Convective dynamo in a stratified plane layer

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Outline

1. Introduction and the anelastic approximation
2. Weakly compressible convection at threshold
3. Influence of weak compressibility on stationary dynamo solutions at dynamo threshold
4. Derivation of nonlinear equations governing the evolution of rapidly rotating anelastic convection
5. Strongly compressible dynamo
6. Summary and conlusions

Governing equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\,\mathbf{u} = -\nabla\left(\frac{p}{\bar\rho}\right) + \mathcal{R}\sigma_m\sigma_\eta^{-1}s\hat{\mathbf{e}}_z - \tau^{1/2}\sigma_m\hat{\mathbf{e}}_z\times\mathbf{u} + M^2\sigma_m\frac{1}{\bar\rho}(\nabla\times\mathbf{B})\times\mathbf{B} + \sigma_m\frac{1}{\bar\rho}\nabla\cdot\widehat{\sigma}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}) + \nabla^2\mathbf{B}$$

$$\boxed{\sigma_{ij} = \bar\rho\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\nabla\cdot\mathbf{u}\delta_{ij}\right)}$$

$$\nabla\cdot(\bar\varrho\mathbf{u}) = 0 \qquad\qquad \nabla\cdot\mathbf{B} = 0$$

$$\bar\rho\bar T\left[\frac{\partial s}{\partial t} + \mathbf{u}\cdot\nabla(\bar s + s)\right] = \sigma_\eta^{-1}\nabla\cdot\left[\lambda(\bar\rho)\,\bar T\nabla(\bar s + s)\right] - \frac{\theta M^2\sigma_\eta}{\mathcal{R}}(\nabla\times\mathbf{B})^2 - \frac{\theta\sigma_\eta}{2\mathcal{R}}\frac{1}{\bar\rho}\widehat{\sigma}:\widehat{\sigma}$$

Braginsky & Roberts (1995)

Jones et al. (2009)

$$\frac{p}{\bar p} = \frac{T}{\bar T} + \frac{\rho}{\bar\rho} \qquad\qquad s = \frac{1}{\gamma}\frac{p}{\bar p} - \frac{\rho}{\bar\rho}$$

Constant molecular conductivity

Constant molecular diffusivity

case 1

$$\bar T = 1 + \theta z, \quad \bar\rho = (1+\theta z)^m$$
$$\bar p = -\frac{\mathcal{R}\sigma_m}{\sigma_\eta\theta(m+1)}(1+\theta z)^{m+1}$$
$$\bar s = \frac{m+1-\gamma m}{\gamma\epsilon}\,ln\,(1+\theta z) + const$$
$$\frac{m+1-\gamma m}{\gamma} = -\frac{\epsilon}{\theta} = O(\epsilon)$$

and

case 2

$$\bar T = 1 + \theta z + \epsilon\frac{\gamma-1}{\theta(\gamma-2)}\left[1 - (1+\theta z)^{\gamma-2/\gamma-1}\right]$$
$$\bar\rho = (1+\theta z)^{1/\gamma-1} + \epsilon\frac{1}{\theta(\gamma-2)}\left[(1+\theta z)^{2-\gamma/\gamma-1} - (\gamma-1)^2\right] \qquad (6)$$
$$\bar p = -\frac{\mathcal{R}\sigma_m(\gamma-1)}{\sigma_\eta\theta\gamma}\left\{(1+\theta z)^{\gamma/\gamma-1} + \epsilon\frac{\gamma(1+\theta z)}{\theta(\gamma-2)}\left[(1+\theta z)^{2-\gamma/\gamma-1} - (\gamma-1)\right]\right\}$$
$$\bar s = \frac{\gamma-1}{\theta}(1+\theta z)^{1/1-\gamma} + const$$

BASIC STATE

Parameters

$\mathcal{R} = \frac{g d^3 \epsilon}{\kappa \nu}$	Rayleigh number
$\sigma_m = \frac{\nu}{\eta}$	Prandtl number
$\sigma_\eta = \frac{\eta}{\kappa}$	diffusivity ratio
$T = \frac{4\Omega^2 d^4}{\nu^2}$	Taylor number
$M = \frac{B_r d}{\sqrt{\mu \varrho_r \nu \eta}}$	Hartmann number
$\theta = \frac{\triangle T}{T_0}$	temperature gradient in the basic state
m	polytropic index

scales:

$$t_D = \frac{d^2}{\eta} t \qquad \mathbf{x}_D = d\mathbf{x} \qquad \mathbf{u}_D = \frac{\eta}{d}\mathbf{u}$$

departure from adiabaticity:

$$\epsilon = -\frac{d}{T_r}\left[\left(\frac{d\bar{T}}{dz}\right)_r + \frac{g}{c_p}\right] = -\frac{d}{c_p}\left(\frac{d\bar{s}}{dz}\right)_r \ll 1$$

$$-1 < \theta = \frac{\triangle T}{T_0} < 0$$

$$\nabla \cdot \mathbf{u} = -\frac{m\theta}{1+\theta z} u_z$$

The exact Boussinesq limit is obtained by simply setting θ=0

The boundaries are assumed to be impermeabe, stress-free, perfectly conducting and isentropic

Weakly compressible ($|\theta|<<1$) convection at thereshold

$$\mathbf{u} = \hat{\mathbf{u}}(z)\, e^{i(k_1 x + k_2 y)} e^{\mu t}$$

$$\hat{\mathbf{u}} = \hat{\mathbf{u}}^0 + \theta \hat{\mathbf{u}}^1 + O\left(\theta^2\right)$$

Assuming additionally rapid rotation $\tau^{-1/12}<<\theta<<1$, and imposing the following BC:

$$\hat{u}_z\big|_{z=0,1} = 0 \qquad \hat{s}\big|_{z=0,1} = 0 \qquad \left.\frac{d\hat{\xi}}{dz}\right|_{z=0,1} = 0$$

where $\xi = \partial_x u_y - \partial_y u_x$ is the 'z' component of the vorticity

leads to:

$$\mathcal{R}^0 = \tau^{2/3} \tilde{\mathcal{R}}^0 \qquad k = \tau^{1/6} \tilde{k} \quad \text{and} \quad \tilde{\mathcal{R}}^0 = \frac{1}{\tilde{k}^2}\left[\tilde{k}^6 + n^2 \pi^2\right]$$

$$\boxed{\mathcal{R} \approx \tau^{2/3} \tilde{\mathcal{R}}^0 + \theta \tau^{2/3} \tilde{\mathcal{R}}^1}$$

case (1)

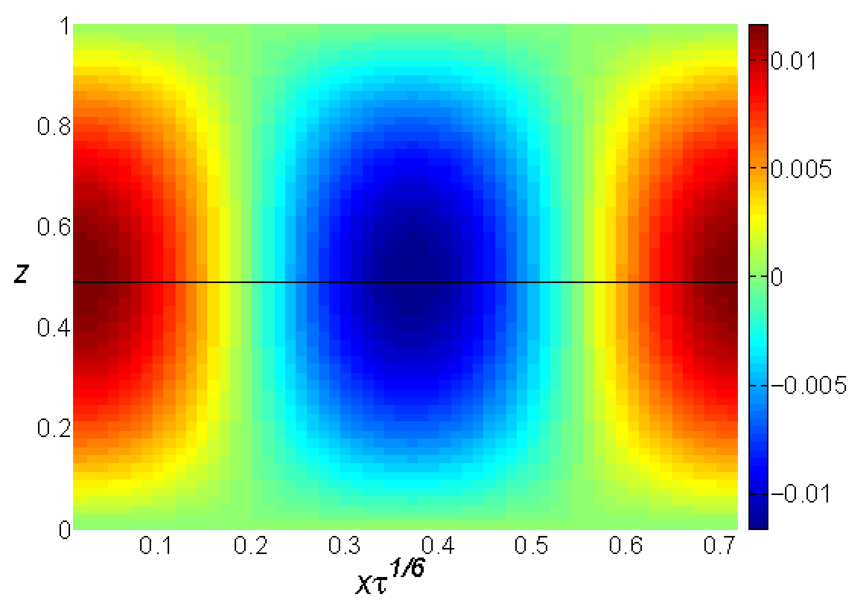
$$\boxed{\tilde{\mathcal{R}}^1 = -\frac{1}{2}(m-1)\tilde{\mathcal{R}}^0}$$

case (2)

$$\boxed{\tilde{\mathcal{R}}^1 = \frac{\gamma}{2(\gamma-1)}\tilde{\mathcal{R}}^0}$$

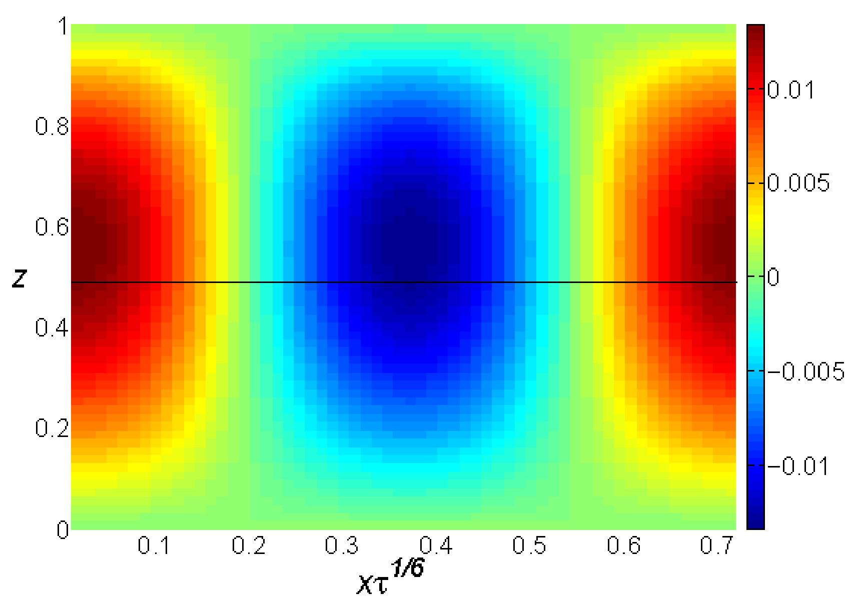
Boussinesq symmetry breaking
(introduction of smaller length scales)

Case 1 (constant k)



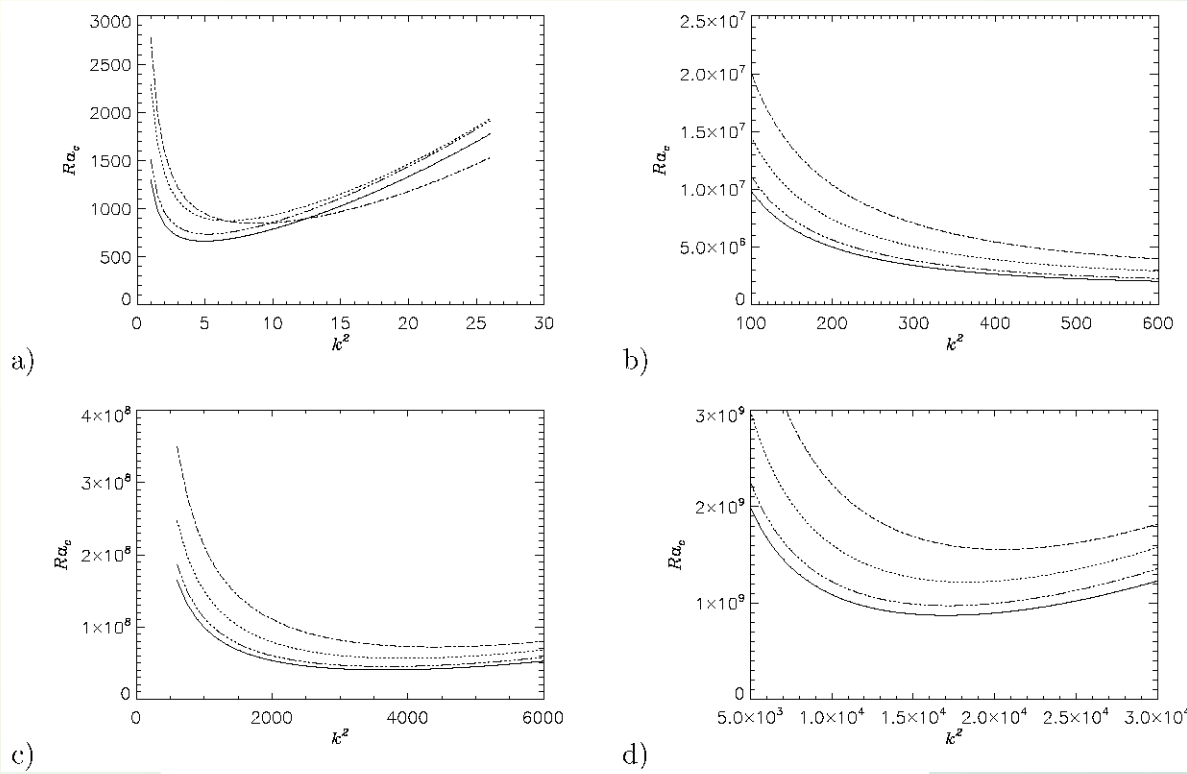
Small downward shift: $\Delta z \approx -0.0056$

Case 2 (constant κ)



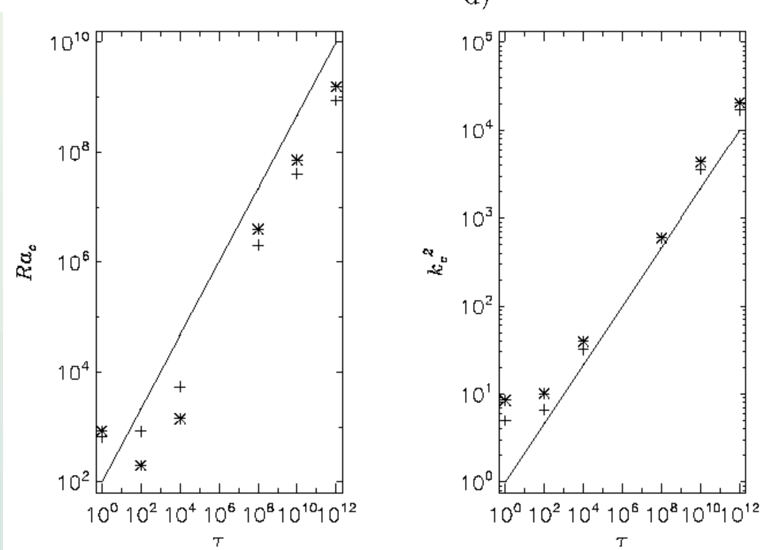
Upward shift: $\Delta z \approx 0.057$

Numerical linear results



Ra_{crit} vs k in case 1 for

θ from 0 to –0.99898



Test of anaytical scaling with the Taylor number $Ra_{crit}(\tau)$ and $k(\tau)$:

* for θ=0

+ for θ=-0.99

Weakly compressible, rapidly rotating $\tau^{-1/12} << \theta << 1$ dynamo at threshold

Defining: $\varepsilon = \tau^{-1/6}$ we introduce the following scaling $\mathbf{x} = (\varepsilon x,\ \varepsilon y,\ z)$

$$\mathbf{u} = \varepsilon^{-1/2}\left[\varepsilon^{3}\mathbf{U}_h(z,t) + \mathbf{u}'(\mathbf{x}_h,z,t)\right]$$

$$\mathbf{B} = \mathbf{B}_h(z,t) + \varepsilon^{1/2}\mathbf{B}'(\mathbf{x}_h,z,t)$$

$$s = \varepsilon\left[S(z,t) + \varepsilon^{1/2}s'(\mathbf{x}_h,z,t)\right]$$

Soward (1974) (Separation of horizontal averages)

This leads to:

$$\frac{\partial \mathbf{B}_h}{\partial t} = \frac{\partial}{\partial z}\left[\hat{\mathrm{e}}_z \times \langle \mathbf{u}' \times \mathbf{B}' \rangle\right] + \frac{\partial^2 \mathbf{B}_h}{\partial z^2}$$

We expand all the depended variables inpowers of $\varepsilon^{1/2}$ and obtain an equation for the Fourier amplitude of the zeroth-order vertical velocity $w = \hat{w}e^{i\mathbf{k}\cdot\mathbf{x}}$:

$$\frac{\partial^2 \hat{w}^0}{\partial z^2} + \frac{m\theta}{1+\theta z}\frac{\partial \hat{w}^0}{\partial z} + \left[\tilde{\mathcal{R}}k^2(1+\theta z)^{m-1} - k^6 - \frac{m\theta^2}{(1+\theta z)^2}\right]\hat{w}^0 = 0$$

which in the limit $\theta << 1$ has the same solution as the one obtained previously in the non-magnetic case

Weakly compressible, rapidly rotating $\tau^{-1/12}<<\theta<<1$ dynamo at threshold

To exclude a uniform magnetic field we now assume the following:

$$\boxed{\int B_{x,y}dz = 0}$$

The average kinetic energy satisfies

$$\mathcal{T} = \mathcal{T}^0\left[1+\theta\mathcal{D}_2\left(1-\frac{\alpha^2}{\chi^2}\right)\right]$$

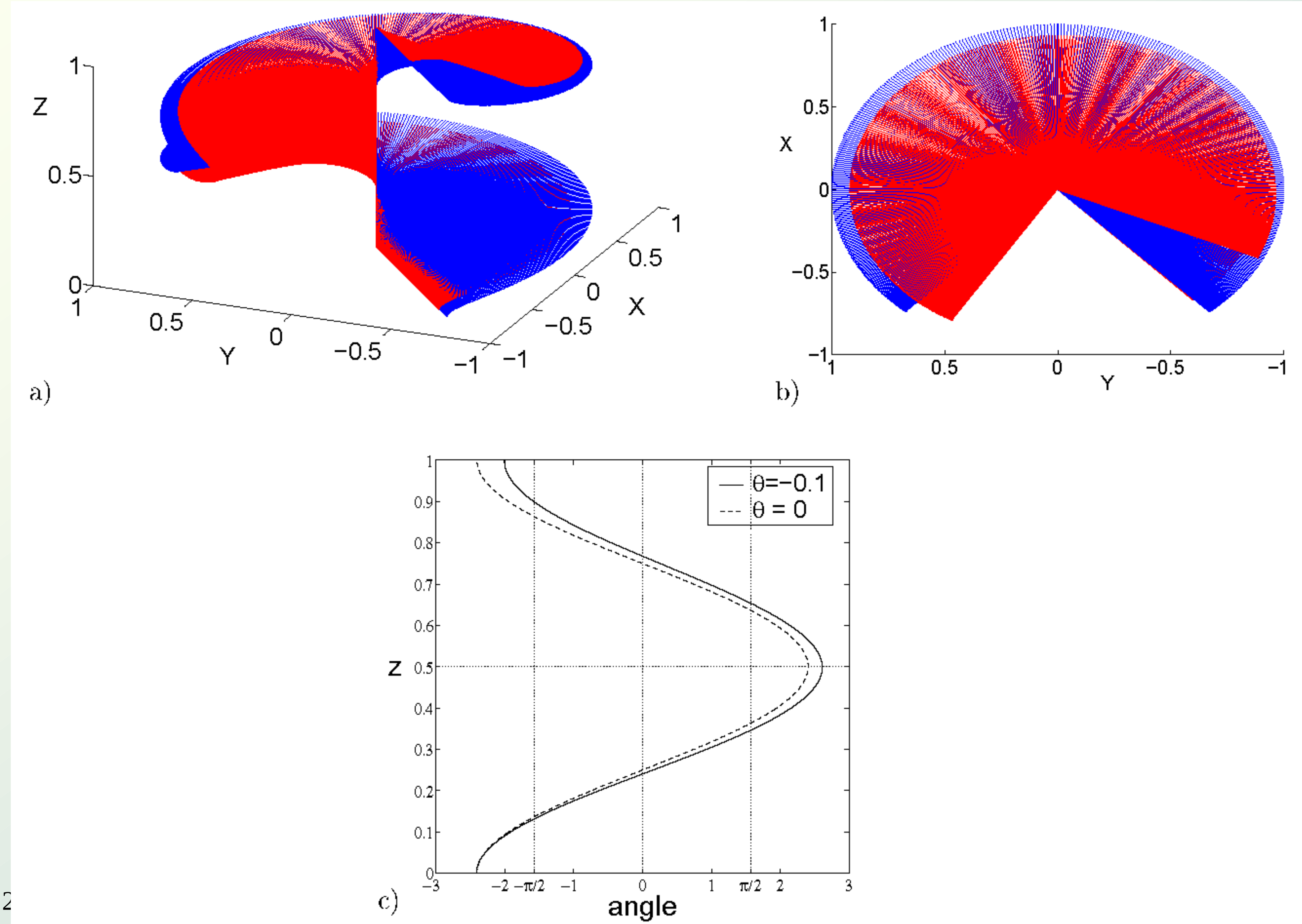
and α^2 , χ^2 depend on the planform of convective modes

- The constant D_2 typically has a different sign in cases 1 and 2.

- We do not study the stability of those solutions, however, the physical intuition suggests, in a compressible medium larger average kinetic energy (larger Rm) is necessery to maintain a stationary magnetic field, since copmpressibility tends to introduce smaller legth scales into the flow and thus enhance dissipative effects.

- The chosen scaling excludes the Lorentz force from dynamics at leading order, and hence the dynamo solutions are kinematic and valid only for small magnetic fields i.e. at dynamo threshold.

Magnetic spiral – the mean field solution



Rapidly rotating, anelastic convection – nonlinear regime

Introducing new scaling ($\varepsilon = \tau^{-1/6}$) (after Julien & Knobloch 1999):

$$\mathbf{x} = (\varepsilon x', \varepsilon y', z) \qquad t = \varepsilon^2 t'$$

$$\mathbf{u} = \varepsilon^{-1} \mathbf{u}' (\mathbf{x}_h, z, t)$$

$$s = S(z) + \varepsilon s' (\mathbf{x}_h, z, t)$$

$$\mathcal{R} = \varepsilon^{-4} \tilde{\mathcal{R}}$$

and assuming square planform of solutions:

$$[u_z, \omega_z, s] = [w(z), \xi(z), \mathfrak{s}(z)] \, e^{i\varpi t} (\cos kx + \cos ky) + c.c. + O(\varepsilon)$$

$$[u_x, u_y] = \frac{1}{k} [-\sin ky, \sin kx] \, \xi(z) \, e^{i\varpi t} + c.c. + O(\varepsilon)$$

which results in vanishing of nonlinear terms in the Navier-Stokes equations

Rapidly rotating, anelastic convection – nonlinear regime

and defining the entropy gradient $g\left(z\right)=\frac{1}{1+\theta z}-d_z\left\langle s\right\rangle_{x,y,t}$ we obtain:

$$d_z^2 w+\frac{m\theta}{1+\theta z}d_z w+\left[\tilde{\mathcal{R}}k^2 g\left(z\right)\frac{i\varpi/\,\sigma+k^2}{i\varpi+k^2/\left(1+\theta z\right)^m}-k^2\left(i\varpi/\,\sigma+k^2\right)^2-\frac{m\theta^2}{\left(1+\theta z\right)^2}\right]w=0$$

$$d_z\left[g\left(z\right)\varsigma\left(w,z\right)\right]+\frac{\theta}{\left(1+\theta z\right)}g\left(z\right)=-\mathcal{V}\left(w,z\right)$$

with:

$$\int_0^1 g\left(z\right)dz=-\triangle\,\bar{s}=\frac{1}{\theta}\ln\left(1+\theta\right)$$

where:

$$\varsigma\left(w,z\right)=1+\frac{2k^2\left|w\left(z\right)\right|^2}{\varpi^2+k^4/\left(1+\theta z\right)^{2m}}$$

$$\mathcal{V}\left(w,z\right)=\frac{2\theta}{\tilde{\mathcal{R}}}\left(1+\theta z\right)^{m-1}\left[k^2\left|w\right|^2+\frac{\sigma^2}{\sigma^2k^4+\varpi^2}\left(\left|d_z w\right|^2+\frac{m\theta}{1+\theta z}d_z\left|w\right|^2+\frac{m^2\theta^2}{\left(1+\theta z\right)^2}\left|w\right|^2\right)\right]$$

The amplitude equation, in the small amplitude limit, reduces to the previously obtained equation in the linear regime !

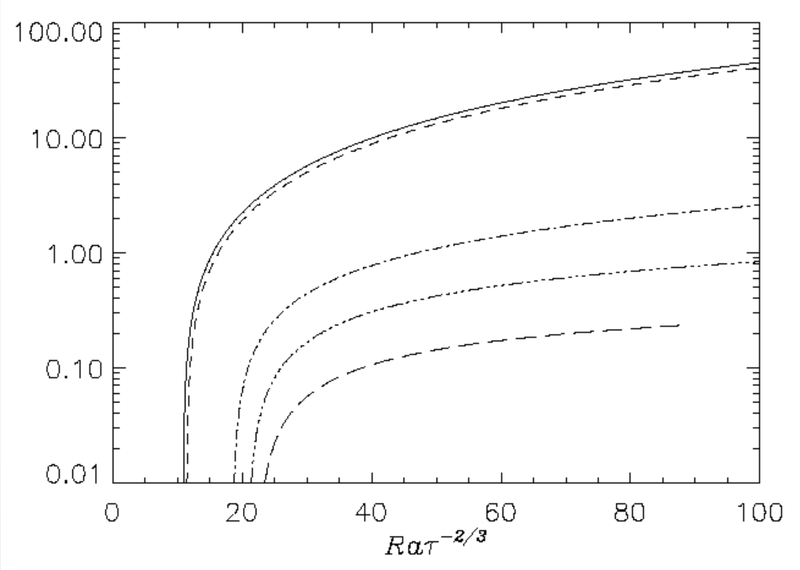
$$\text{Nusselt number} = \frac{\text{Total convective heat flux}}{\text{Conductive heat flux in the Basic State}}$$

$$Nu = 1 + 2a^2 \frac{\tilde{\mathcal{R}}_{2,0}}{\tilde{\mathcal{R}}_{c,0}} + \theta\,(m-1) \qquad \text{in Case 1,}$$

$$Nu = 1 + 2a^2 \frac{\tilde{\mathcal{R}}_{2,0}}{\tilde{\mathcal{R}}_{c,0}} - \theta \frac{\gamma}{\gamma - 1} \qquad \text{in Case 2.}$$

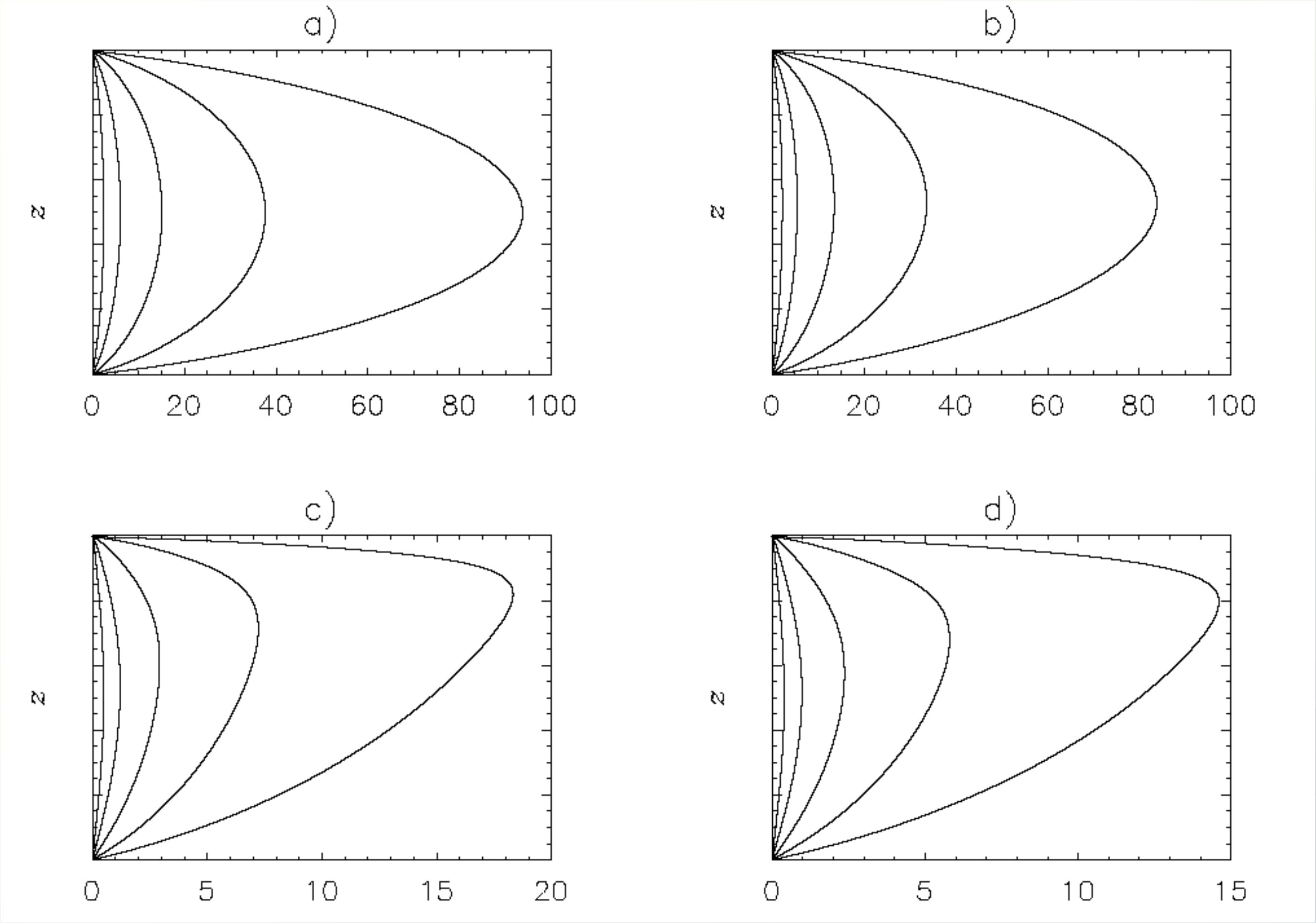
a is the amplitude of the perturbation

Weakly nonlinear theory $\Rightarrow$ In case 2, for the same distance from criticality (i.e. $R-R_c$) the Nusselt number is greater for compressible convection (in case 1 only if m<1)



Nu – 1 vs the 'compensated' Rayleigh number in case 1, for m=1.4 and θ from 0 to –0.998

Numerical nonlinear results



The ‘z’-dependence profiles of the vertical velocity for different values of the Rayleigh number and for

θ = 0, -0.2, -0.965, and -0.977

Rapidly rotating, convective dynamo for higher compressibilities

<u>having:</u>

$$\boxed{w^0 = \sum_{|k|=k} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{A}\left(t,\mathbf{k}\right) \mathcal{W}\left(z\right)}$$

we may solve the following kinematic dynamo problem:

$$\frac{\partial B_i}{\partial t} = -2\frac{\partial}{\partial z}\left\{\mathcal{W}\left(z\right)\left(\mathcal{W}'\left(z\right) + \frac{m\theta}{1+\theta z}\mathcal{W}\right)\mathbb{M}_{ij}B_j\right\} + \frac{\partial^2 B_i}{\partial z^2} = 0 + O\left(\theta^2\right)$$

where $'$ denotes a derivative with respect to 'z' and

$$\mathbb{M} = \begin{bmatrix} -\alpha_{21} & -\alpha_{22} \\ \alpha_{11} & \alpha_{12} \end{bmatrix}; \qquad \alpha_{ij} = \sum_{|k|=k} \frac{k_i k_j}{k^6}\left|\hat{A}\left(t,\mathbf{k}\right)\right|^2;$$

The average kinetic energy is

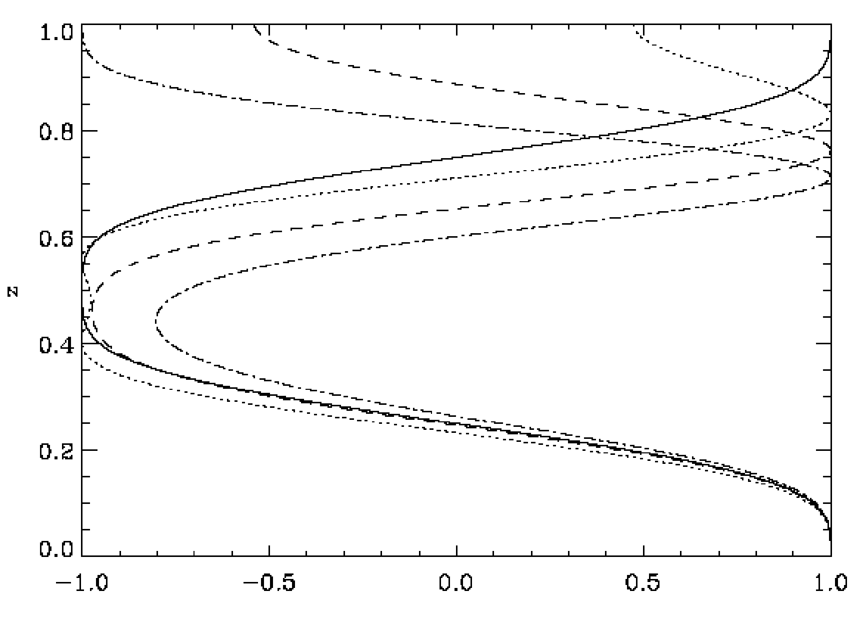
$$\mathcal{T} = \frac{1}{2}\left\langle \left(\mathbf{u}_h'^0\right)^2 + \left(w^0\right)^2 \right\rangle_C = \frac{1}{2}\left[\frac{1}{k^6}\overline{\left(\mathcal{W}' + \frac{m\theta}{1+\theta z}\mathcal{W}\right)^2} + \overline{\mathcal{W}^2}\right]\sum_{|k|=k}\left|\hat{A}\left(\mathbf{k}\right)\right|^2$$

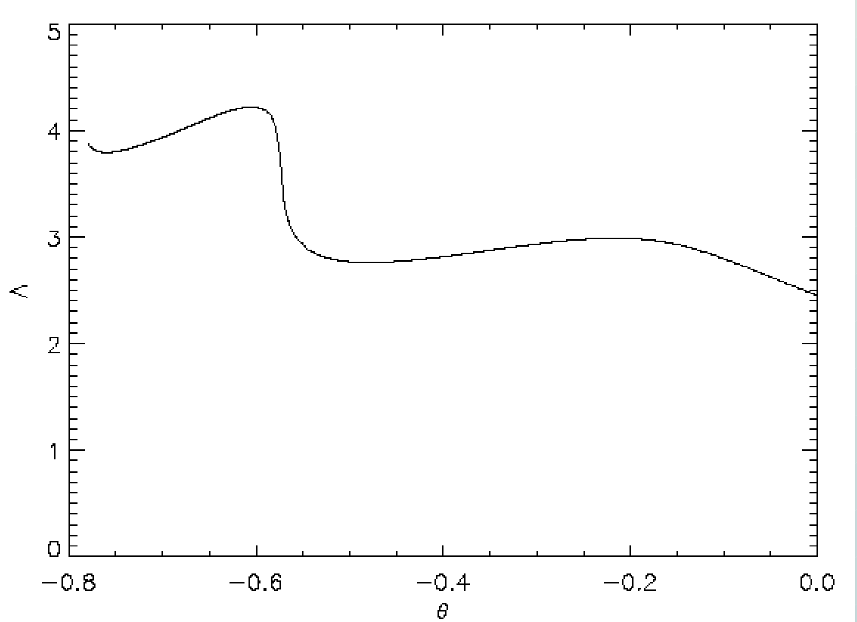
and the overbar indicates a vertical average.

For square planform a stationary dynamo solution has the same average kinetic energy as in the Boussinesq case studied by Soward (1974)

$$\mathcal{T} = \mathcal{T}^0 + O\left(\theta^2\right)$$

Rapidly rotating, convective dynamo for higher compressibilities





The ‘z’-dependence of the ‘x’ component of the mean magnetic field <B>$_x$

The ‘magnetic Reynolds number’ (the average kinetic energy of the flow) vs the compressibility θ

Conclusions

I. Depending on the k=const or κ=const formulation the critical Rayleigh number for convection in compressible case is greater or smaller then in Boussinesq case, respectively.

II. Analytical stationary dynamo solutions were obtained. We reported that a stationary large scale magnetic field could be sustained by a compressible flow with either smaller or larger average kinetic energy than that, which would be necessary to sustain an analogous magnetic field by an incompressible flow.

III. The results suggest that Rm_{crit} for dynamo action would typically be increased by compressibility since it introduces smaller length scales into the dynamics and thus enhances diffusive effects.

Disadvantages

I. We do not perform higher order analysis in θ and τ and hence we do not know the actual critical Rayleigh number for dynamo action (order θ^4 and τ^8 would be necessary).

II. We do not study the stability of obtained stationary dynamo solutions.

Weakly compressible, rapidly rotating $\tau^{-1/12} << \theta << 1$ dynamo at threshold

we postulate:

$$w^0 = \sum_{|k|=k} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{A}(t,\mathbf{k})\left[\sin(\pi z) + \theta f(z)\right] + \theta \sum_{|k|=k} e^{i\mathbf{k}\cdot\mathbf{x}} \hat{C}(t,\mathbf{k}) \sin(\pi z)$$

and k is close to k_c

which leads to:

($f(z)$ and $h(z)$ are known functions)

$$\boxed{\frac{\partial B_i}{\partial t} + 2\pi\Lambda \frac{\partial}{\partial z}\left\{\sin(2\pi z)\mathbb{M}_{ij}B_j + \theta\left[h(z)\mathbb{M}_{ij}B_j + 2\frac{\Xi}{\Lambda}\sin(2\pi z)\mathbb{N}_{ij}B_j\right]\right\} - \frac{\partial^2 B_i}{\partial z^2} = 0 + O\left(\theta^2\right)}$$

$$\mathcal{T} = \frac{1}{2}\left\langle\left(\mathbf{u}_h^{\prime 0}\right)^2 + \left(w^0\right)^2\right\rangle_C = \frac{\tilde{\mathcal{R}}^0}{4k^4}\left[\left(1+\theta\mathcal{D}_2\right)\sum_{|k|=k}\left|\hat{A}(\mathbf{k})\right|^2 + 2\theta\sum_{|k|=k}\left(\hat{A}(\mathbf{k})\hat{C}^*(\mathbf{k})\right)\right] \doteq \mathcal{T}^0\left[1+\theta\left(\mathcal{D}_2+2\mathcal{Q}\right)\right]$$

$$\Lambda = \frac{\mathcal{T}^0}{\tilde{\mathcal{R}}^0}; \qquad \Xi = \frac{\mathcal{T}_{ac}}{\tilde{\mathcal{R}}^0}; \qquad \mathcal{T}^0 = \frac{\tilde{\mathcal{R}}^0}{4k^4}\sum_{|k|=k}\left|\hat{A}(\mathbf{k})\right|^2; \qquad \mathcal{T}_{ac} = \frac{\tilde{\mathcal{R}}^0}{4k^4}\sum_{|k|=k}\left(\hat{A}(\mathbf{k})\hat{C}^*(\mathbf{k})\right); \qquad \mathcal{Q} = \mathcal{T}_{ac}/\mathcal{T}^0$$

$$\mathbb{M} = \begin{bmatrix} -\alpha_{21} & -\alpha_{22} \\ \alpha_{11} & \alpha_{12} \end{bmatrix}; \qquad \mathbb{N} = \begin{bmatrix} -\beta_{21} & -\beta_{22} \\ \beta_{11} & \beta_{12} \end{bmatrix}$$

$$\alpha_{ij} = \frac{1}{2k^6\Lambda}\left\langle\nabla A \otimes \nabla A\right\rangle_{ij} = \sum_{|k|=k}\frac{k_i k_j}{k^2}q(\mathbf{k}); \qquad \beta_{ij} = \frac{1}{2k^6\Xi}\left\langle\left(\nabla A \otimes \nabla C\right)_S\right\rangle_{ij} = \sum_{|k|=k}\frac{k_i k_j}{k^2}p(\mathbf{k})$$

$$q(\mathbf{k}) = \frac{1}{2k^4\Lambda}\left|\hat{A}(\mathbf{k})\right|^2; \qquad p(\mathbf{k}) = \frac{1}{2k^4\Xi}\left(\hat{A}(\mathbf{k})\hat{C}^*(\mathbf{k})\right)$$