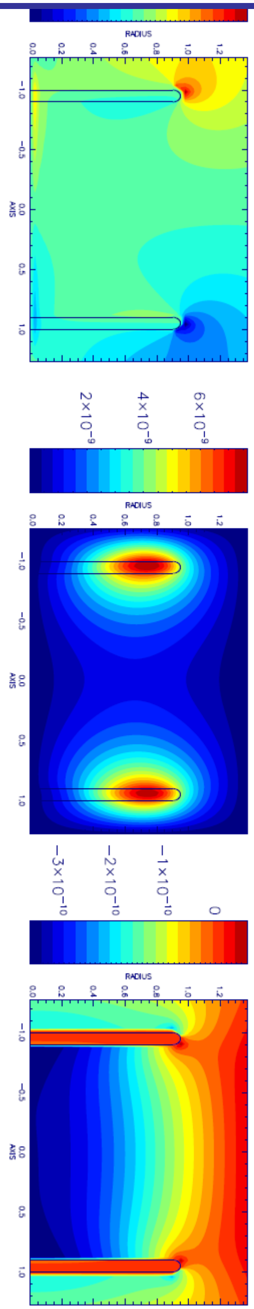
SFEMaNS

Giesecke, Nore et al, GAFD, 2010

σ or μ jump : Ohmic decay for d=0.6 and m=0



Ohmic decay for d=0.1 and m=0

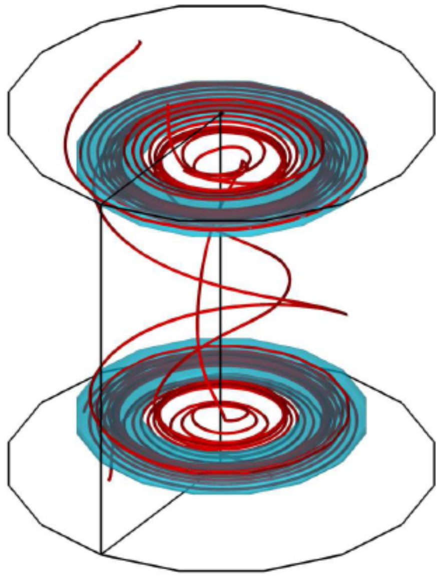


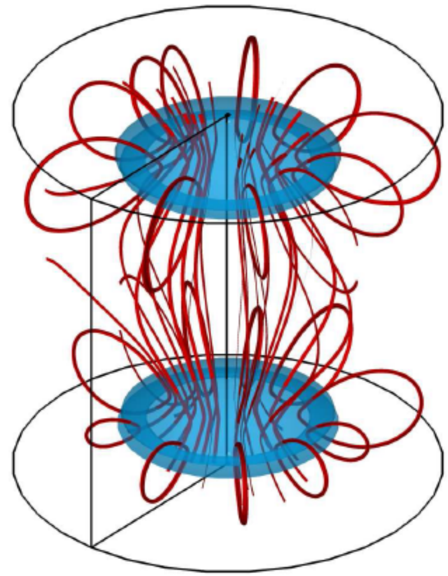
H_r

H_θ

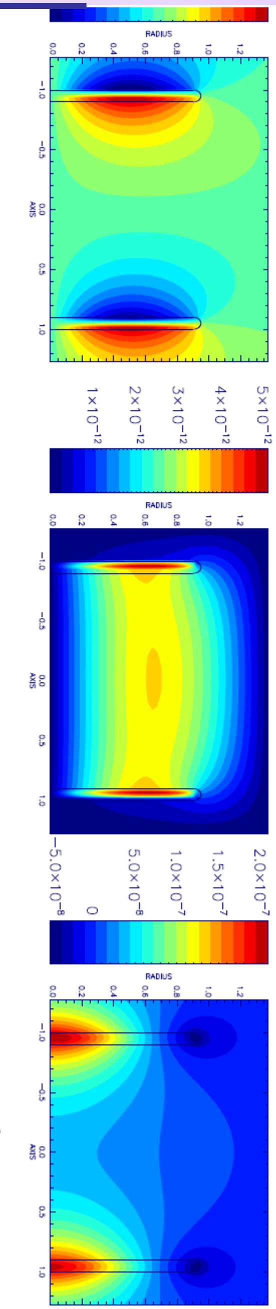
H_z

tor m=0 dominates **pol m=0 dominates**





μ=100→μeff=4.4 **σ=100→ σeff=4.4**



H_r

H_θ

H_z

pol m=0

$$H_r(r,z,t) = H_r(r,z)e^{\gamma_{pol}t}$$
$$H_z(r,z,t) = H_z(r,z)e^{\gamma_{pol}t}$$

tor m=0

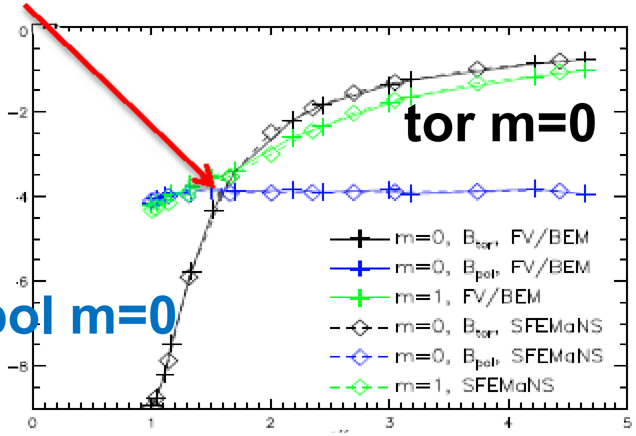
$$H_\theta(r,z,t) = H_\theta(r,z)e^{\gamma_{tor}t}$$

m=1

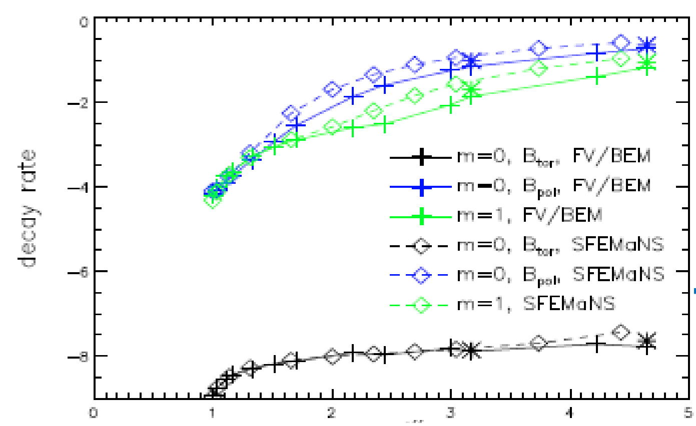
$$\boldsymbol{H}(r,\theta,z,t) = \boldsymbol{H}(r,\theta,z)e^{\gamma_{m=1}t}$$

mode crossing for m=0 at **μeff=1.5** (μ≈20 for d=0.1)

decay rate γ



μeff



σeff

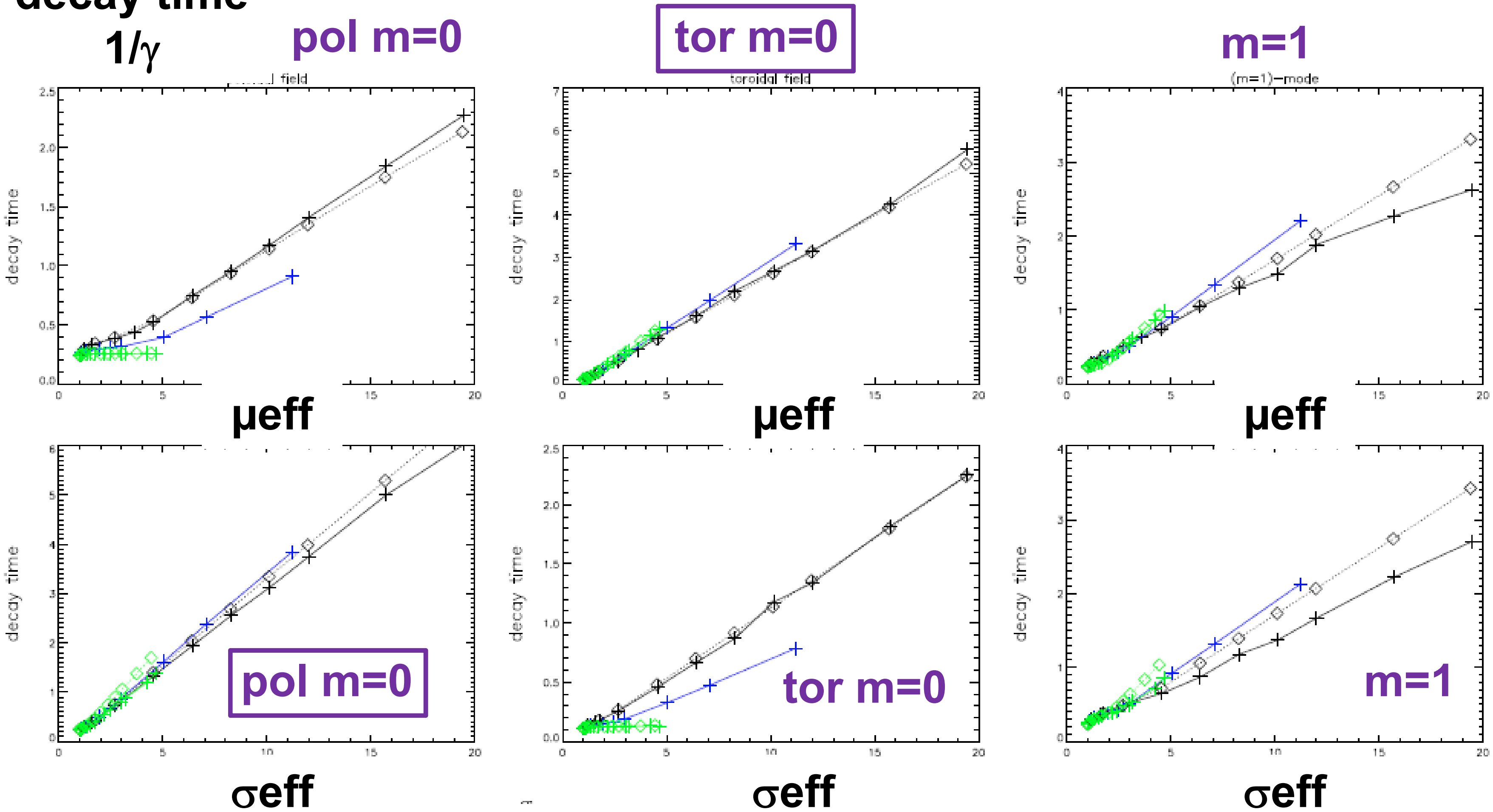
pol m=0 always for σ jump

$$\frac{\mu_r^{eff}}{\mu} - 1 = (\mu_r - 1)\frac{2\pi R_d^2 d}{V_{cyl}}$$

μeff or σeff are proportional to d

Ohmic decay for d=0.1, 0.3, 0.6

decay time 1/γ



d=0.1
d=0.3
d=0.6

scaling laws: decay time 1/|γ| ∝ cμeff or ∝ c σeff

μ or σ ↑, decay time ↑

variations with d → not only disk volume matters

σ or μ jump in a von Kármán geometry: kinematic dynamo

$$\frac{\partial \mu^c \boldsymbol{H}}{\partial t} = \nabla \times (\boldsymbol{u} \times \mu^c \boldsymbol{H}) - \nabla \times \left(\frac{\nabla \times \boldsymbol{H}}{\sigma} \right)$$

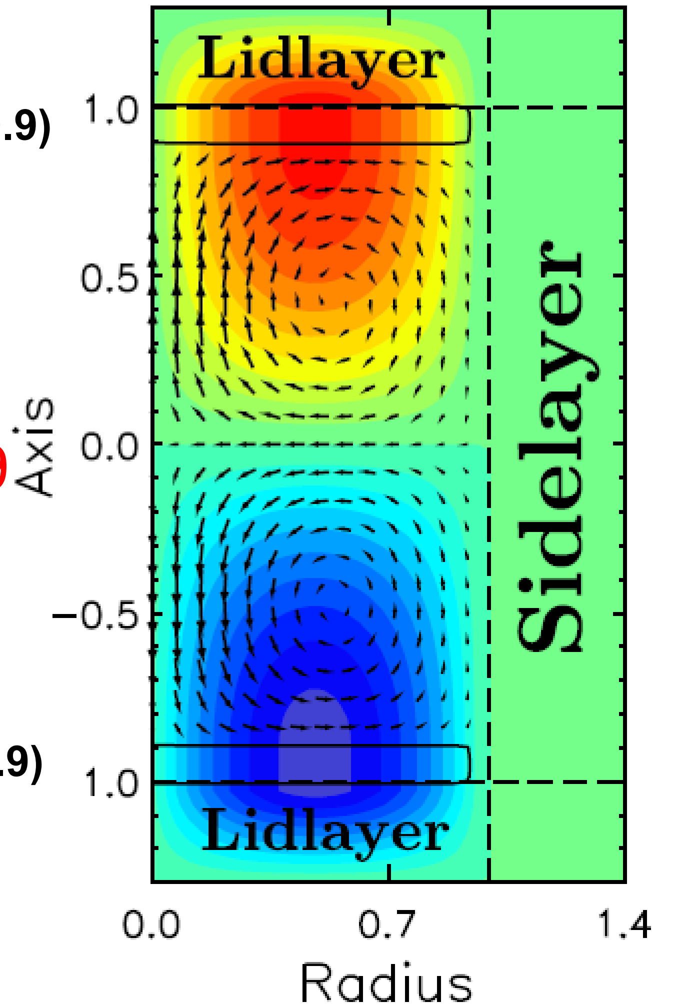
disk velocity= **U_θ(r,z=0.9)**
from MND

$Rm^{fluid} = \mu_0\, \sigma_0\, U\, R$

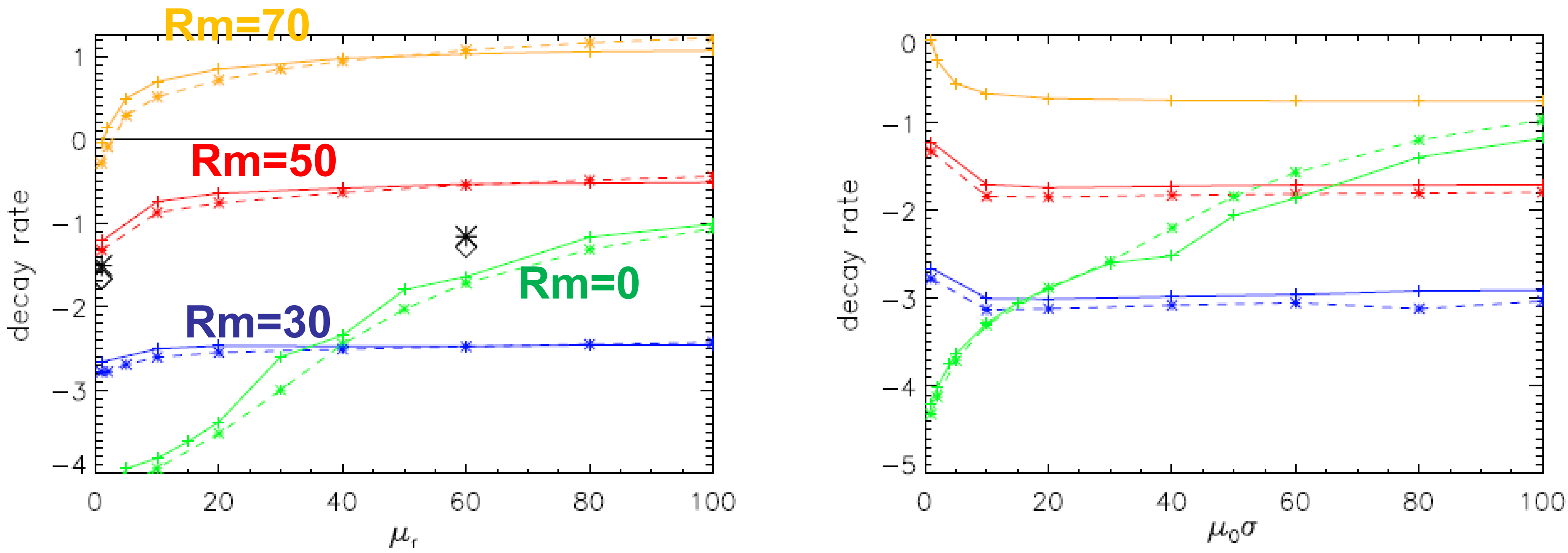
MND flow between $0 \le r \le 1$, $-0.9 \le z \le 0.9$

lid-layer U_θ linear interpolation

disk velocity= **U_θ(r,z=-0.9)**
from MND



Kinematic dynamo (m=1) with MND

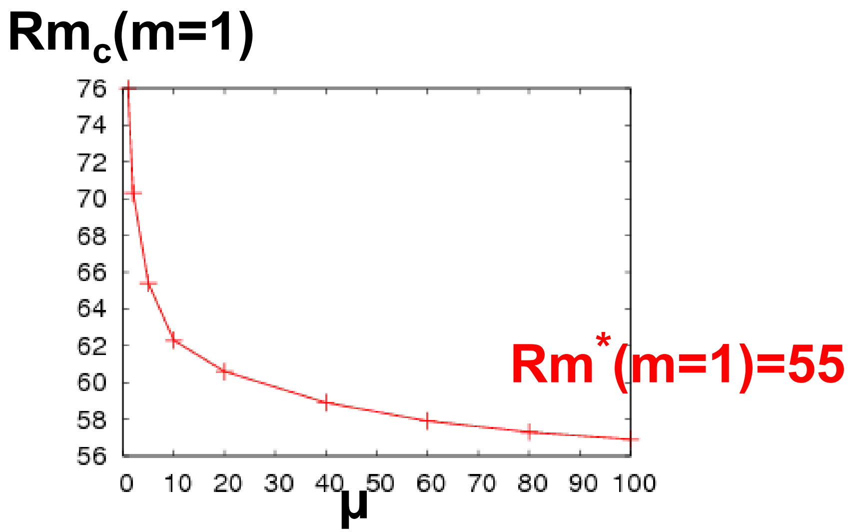


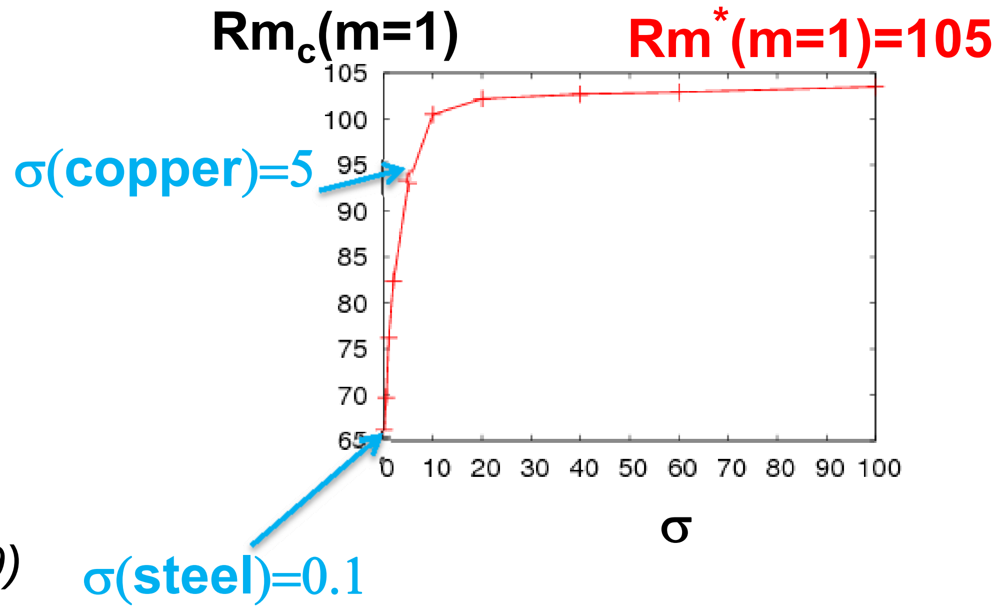
U MND axi → dynamo for H(m=1)

reduction of Rm_c(m=1) with **μ jump** ↑: Rm_c(μ=1)≈76 vs Rm_c(μ=100) ≈57

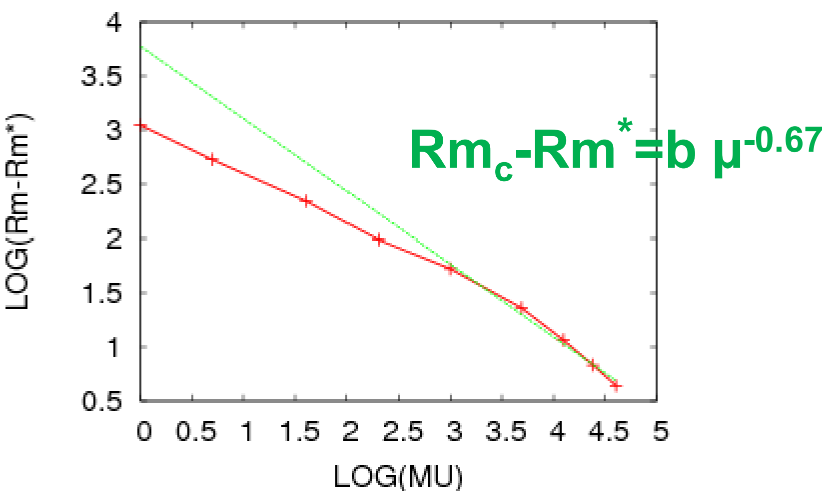
(with Hxn=0 condition and side-layer of 0.4, Rm_c(m=1)=39)

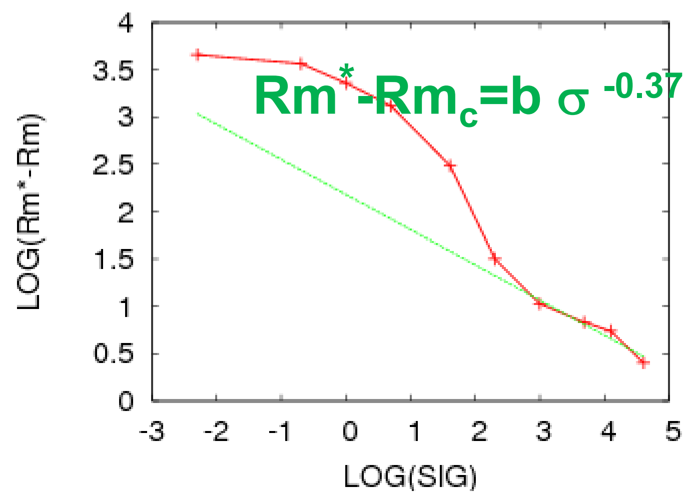
increase of Rm_c(m=1) with **σ jump** ↑ : Rm_c(σ =0.1)≈66 vs Rm_c(σ =100) ≈104





(NB: with Hxn=0 condition, $Rm_c(m=1)=39$)

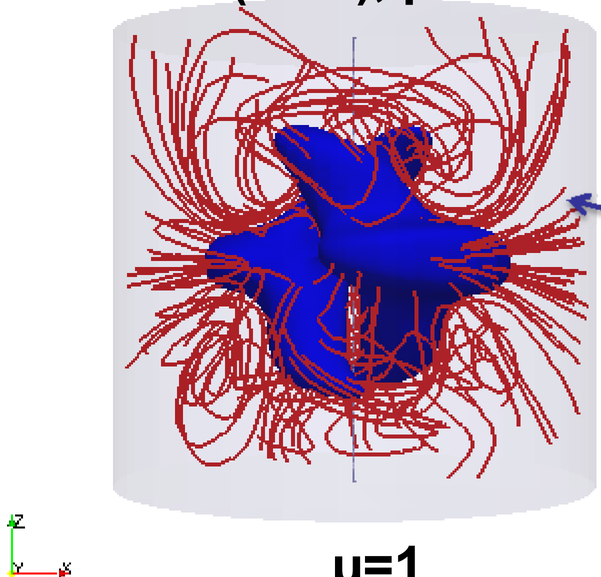




Conclusion: better to use ferromagnetic disks than steel or copper disks for the m=1 mode

Kinematic dynamo (m=1) with MND

H(m=1), μ=1



μ=1

Rm=50

$25\%\ E_{mag}^{max}$

H lines

j lines

$25\%\ j^{2\ max}$

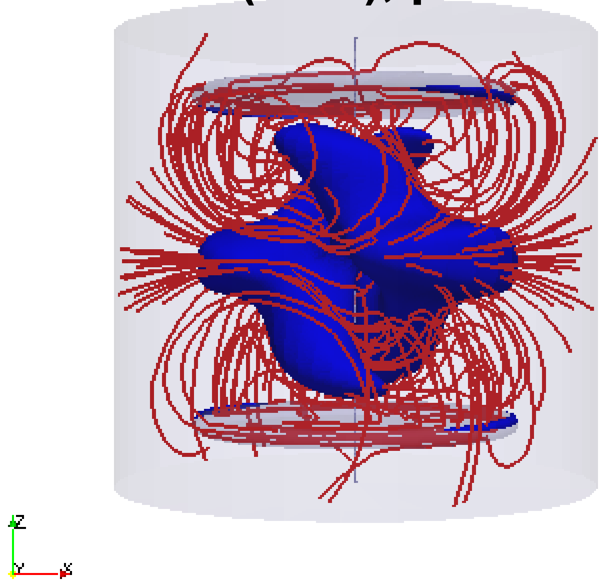
j(m=1) concentrated

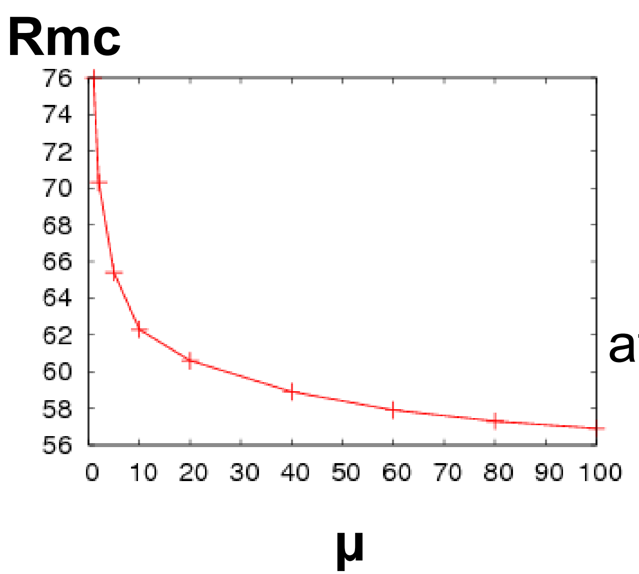
between the disks and

at the surface of the disks

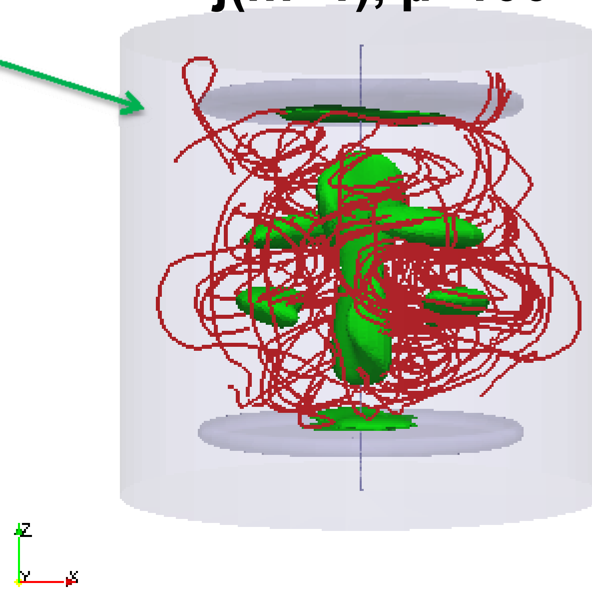
when **μ jump** ↑

H(m=1), μ=100



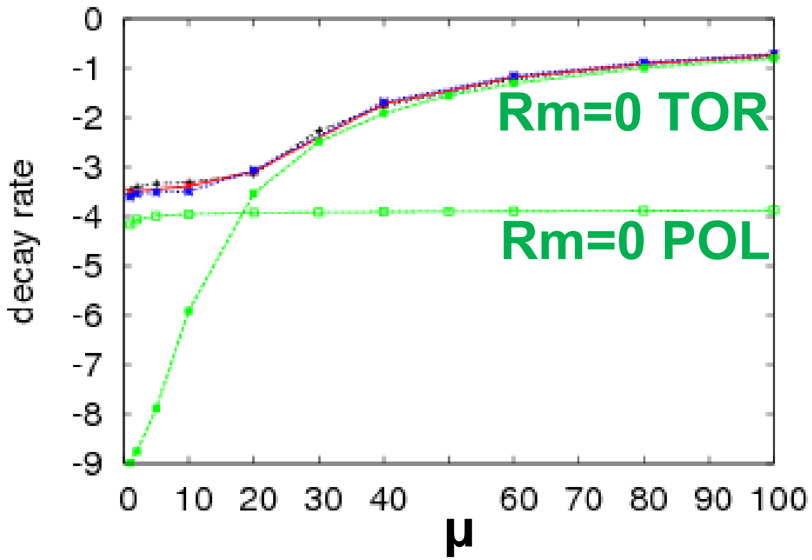


j(m=1), μ=100

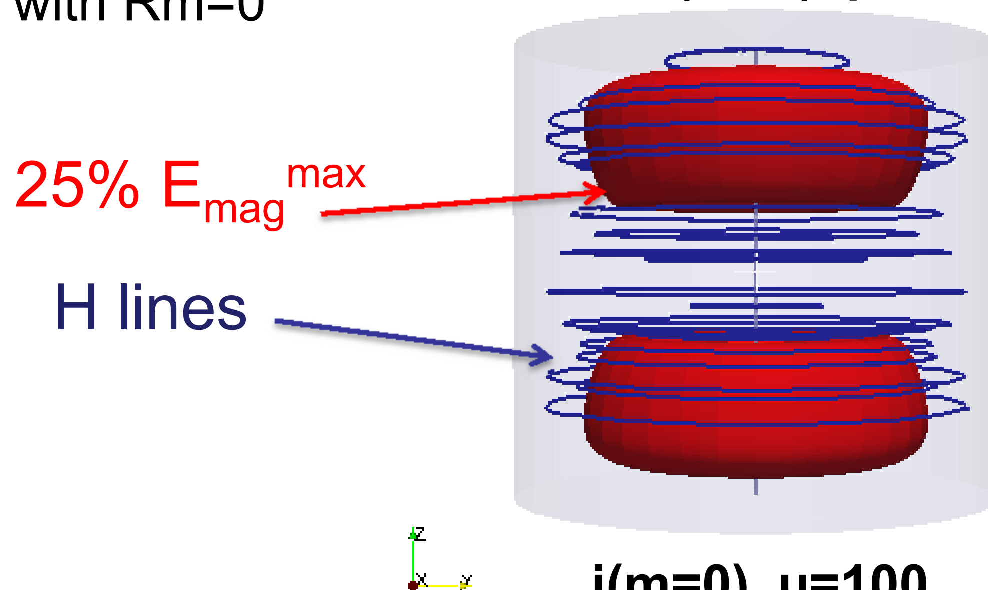


Kinematic runs (m=0) with MND

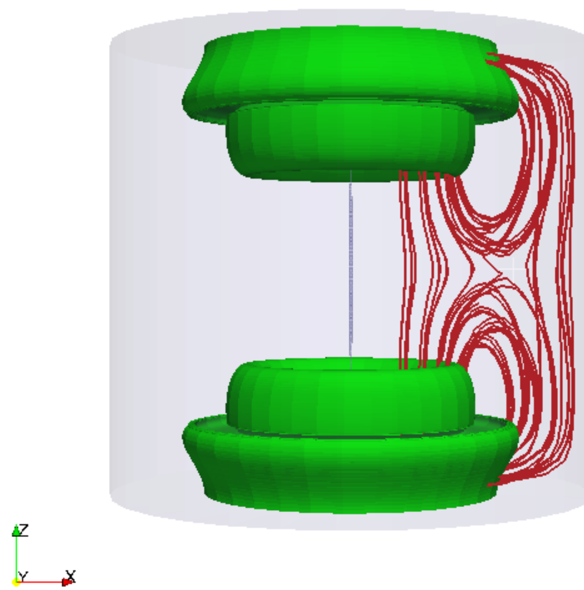
decay rates for Rm=30,50,70 merge with Rm=0



Rm=50,H(m=0), μ=100



j(m=0), μ=100



❖ decay rate decreases with **μ jump** ↑

$|\gamma|(\mu=1, Rm=50)\approx 3.3$ to $|\gamma|(\mu=100, Rm=50)\approx 0.7$

❖ same decay rates for Rm=30, 50, 70 →

tor comp. dominates → 1st step of α-Ω mechanism of

Pétrélis et al. 2007, Laguerre et al. 2008 ?

flow has little influence → need only a small asymmetry

to loop the mechanism?

Summary of σ or μ jump results

- Ohmic decay
  - σeff or μeff variations lead to different H geometries:
    it is not the same to vary σ or μ (geom. constraints)

  - 'loose' scaling laws : decay time $1/|\gamma| \propto c\mu eff$ or $\propto c\,\sigma eff$
    μ or σ ↑, decay time ↑ for m=0 and m=1 modes,
    variations with d → not only disk volume matters

- Kinematic dynamo with MND
  - reduction of $Rm_c(m=1)$ with **μ jump** ↑
  - increase of $Rm_c(m=1)$ with **σ jump** ↑

  → not only magnetic diffusivity 1/(σeff μeff) matters
  - toroidal m=0 mode enhanced by μ and located in disks

  → enhancement of efficiency of α-Ω mechanism ?

- What next? Add α effect (need helicity measurements) ?

  Nonlinear computations (need comparisons with real flow) ?

Navier-Stokes equations

$$\frac{\partial \vec{U}}{\partial t} + (\vec{U}.\nabla)\vec{U} = -\nabla p + \nu \Delta \vec{U} + (\nabla \times \vec{H}) \times \mu \vec{H} \ \text{ in } \ \Omega_c$$

Lorentz Force

$$\nabla . \vec{U} = 0 \text{ in } \Omega_c$$

$$\vec{U}\Big|_{\partial\Omega_c} = f \ \ (\text{b.c.}) \qquad \vec{U}\Big|_{t=0} = \vec{U}_0 \ (\text{i.c.})$$

Maxwell equations (eddy current approximation)

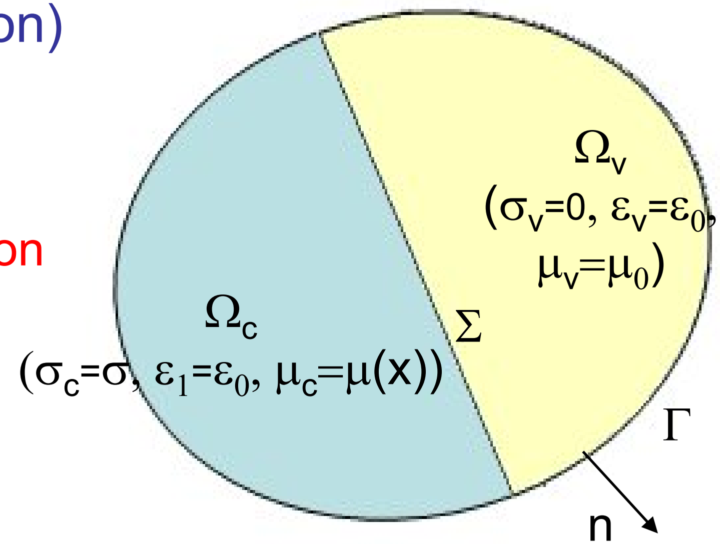
$$\mu \frac{\partial \vec{H}}{\partial t} = -\nabla \times \vec{E} \text{ in } \Omega, \vec{B} = \mu \vec{H}$$

Magnetic induction

$$\vec{j} = \nabla \times \vec{H} = \sigma(\vec{E} + \vec{U} \times \mu \vec{H}) \text{ in } \Omega_c$$

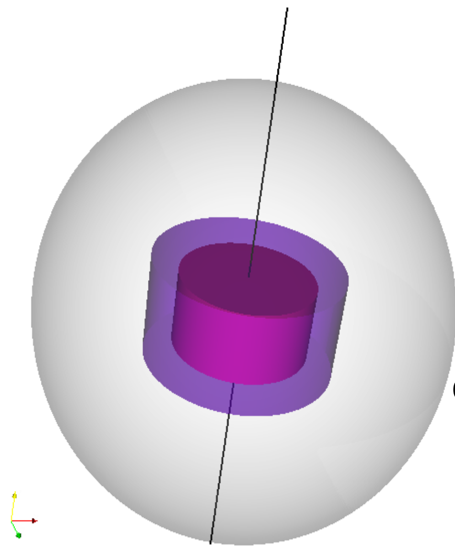
$$\nabla \times \vec{H} = 0 \text{ in } \Omega_v$$

$$\vec{H}\Big|_{t=0} = \vec{H}_0 + (\text{b.c.})$$



Ω_c – *conducting domain*

Ω_v – *insulating domain*



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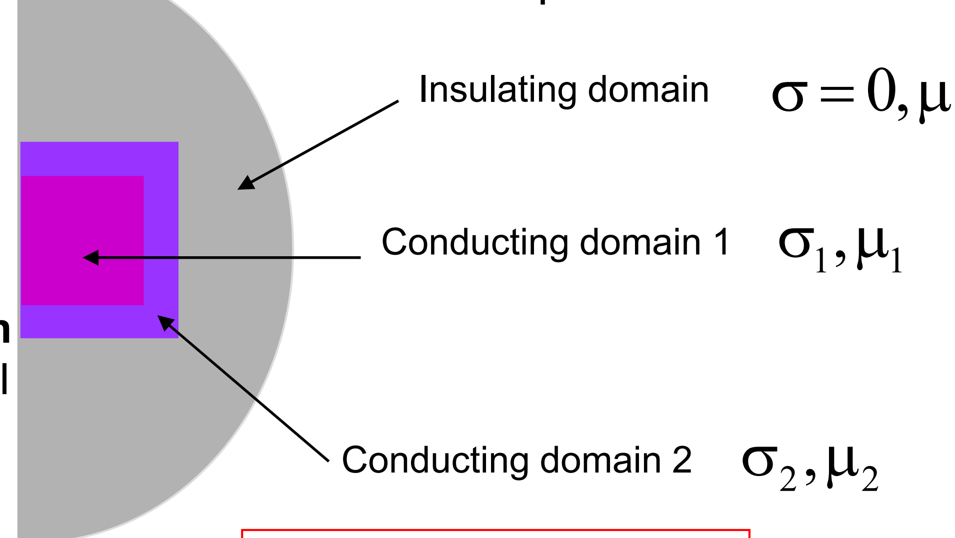
Fourier decomposition in the azimuthal direction with azimuthal **m**

Finite element method in meridian plane



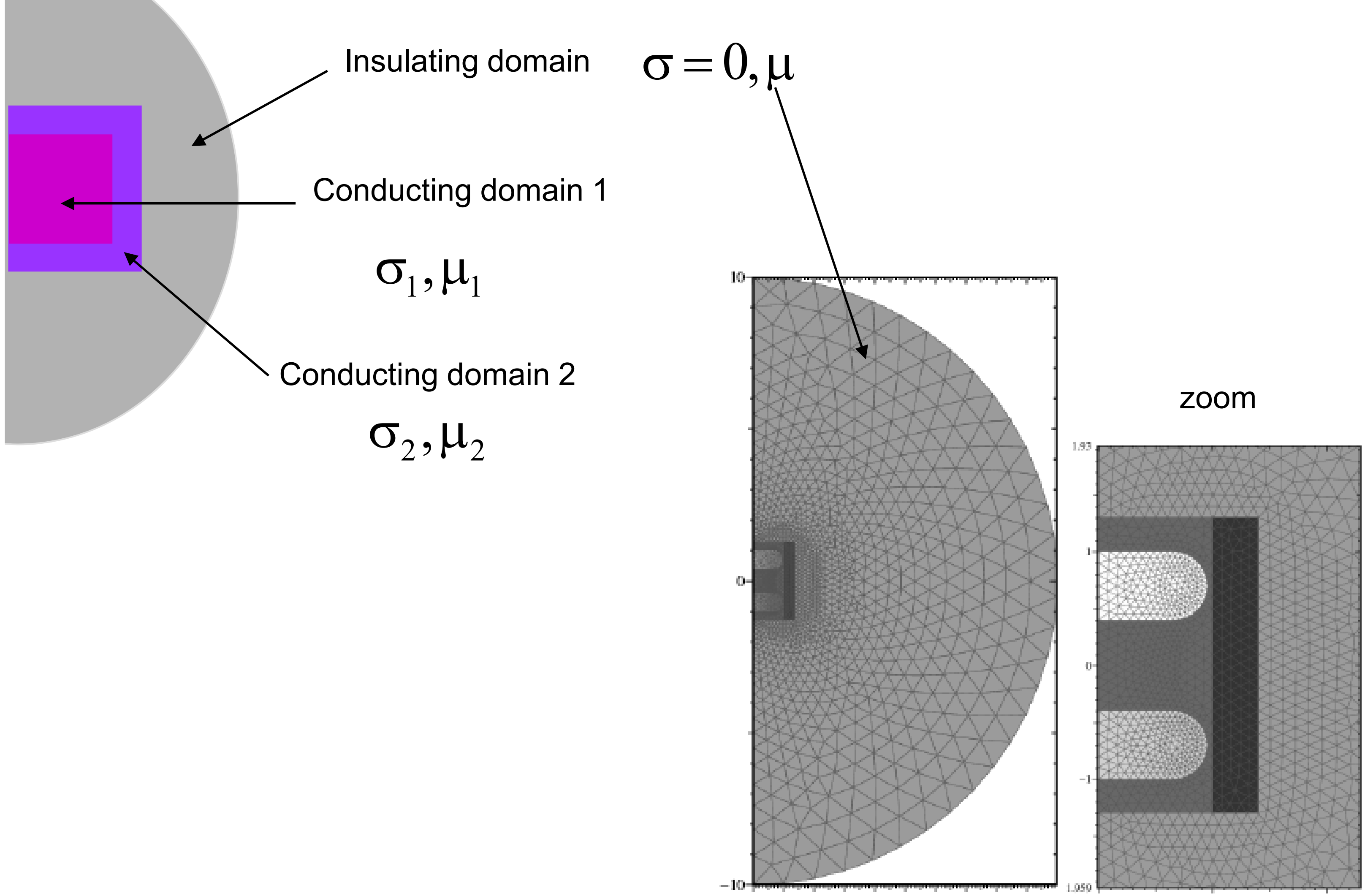
3D initial problem

Spectral-FEM method

- In the insulating domain, the magnetic field is written as the gradient of a **scalar potential** (simply connected vacuum) $\vec{H} = \nabla \phi$
- **Lagrange triangular finite elements** :
  P1 for the pressure and magnetic field in the conducting domain
  P2 for the velocity in the conducting domain and the scalar potential in the insulating domain
- Continuity of the magnetic field on the conducting/insulating domain interface is imposed using a **Penalty method (exact continuity conditions)**

Guermond et al., JCP, 2007, 2009

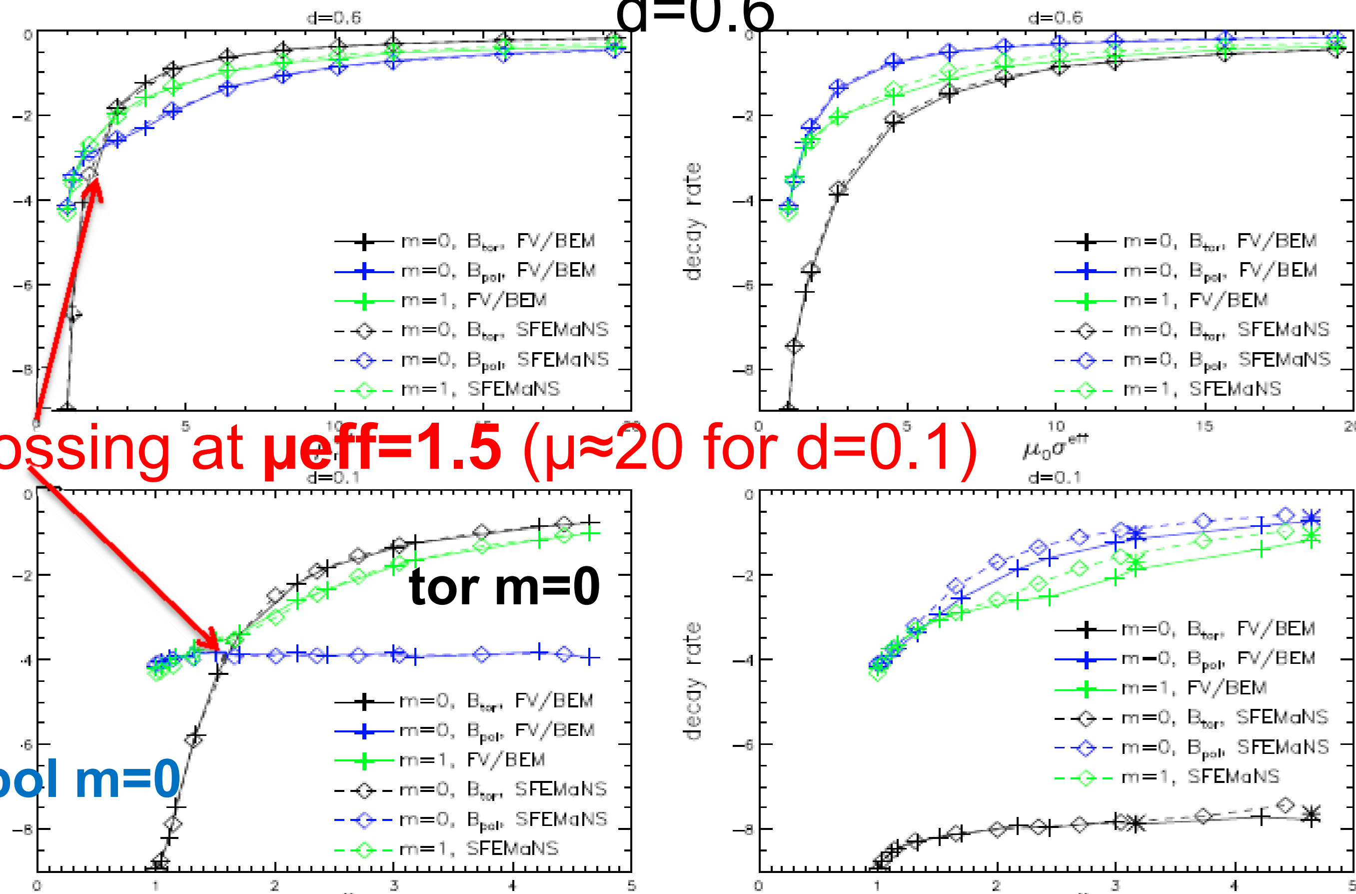
Finite element method
in meridian plane



Ohmic decay for d=0.1, 0.6

d=0.6

decay rate γ



mode crossing at **μeff=1.5** (μ≈20 for d=0.1)

tor m=0

pol m=0 always for σ jump

decay rate γ

pol m=0

μeff d=0.1 **σeff**

pol m=0

$$H_r(r,z,t) = H_r(r,z)e^{\gamma_{pol}t}$$
$$H_z(r,z,t) = H_z(r,z)e^{\gamma_{pol}t}$$

tor m=0

$$H_\theta(r,z,t) = H_\theta(r,z)e^{\gamma_{tor}t}$$

m=1

$$\boldsymbol{H}(r,\theta,z,t) = \boldsymbol{H}(r,\theta,z)e^{\gamma_{m=1}t}$$

Find $(\vec{H}^c, \phi)$ in $(L^2(]0,T[; H_{curl}(\Omega_c)^d) \times L^2(]0,T[; H^1(\Omega_v))$ such that

$\forall \vec{b} \in H_{curl}(\Omega_c)^d$ and $\forall \psi \in H^1(\Omega_v); \vec{b} \times \vec{n}^c + \nabla \psi \times \vec{n}^v |\Sigma = 0$

$$(\mu^c \partial_t \vec{H}^c, \vec{b})_{\Omega_c} + (\frac{1}{\sigma} \nabla \times \vec{H}^c, \nabla \times \vec{b})_{\Omega_c} + (\mu^v \partial_t \nabla \phi, \nabla \psi)_{\Omega_v} +$$

$$Consis(\vec{H}^c, \vec{b}, \psi) = (\vec{U} \times \mu^c \vec{H}^c, \nabla \times \vec{b})_{\Omega_c} + CL(\vec{a}, \vec{b}, \psi)$$

Integration by parts

$$Consis(\vec{H}^c, \vec{b}, \psi) = \left[\frac{1}{\sigma}(\nabla \times \vec{H}^c - \sigma \vec{U} \times \mu^c \vec{H}^c), \vec{b} \times \vec{n}^c + \nabla \psi \times \vec{n}^v \right]_\Sigma$$

$$CL(\vec{a}, \vec{b}, \psi) = (\vec{a}, \vec{b})_{\Gamma_c} + (\vec{a}, \nabla \psi)_{\Gamma_v}, \text{ with given } \vec{a}$$

Weak formulation with penalty method (IPG)

Find $(\vec{H}^c, \phi)$ in $(L^2(]0,T[;C^0(\overline{\Omega_c})^d) \times L^2(]0,T[;C^0(\overline{\Omega_v}))$ such that

$\forall\, \vec{b} \in C^0(\overline{\Omega_c})^d$ and $\forall\, \psi \in C^0(\overline{\Omega_v})$ Continuity constraint is relaxed

$$(\mu^c \partial_t \vec{H}^c, \vec{b})_{\Omega_c} + (\frac{1}{\sigma} \nabla \times \vec{H}^c, \nabla \times \vec{b})_{\Omega_c} + (\mu^v \partial_t \nabla \phi, \nabla \psi)_{\Omega_v} +$$

$$Consis\,(\vec{H}^c, \vec{b}, \psi) + Penal\,(\vec{H}^c, \vec{b}, \phi, \psi) = (\vec{U} \times \mu^c \vec{H}^c, \nabla \times \vec{b})_{\Omega_c} + CL\,(\vec{a}, \vec{b}, \psi)$$

Integration by parts

$$\mathrm{Consis}\,(\vec{H}^c, \vec{b}, \psi) = \left[\frac{1}{\sigma}(\nabla \times \vec{H}^c - \sigma \vec{U} \times \mu^c \vec{H}^c), \vec{b} \times \vec{n}^c + \nabla \psi \times \vec{n}^v \right]_{\Sigma}$$

$$\mathrm{Penal}\,(\vec{H}^c, \vec{b}, \phi, \psi) = \frac{\beta}{h}(\vec{H}^c \times \vec{n}^c + \nabla \phi \times \vec{n}^v, \vec{b} \times \vec{n}^c + \nabla \psi \times \vec{n}^v)_{\Sigma}$$

➡ Continuity of tangential magnetic field at the conducting-vacuum interface weakly imposed

$$CL\,(\vec{a}, \vec{b}, \psi) = (\vec{a}, \vec{b})_{\Gamma_c} + (\vec{a}, \nabla \psi)_{\Gamma_v}, \text{ with given } \vec{a}$$

Conclusions

➢ Kinematic dynamo in a periodic cylinder: validation with Ponomarenko

➢ Alpha dynamo in VKS: role of the bladed impellers with Alpha-Omega dynamo, dynamo action is favoured by VTF BC

➢ Jump in permeabilities: infinite μ_r (VTF) ➡ soft iron, $\mu_r \approx 100$ some fundamental questions left (cf. new results in April 2010)

Prospects

❖ LES model for Navier-Stokes ($Re \approx 10^4$) + DNS for Maxwell ($Rm \approx 50$)

❖ Alfvén waves in a finite cylinder (LGIT, Grenoble)

❖ MRI in a finite cylinder

❖ Precessional spheroid