Wavelets to study 3D homogeneous MHD turbulence

Marie Farge,
LMD-CNRS, ENS-Paris,
Wissenschaftskolleg zu Berlin

Katsunori Yoshimatsu, Naoya Okamoto,
Yuji Kondo, Yasuhiro Kawahara and Hiroyuki Hagiwara,
Computational Sciences, Nagoya University
&
Kai Schneider, CMI, Université de Provence, Marseille

To Mike Proctor, Cargèse, September 23rd 2010

Turbulence and Wavelets

Coherent Vorticity and Current sheet Extraction (CVCE)

Coherent Vorticity and Current sheet Simulation (CVCS)

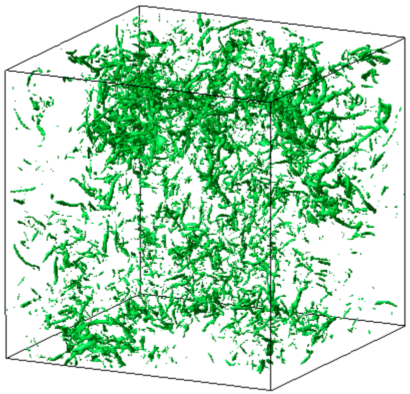
Turbulence

Turbulence { Eddies excited on a wide range of scales; Fluctuations present coherent structures; Strong spatial and temporal intermittency

- Eddies excited on a wide range of scales
- Fluctuations present coherent structures
- Strong spatial and temporal intermittency

Hydrodynamic turbulence

$|\vec{\omega}|$

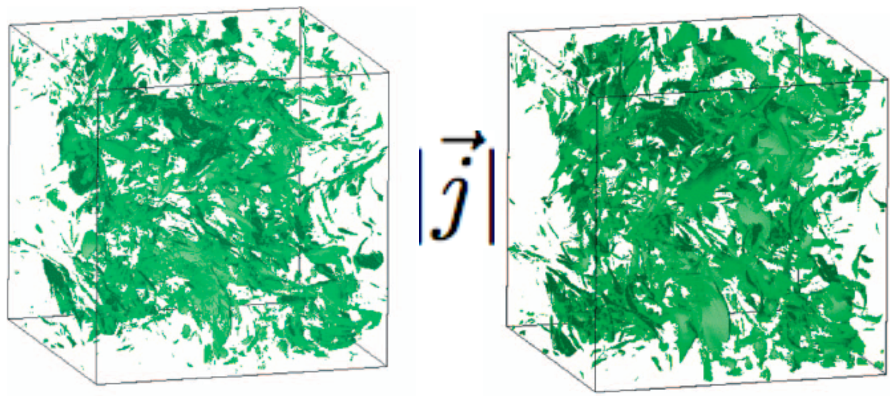


Vorticity tubes

*Siggia., J. Fluid Mech.., **107**, 375, 1981*

MHD turbulence

$|\vec{\omega}|$ $|\vec{j}|$



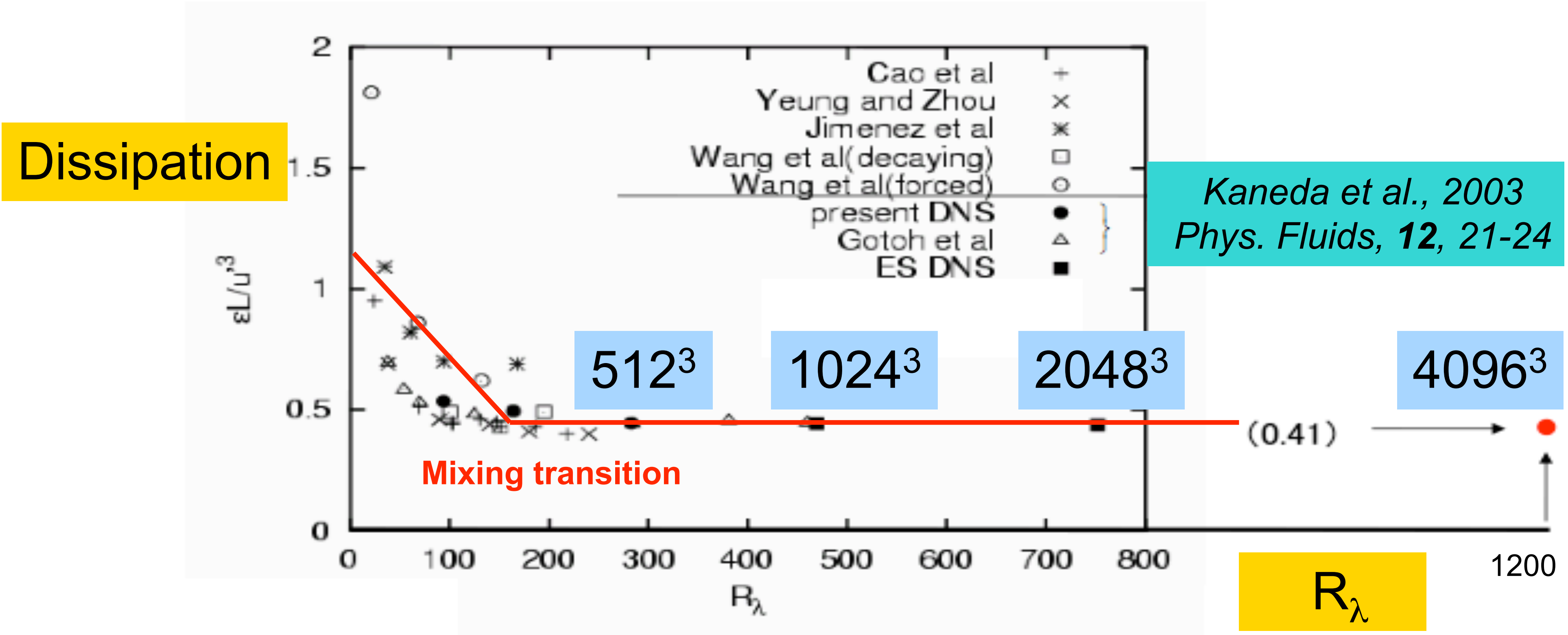
Vorticity sheets and current sheets

*Politano et al., Phys. Plasmas, **2**, 2931, 1995*

Fully-developed turbulence

Normalized energy dissipation α → ?

$$\alpha = \epsilon L / u'^3 \qquad \text{as } \nu \to 0, \text{ or } Re \to \infty$$



Kaneda et al., 2003
Phys. Fluids, ***12***, *21-24*

Dissipation independent of viscosity $\Rightarrow$ turbulent dissipation

Why? how turbulent dissipation differs from viscous dissipation?

How to decompose turbulent flows?

Reynolds averaging (1883)

Field f **= Mean** $\langle f \rangle$ **+ Fluctuations** f'

such that

$$\langle f' \rangle = 0 \qquad \langle\langle f \rangle\rangle = \langle f \rangle$$

$$\langle f + g \rangle = \langle f \rangle + \langle g \rangle \qquad \langle \nabla f \rangle = \nabla \langle f \rangle$$

but nonlinearity is hard to handle since there is no scale separation

$$\Longrightarrow \langle f g \rangle = \langle f \rangle \langle g \rangle + \langle f' g' \rangle$$

Reynolds Stress $\langle f' g' \rangle$

We propose to decompose **Fluctuations** f' *into*
Coherent Fluctuations + **Incoherent Fluctuations**

$$f' = f'_c + f'_i$$

How to decompose turbulent fluctuations?

'In 1938 Tollmien and Prandtl suggested that turbulent fluctuations might consist of two components, a diffusive and a non-diffusive. Their ideas that fluctuations include both random and non random elements are correct, but ***as yet there is no known procedure for separating them.****'*

Hugh Dryden, Adv. Appl. Mech., ***1****, 1948*

turbulent fluctuations
= **non random** + **random**
= **coherent structures** + **incoherent noise**

⟹ Coherent Vorticity Extraction (**CVE**)

turbulent dynamics
= chaotic **non diffusive** + stochastic **diffusive**
= **inviscid nonlinear** dynamics + **turbulent dissipation**

⟹ Coherent Vorticity Simulation (**CVS**)

Farge, Ann. Rev. Fluid Mech., ***24****,1992*

Farge, Schneider, Kevlahan, Phys. Fluids, ***11*** *(8), 1999*

Farge, Pellegrino, Schneider Phys. Rev. Lett., ***87*** *(5), 2001*

How to define coherent structures?

Since there is not yet a universal definition of coherent structures which emerge out of turbulent fluctuations, we adopt an **apophetic method** : **instead of defining what they are, we define what they are not**.

For this we propose the minimal statement:
'Coherent structures are not noise'

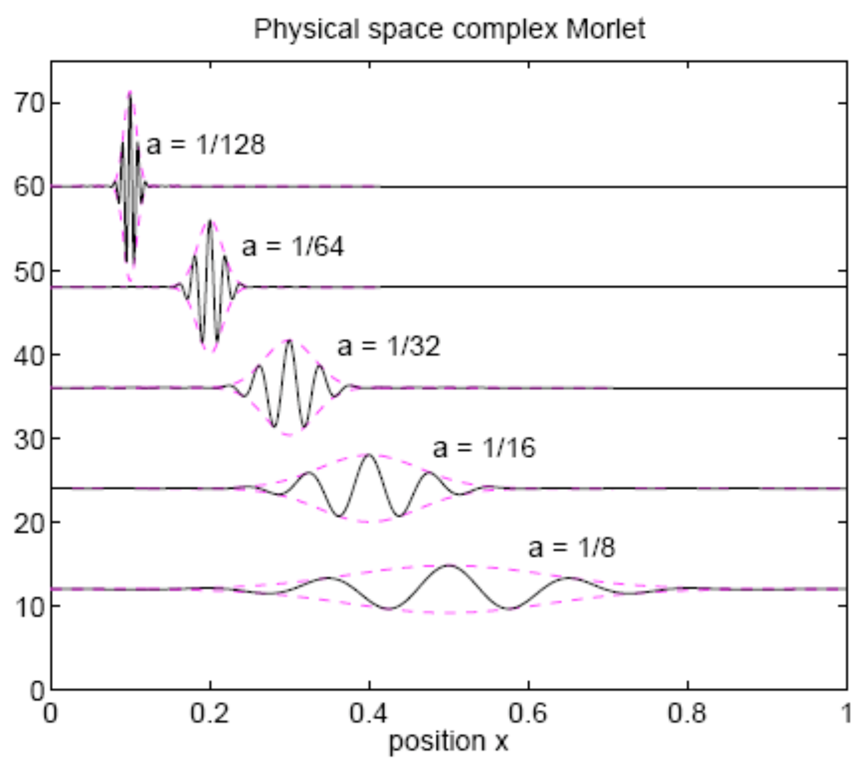
$\Longrightarrow$

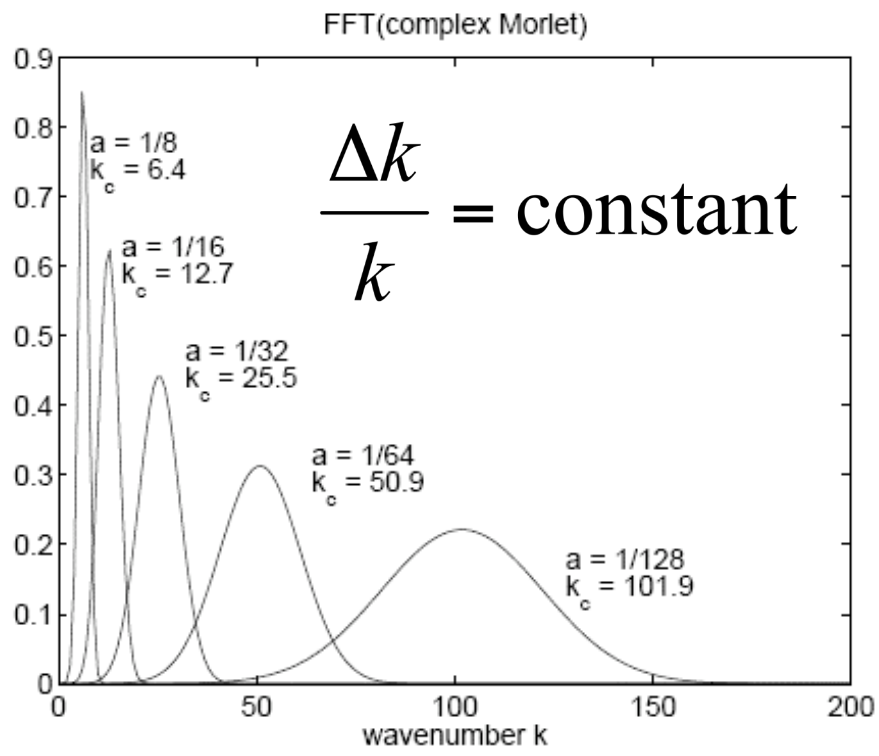
Extracting coherent structures becomes a **denoising problem**, not requiring any hypotheses on the structures themselves but only on the noise to be eliminated.

Choosing the **simplest hypothesis** as a first guess, we suppose we want to eliminate an additive Gaussian white noise, and for this we use a nonlinear wavelet filtering.

Farge, Schneider et al.
Phys. Fluids, ***15*** *(10), 2003*

Wavelet representation





Physical space

Spectral space

A. Grossmann and J. Morlet,
Decomposition of Hardy functions into square integrable wavelets of constant shape,
SIAM J. Math. Anal., ***15****, 1984*

M. Farge
Wavelet transforms and their applications to turbulence
Ann. Rev. Fluid Mech., ***24****, 1992*

3D orthogonal wavelets

- **fast algorithm** with **linear complexity**
- **no redundancy** between the coefficients

A 3D vector field **v(x)** sampled on $N = 2^{3J}$ equidistant grid points

$\psi_\lambda(\boldsymbol{x})$ 3D wavelet → orthogonal wavelet series

$$\boldsymbol{v}(\boldsymbol{x}) = \sum \widetilde{\boldsymbol{v}}_\lambda \psi_\lambda(\boldsymbol{x}), \quad \widetilde{\boldsymbol{v}}_\lambda = \langle \boldsymbol{v}, \psi_\lambda \rangle$$

$$\Lambda = \{\lambda = (j, i_n, \mu),\ j = 0, ..., J-1,\ i_n = 0, ..., 2^j - 1, n = 1, 2, 3,\ \text{and}\ \mu = 1, ..., 7\}$$

$N_j = 7 \times 2^{3j}$, wavelet coefficients at a scale indexed by *j*

We use here **the Coifman 12 wavelet** which is compactly supported, four vanishing moments, quasi-symmetric.

Wavelet denoising

- **Apophatic method**
  - **no hypothesis on the structures**,
  - **only hypothesis on the noise**,
  - simplest hypothesis as our first choice.

- **Hypothesis on the noise**

  $f_n = f_d + n$

  n : Gaussian white noise,

  $<f_n^2>$: variance of the noisy signal,

  N : number of coefficients of f_n.

- **Wavelet decomposition**

  $$\tilde{f}_{ji} = < f \mid \psi_{ji} >$$

  j scale

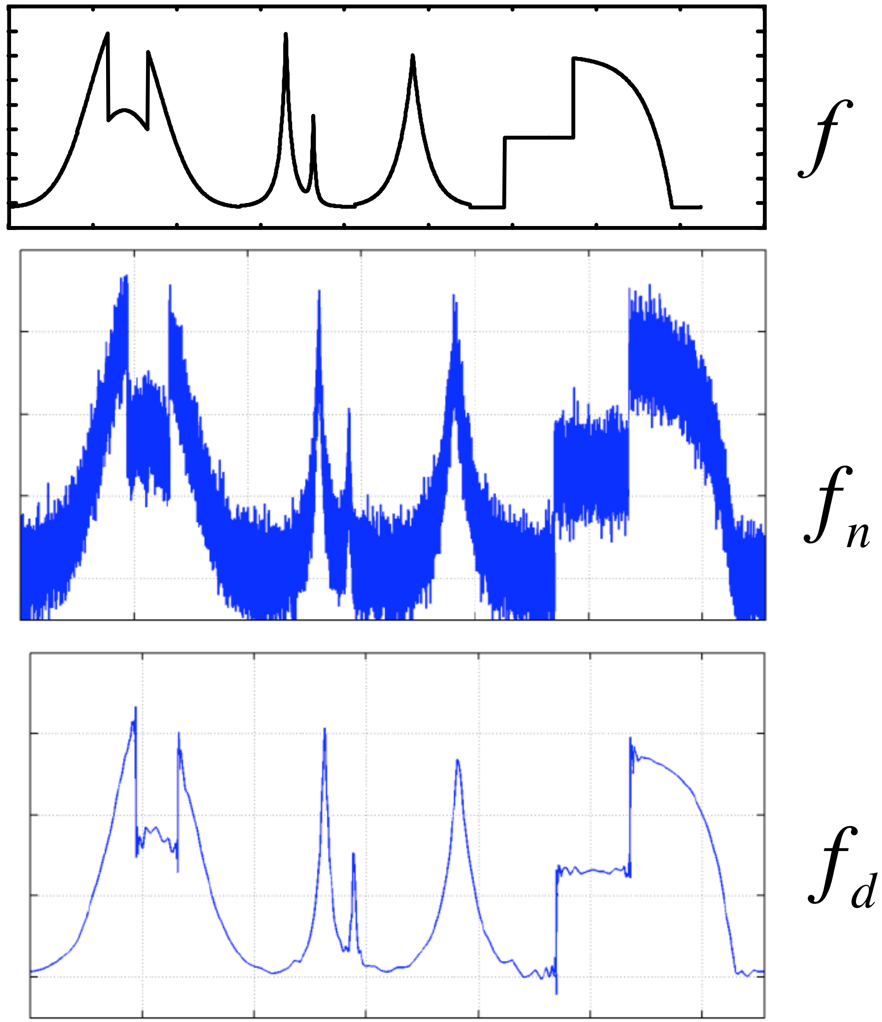
  i position

- **Estimation of the threshold**

  $$\varepsilon_n = \sqrt{2 < f_n^2 > \ln(N)}$$

- **Wavelet reconstruction**

  $$f_d = \sum_{ji : |\tilde{f}_{ji}| < \varepsilon_n} \tilde{f}_{ji} \psi_{ji}$$



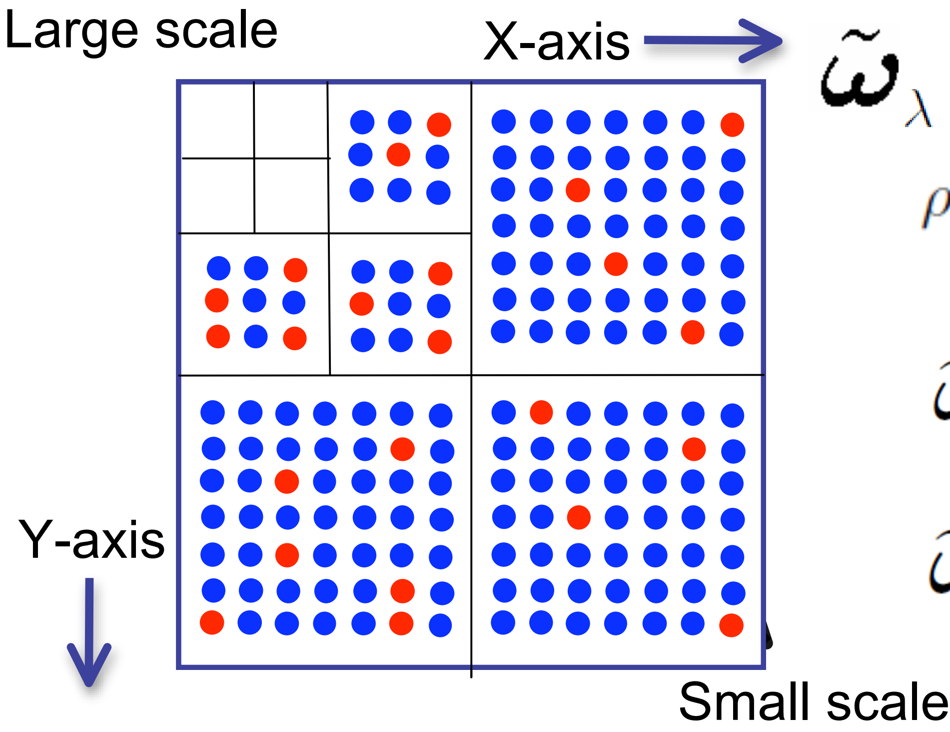
Donoho and Johnstone
*Biometrika, **81**, 1994*

Azzalini, Farge and Schneider
*ACHA, **18** (2), 2005*

Vorticity field in physical space

$\omega \xrightarrow{\text{FWT}} \widetilde{\omega}_\lambda$ Wavelet coefficients of ω

The threshold value depends on the **enstrophy** and the **resolution** of the field only.



$$\rho_\epsilon(\widetilde{\omega}_\lambda) = \begin{cases} \widetilde{\omega}_\lambda & \text{for} \quad |\widetilde{\omega}_\lambda| > \epsilon, \\ 0 & \text{for} \quad |\widetilde{\omega}_\lambda| \leqslant \epsilon, \end{cases}$$

$$\widetilde{\omega}_c = \rho_\epsilon(\widetilde{\omega}_\lambda)$$ coherent

$$\widetilde{\omega}_i = \widetilde{\omega}_\lambda - \widetilde{\omega}_c$$ incoherent

FWT inverse

ω_c Coherent vorticity in physical space

Coherent vorticity in physical space

$$\omega_i = \omega - \omega_c$$

Turbulence and Wavelets

Coherent Vorticity and Current sheet Extraction (CVCE)

Coherent Vorticity and Current sheet Simulation (CVCS)

Application to MHD turbulence

- Generalization of the **CVE** method to **CVCE**, **C**oherent **V**orticity and coherent **C**urrent **E**xtraction, out of MHD homogeneous and isotropic turbulence,

- Decomposition of vorticity and current density fields into coherent and incoherent contributions,

- Evaluation of the compression thus obtained to assess the feasibility and potential of **CVCS**, **C**oherent **V**orticity and **C**urrent **S**imulation, a deterministic computation of the coherent fields only using an adaptive wavelet basis, while discarding the incoherent contributions.

3D MHD equations

3D incompressible MHD turbulence

without mean magnetic field

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla P + \boldsymbol{j} \times \boldsymbol{b} + \nu \Delta \boldsymbol{u} + \boldsymbol{f}^u$$

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) + \eta \Delta \boldsymbol{b} + \boldsymbol{f}^b$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

$$\nabla \cdot \boldsymbol{b} = 0$$

u	velocity field
b	magnetic field

Magnetic Prandtl number $Pr_m=1$: $\nu = \eta$,

Random forcing imposed on velocity and magnetic fields at low-wavenumbers, k< 2.5.

Direct Numerical Simulation (DNS)

- DNS of 3D incompressible MHD turbulence without mean magnetic field in a periodic box,
- Magnetic Prandtl number $Pr_m=1$,
- The simulations use a dealiased pseudo-spectral method, and a fourth order Runge-Kutta method for time marching,
- Random forcing imposed on velocity and magnetic fields at low-wavenumbers, k< 2.5,
- The DNS was performed until the energy dissipation rate per unit mass remains almost constant to insure that the flow has reached a statistically quasi-stationary state.

Yoshida and Arilitsu,
Phys. Fluids, ***19****, 045106, 2007*

DNS parameters

N	E^u	E^b	Z^u	Z^b	$k_{\max}\eta_{\mathrm{IK}}$	R^u_λ	R^b_λ
512^3	0.386	0.873	96.8	136	2.1	159	304

E^u, E^b: kinetic and magnetic energies

Z^u, Z^b: kinetic and magnetic enstrophies

η_{IK}: The Iroshnikov and Kraichnan (IK) microscale

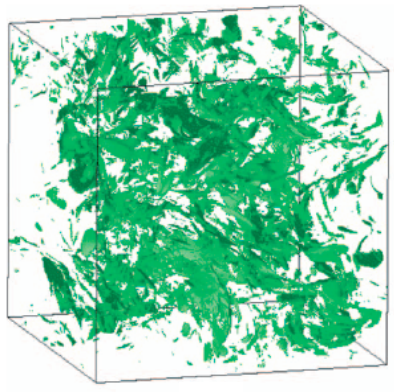
R^u_λ, R^b_λ : kinetic and magnetic Taylor microscale Reynolds numbers

Cross helicity and magnetic helicity are almost zero.

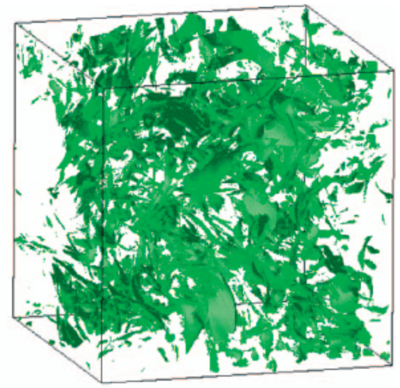
Yoshimatsu, Kondo, Schneider, Okamoto, Hagiwara and Farge
*Phys. Plasmas, **16**, 082306, 2009*

CVCE method

- Extraction coherent vorticity and coherent current from the vorticity field and current density field.



$|\boldsymbol{\omega}|$



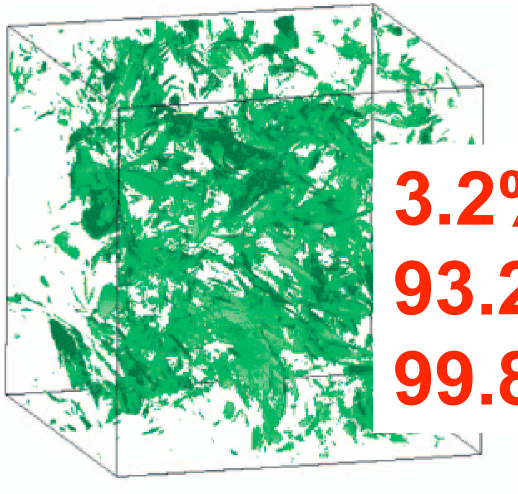
$|\boldsymbol{j}|$

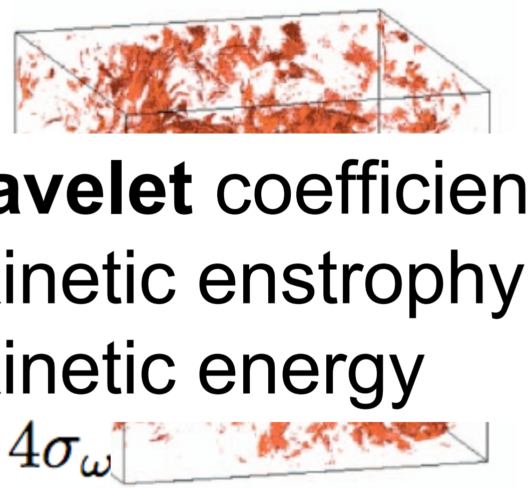
- The same definition of coherent structures as **CVE**,
- Application of nonlinear thresholding to the wavelet coefficients of $\boldsymbol{\omega}$ and $\boldsymbol{j}$, separately.

total

coherent

$|\omega|$

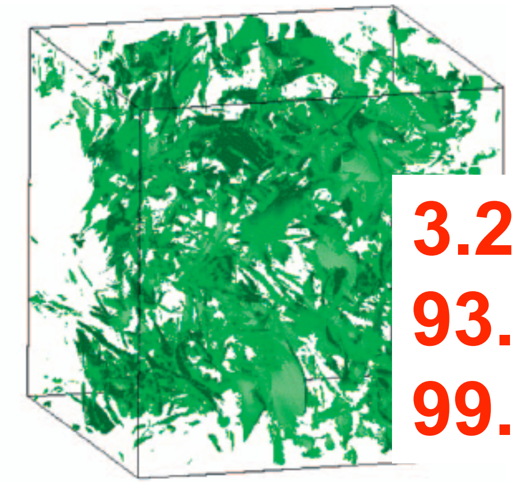


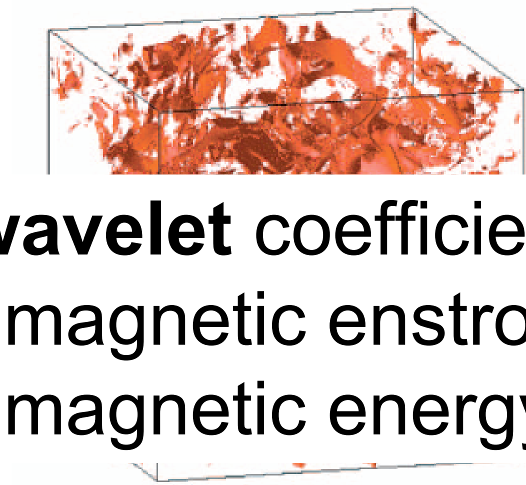


3.2% of the **wavelet** coefficients
93.2% of the kinetic enstrophy
99.8% of the kinetic energy

$$|\omega| = m_\omega + 4\sigma_\omega$$

$|\boldsymbol{j}|$



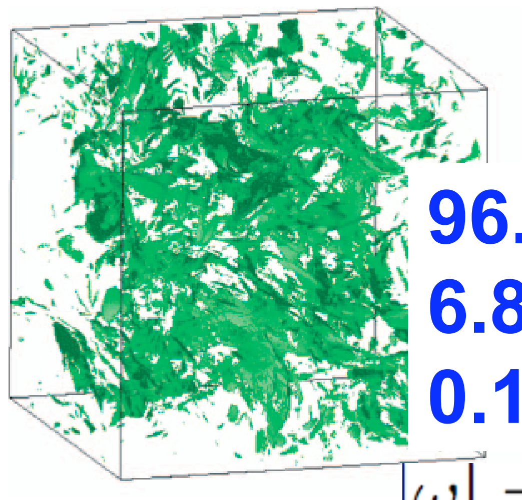


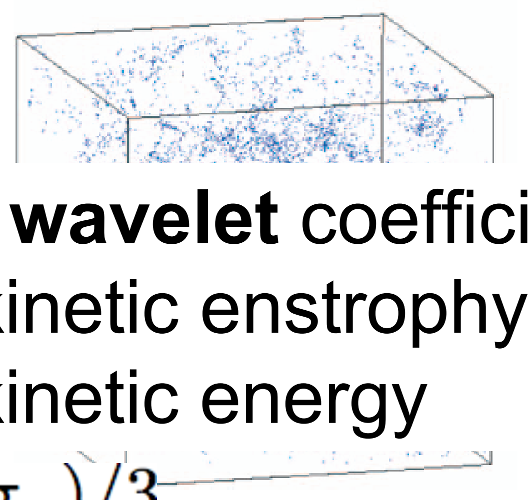
3.2% of the **wavelet** coefficients
93.7% of the magnetic enstrophy
99.9% of the magnetic energy

$$|\boldsymbol{j}| = m_j + 4\sigma_j$$

total | incoherent

$|\omega|$

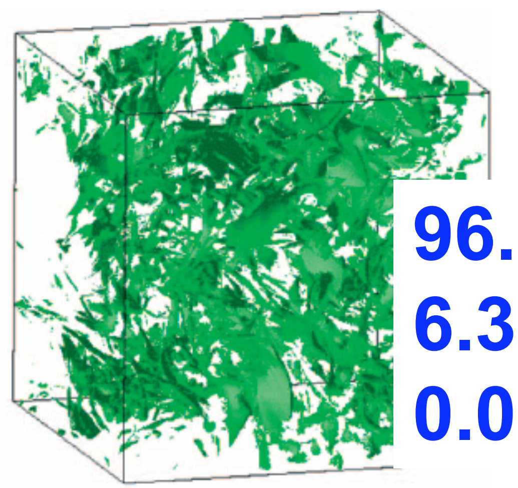


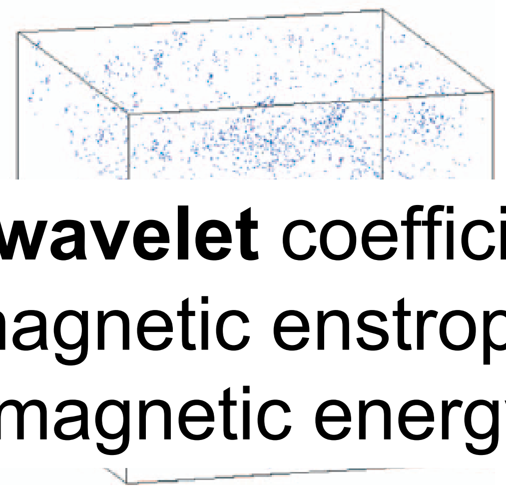


96.8% of the **wavelet** coefficients
6.8% of the kinetic enstrophy
0.1% of the kinetic energy

$$|\omega| = (m_\omega + 4\sigma_\omega)/3$$

$|j|$

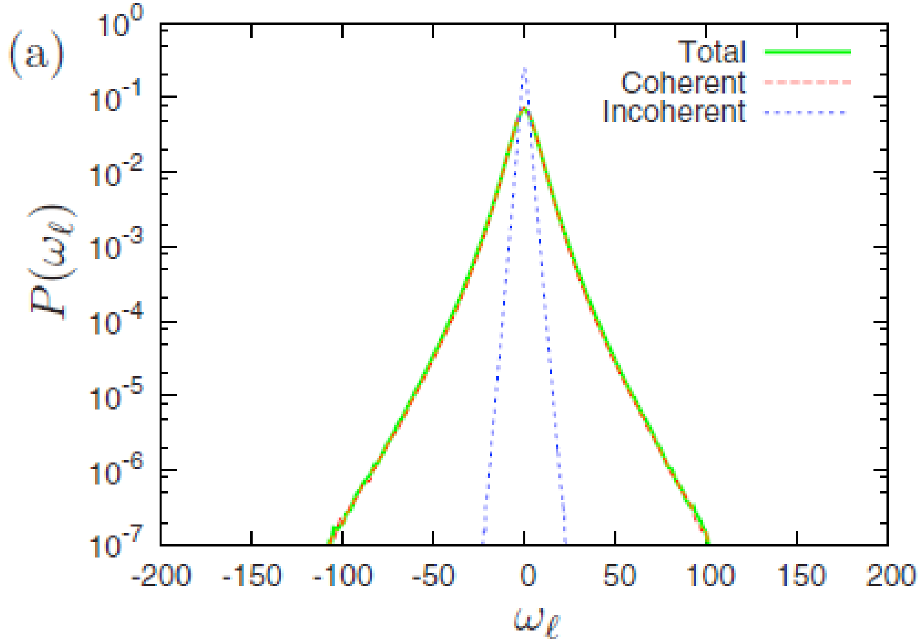




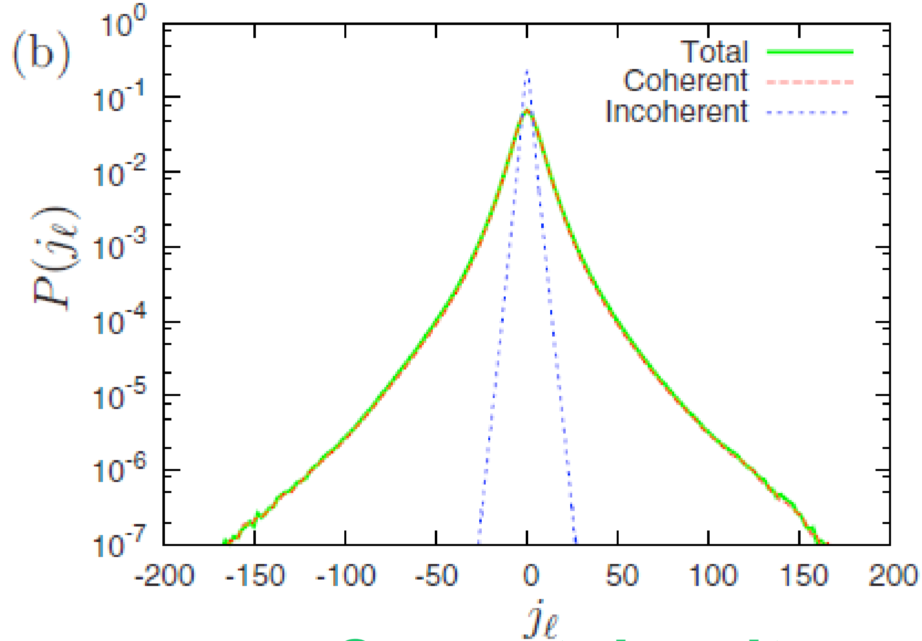
96.8% of the **wavelet** coefficients
6.3% of the magnetic enstrophy
0.06% of the magnetic energy

$$|j| = (m_j + 4\sigma_j)/3$$

PDF of vorticity and current density



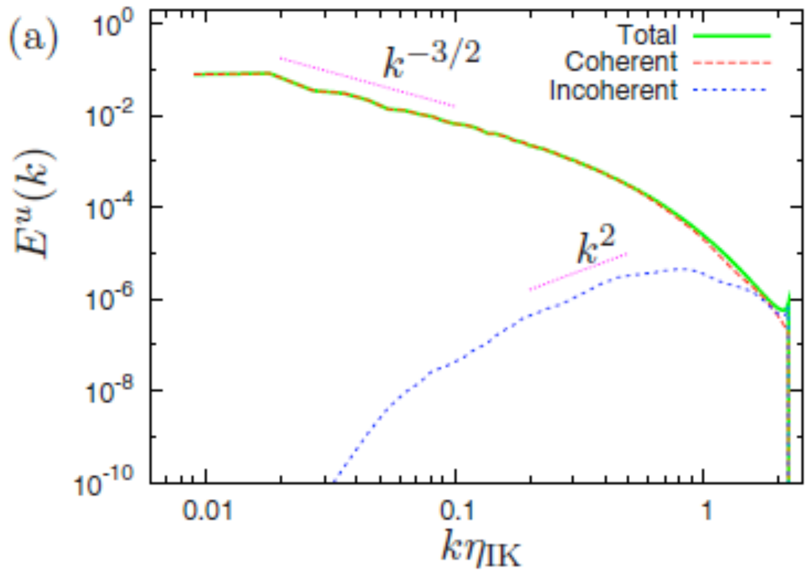
Vorticity



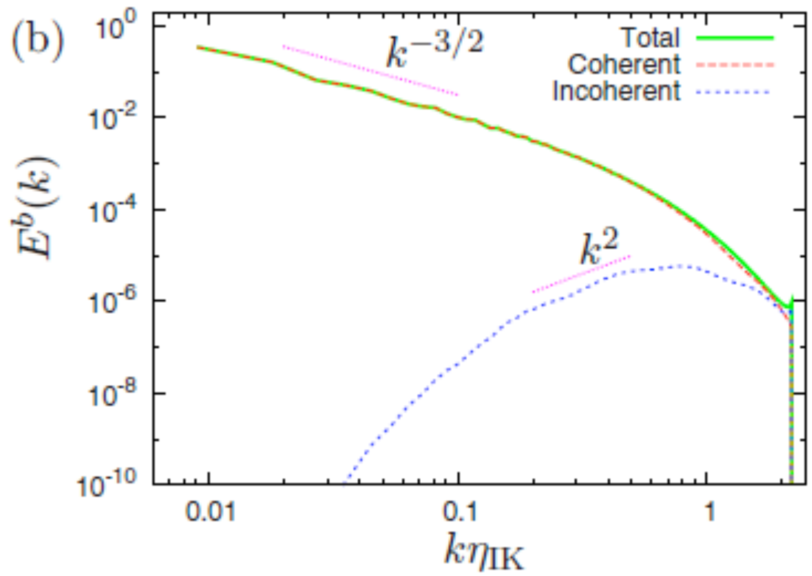
Current density

- The total and coherent PDFs well superimpose.
- The PDFs of the incoherent fields have reduced variances compared to those of the total fields.

Energy spectra



Kinetic energy



Magnetic energy

- **Coherent** contributions:

$$E_C^u(k) \propto k^{-3/2} \quad \textit{(Iroshnikov-Kraichnan)} \quad E_C^b(k) \propto k^{-3/2}$$

- **Incoherent** contributions:

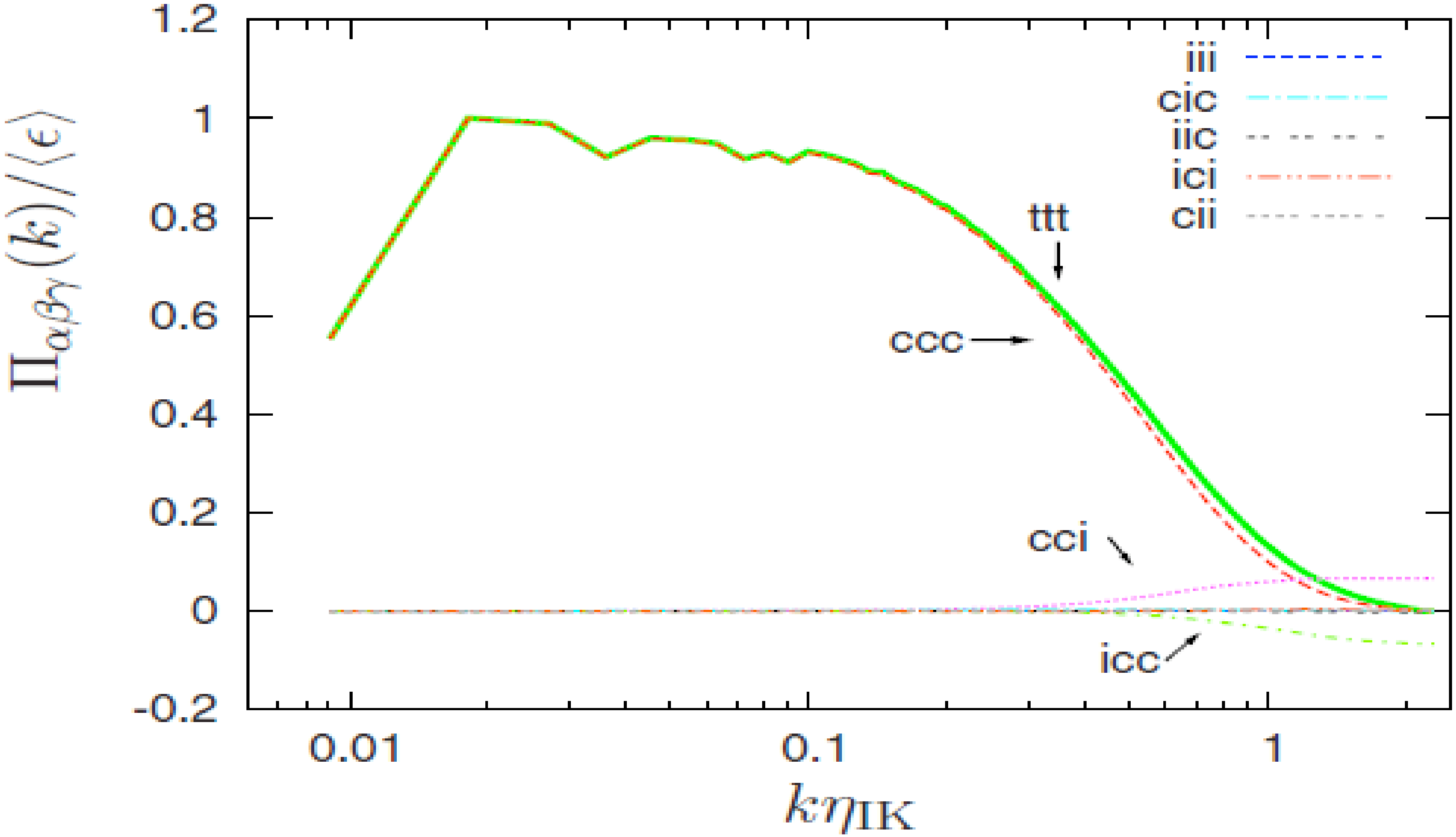
$$E_I^u(k) \propto k^2 \quad \textit{(energy equipartition)} \quad E_I^b(k) \propto k^2$$

Nonlinear transfers and energy fluxes

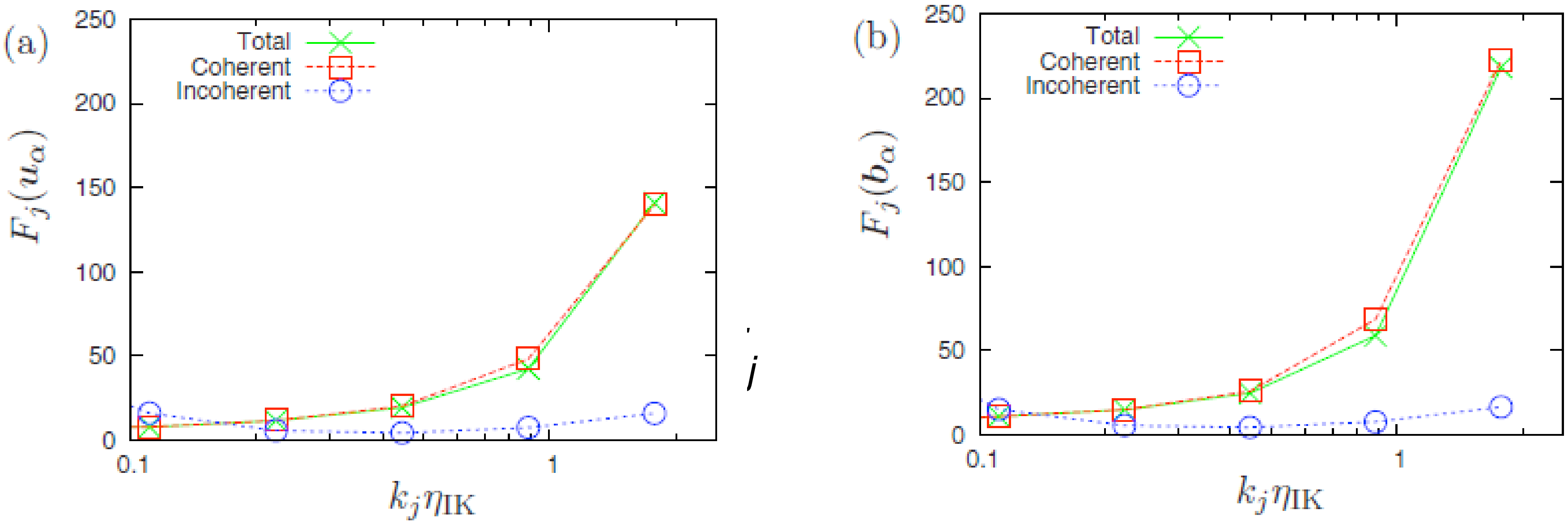
$$T_{\alpha\beta\gamma}(k) = \sum_{k-\frac{1}{2}\leq|\vec{p}|<k+\frac{1}{2}} \widehat{\vec{v}}_\alpha(-\vec{p}) \cdot [(\vec{v}_\beta \cdot \widehat{\nabla})\vec{v}_\gamma](\vec{p})$$

energy flux $\Pi_{\alpha\beta\gamma}(k) = -\int_0^k T_{\alpha\beta\gamma}(k)dk$ for $(\alpha, \beta, \gamma) \in \{c, i\}$

Yoshimatsu, Kondo, Schneider, Okamoto, Hagiwara and Farge
*Phys. Plasmas, **16**, 082306, 2009*



Scale dependent flatness



Velocity

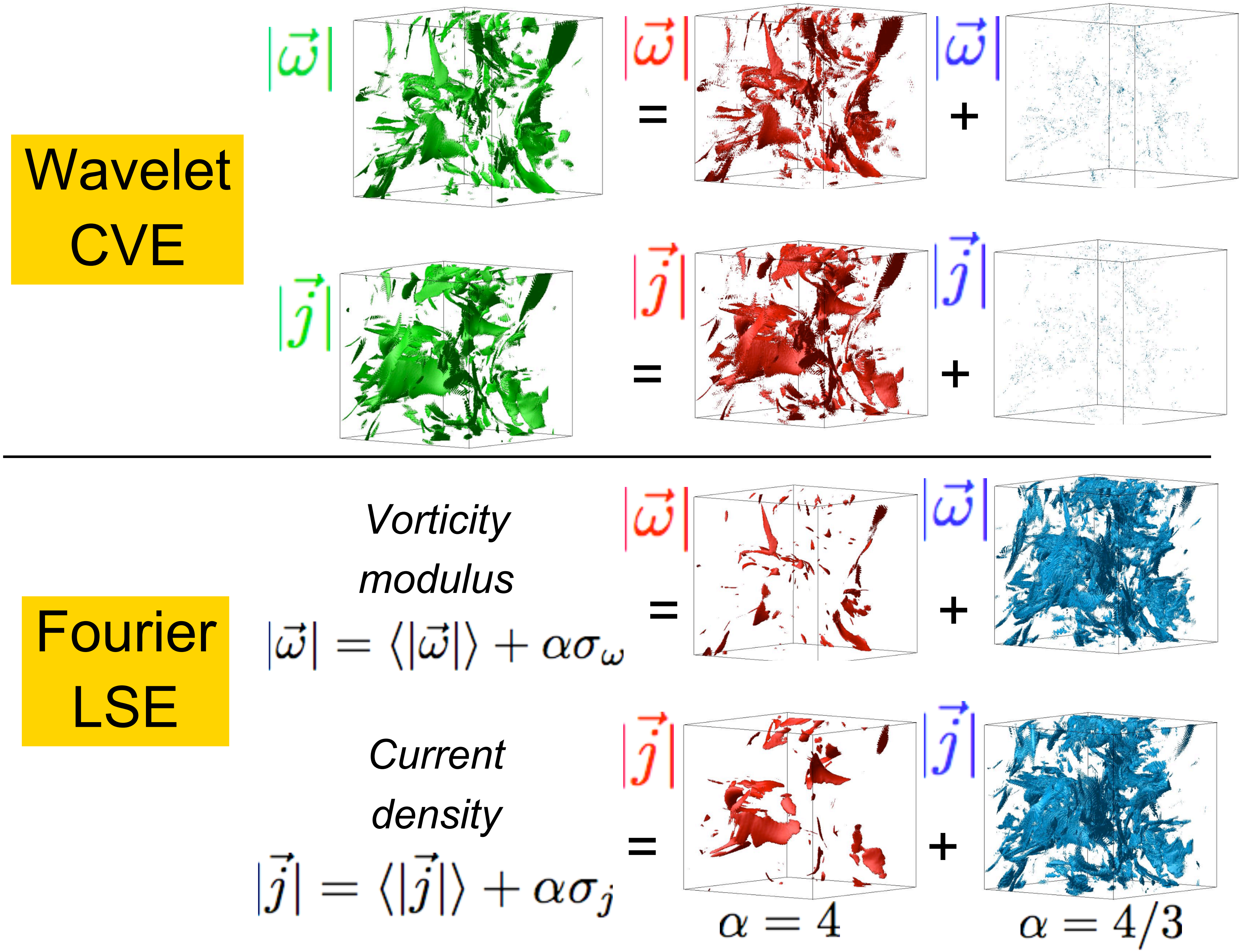
Magnetic field

$$k_j = 2^j/1.3$$

- The magnetic field is more intermittent than velocity.

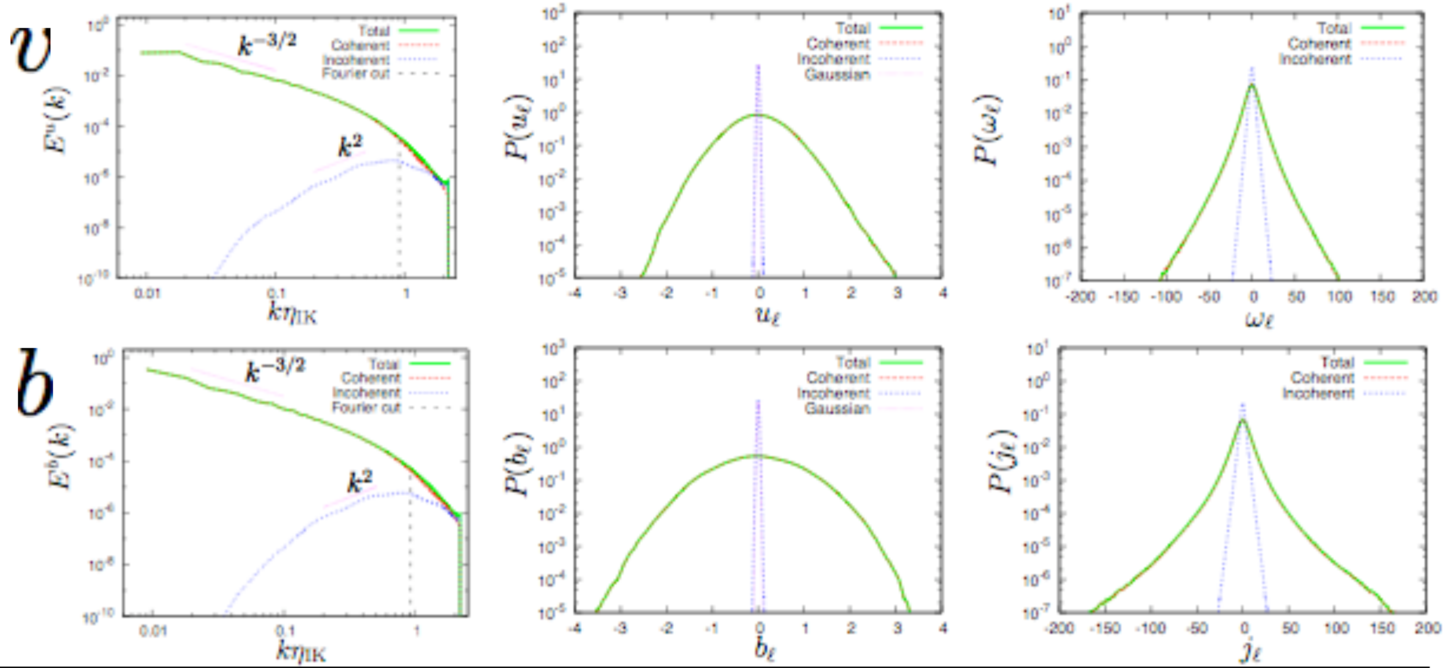
Cho et al., Astrophys. J.,
595*, 812, 2003*

- Coherent structures (vorticity sheets and current sheets) are responsible for the intermittency.

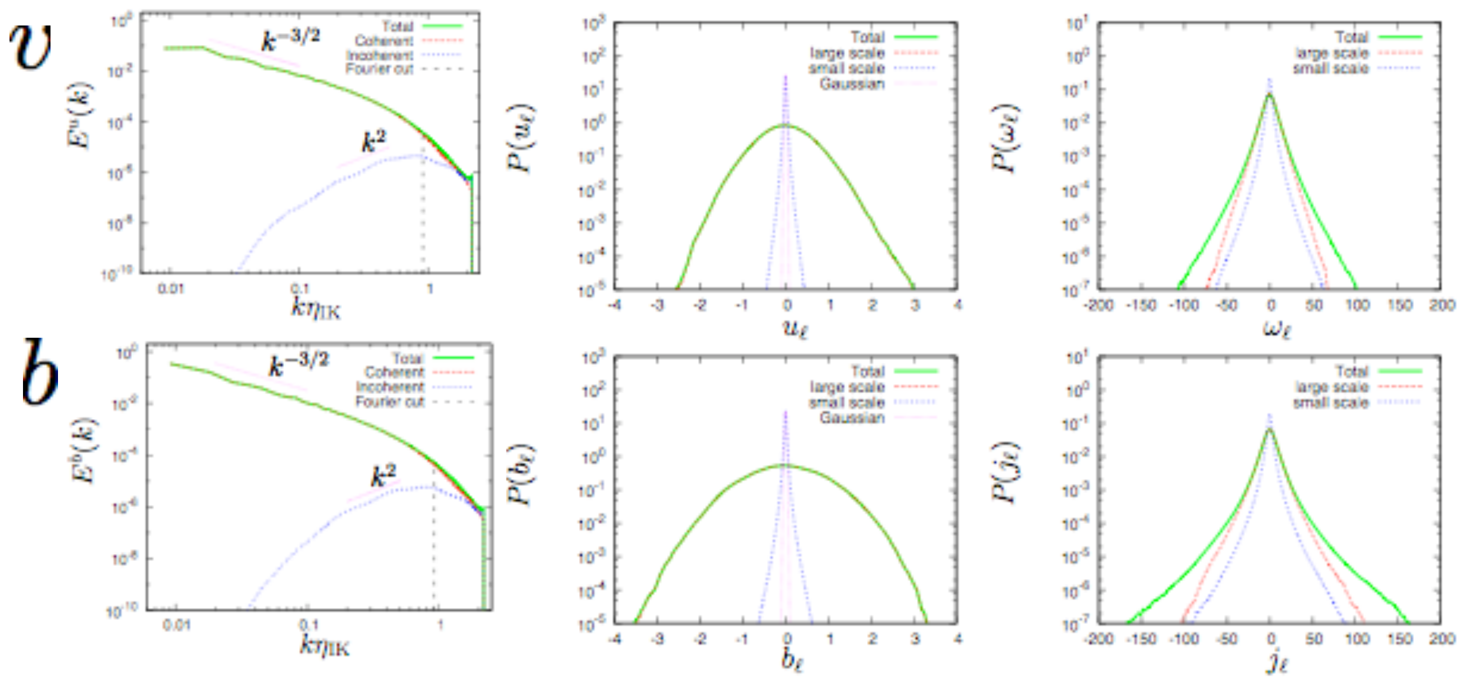


Wavelet CVE
$|\vec{\omega}|$ = $|\vec{\omega}|$ + $|\vec{\omega}|$
$|\vec{j}|$ = $|\vec{j}|$ + $|\vec{j}|$
Fourier LSE
Vorticity modulus
$|\vec{\omega}| = \langle|\vec{\omega}|\rangle + \alpha\sigma_\omega$
$|\vec{\omega}|$ = $|\vec{\omega}|$ + $|\vec{\omega}|$
Current density
$|\vec{j}| = \langle|\vec{j}|\rangle + \alpha\sigma_j$
$|\vec{j}|$ = $|\vec{j}|$ + $|\vec{j}|$
$\alpha = 4$
$\alpha = 4/3$

Wavelet
CVE



Fourier
LSE

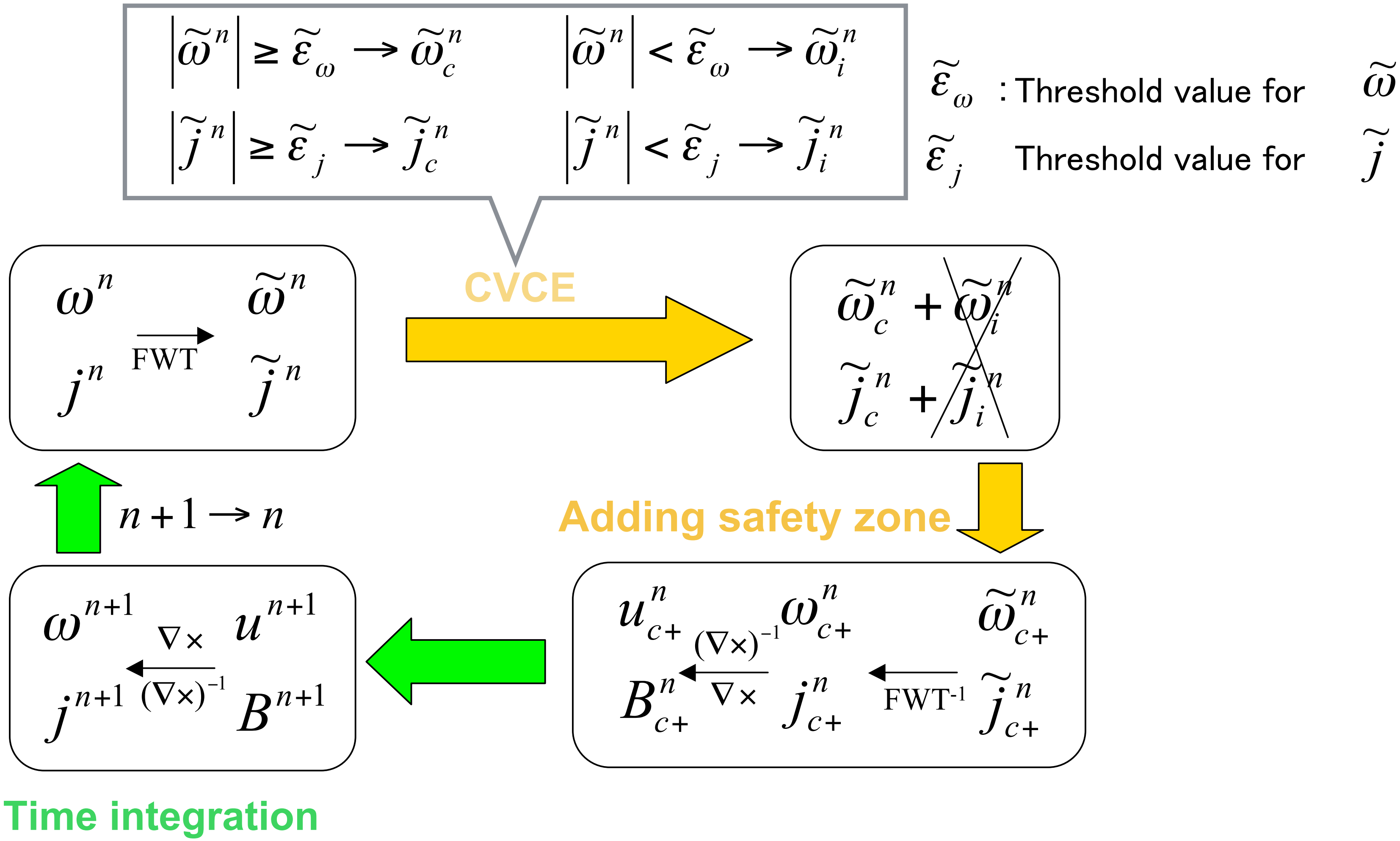


Turbulence and Wavelets

Coherent Vorticity and Current sheet Extraction (CVCE)

Coherent Vorticity and Current sheet Simulation (CVCS)

Flow chart of CVCS



n : time

$\sim$: wavelet coefficients

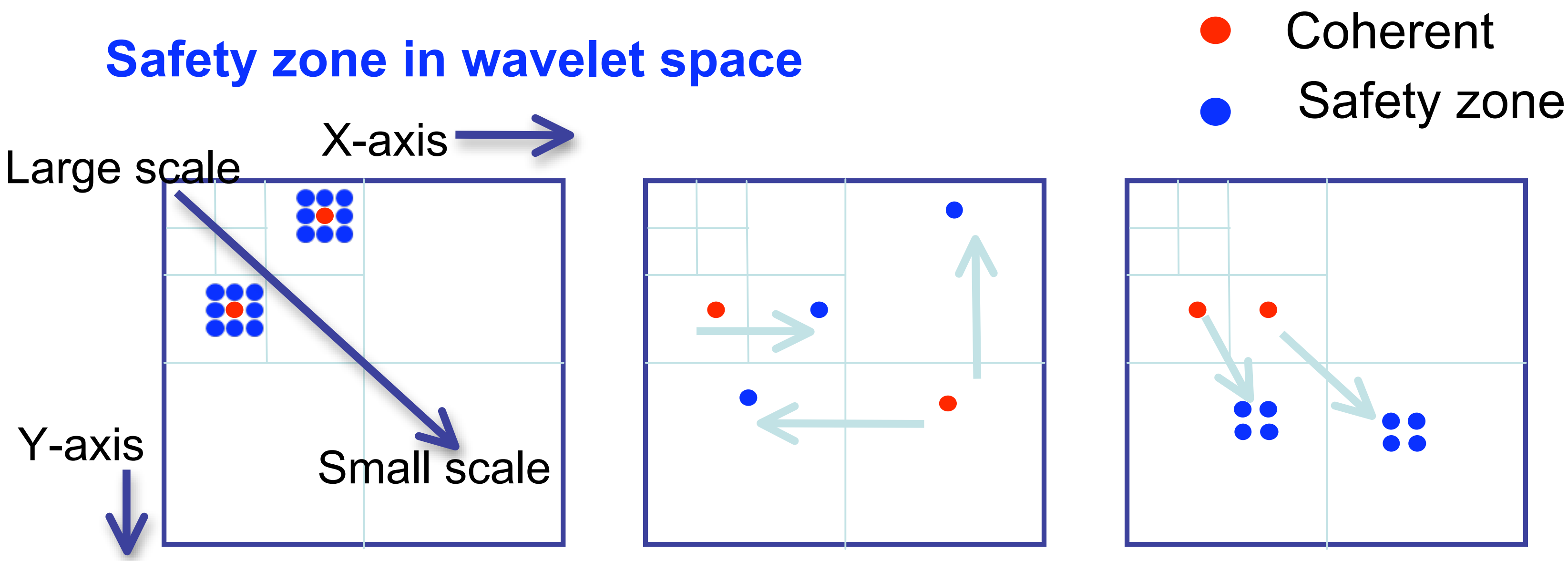
c : coherent

$c+$: coherent + safety zone

Safety zone in 3D

To track and predict the motion of the coherent structures and their generation of small scales, one requires to add a **safety zone** to the retained wavelet coefficients.

Safety zone in wavelet space



In space

2D: 8
3D: 26

In direction

2D: 1 or 2
3D: 2 or 3

In scale

2D: 4
3D: 8

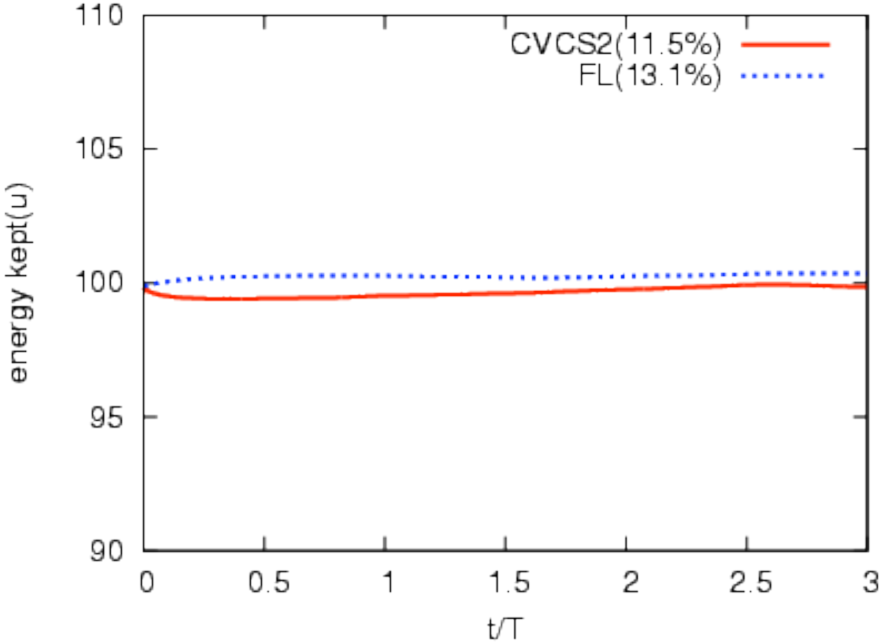
Farge & Schneider, 2001,
Flow, Turb., Comb., ***66****(4), 393*

Schneider, Farge et al., 2005,
J. Fluid Mech., ***534****(5), 39*

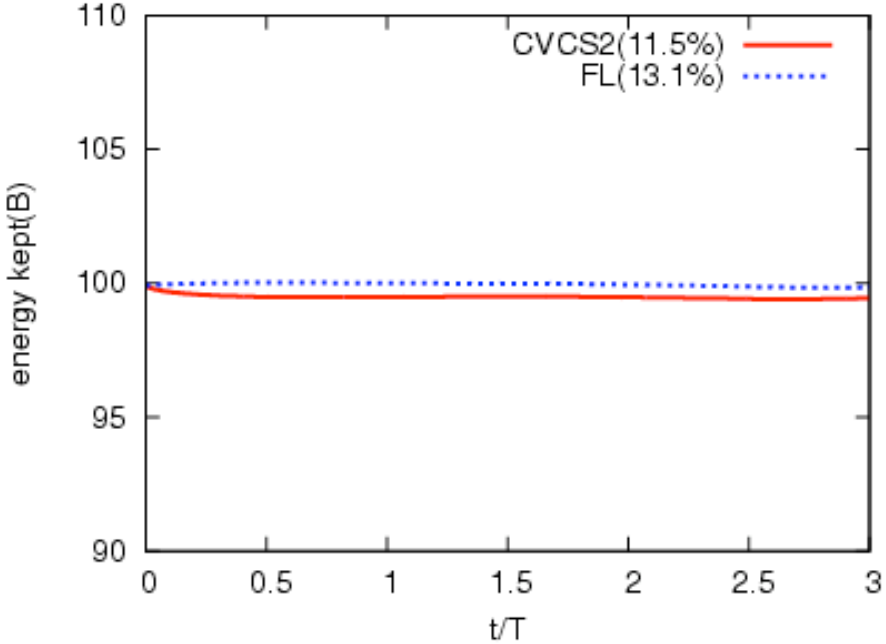
DNS / CVCS / Fourier LES

11.5% N **Retained energy compared to DNS**

Kinetic energy



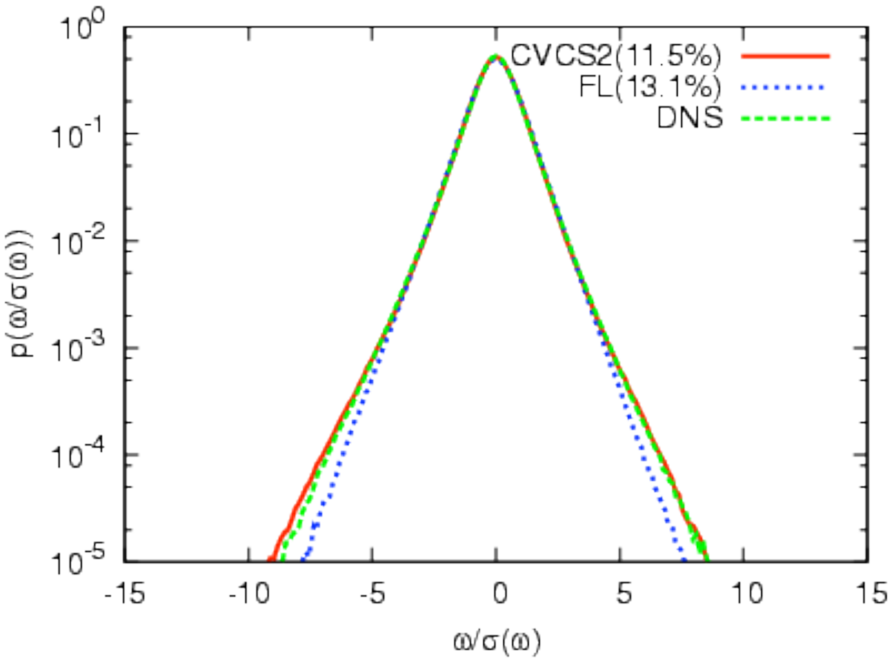
Magnetic energy

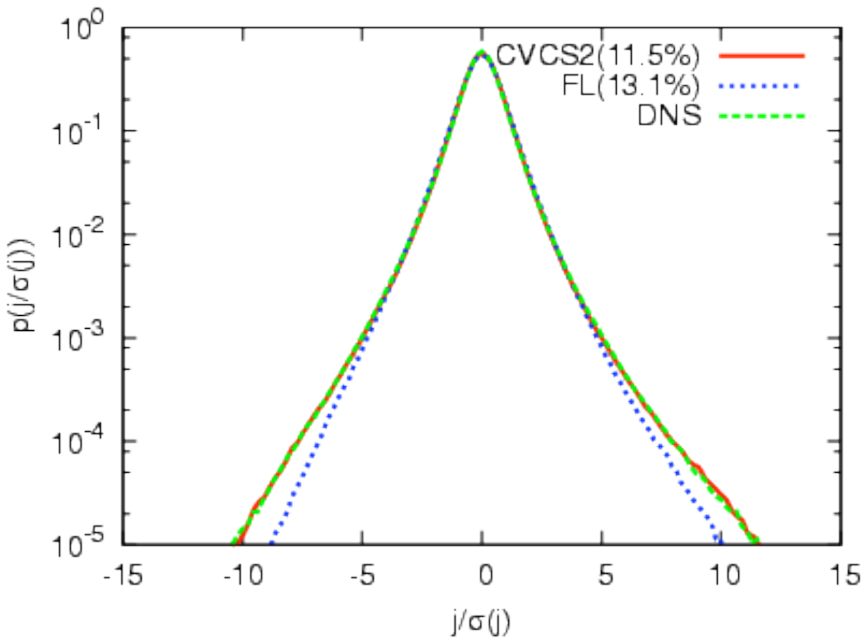


For the ratio of CVS energy over DNS energy,
we find a good agreement within 1.5%.

DNS / CVCS / Fourier LES

11.5% N **PDFs of vorticity and current density**



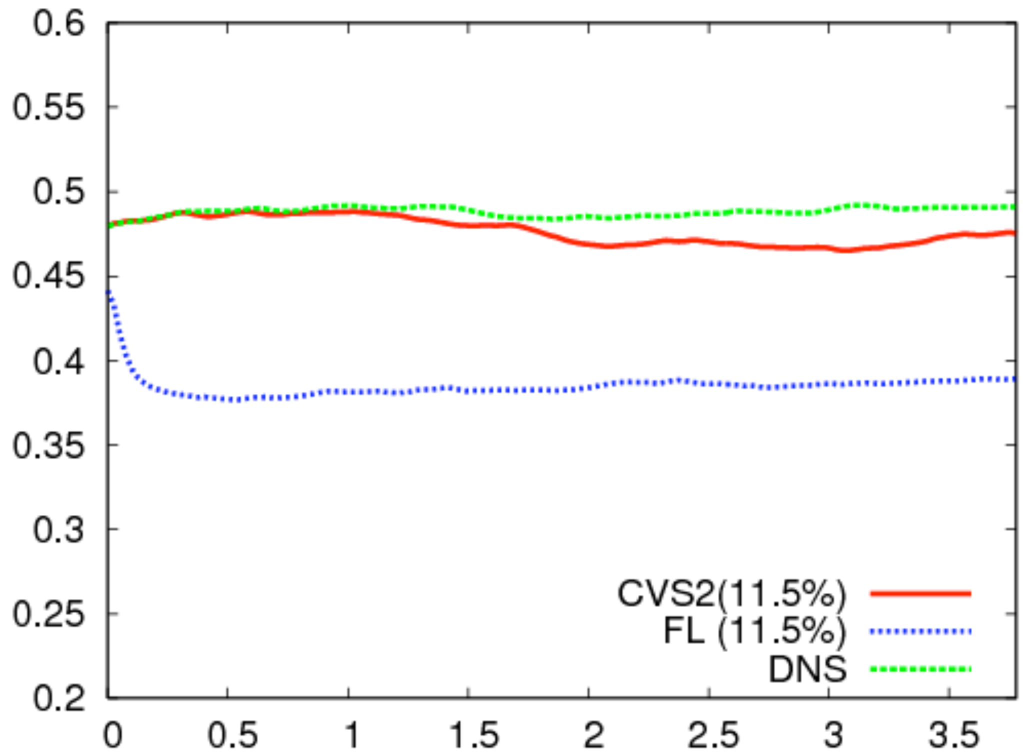


- The PDFs normalized by each standard deviation for DNS and CVCS almost superimpose.
- The PDF for Linear Fourier is slightly narrower compared to that for DNS.

DNS / CVCS / Fourier LES

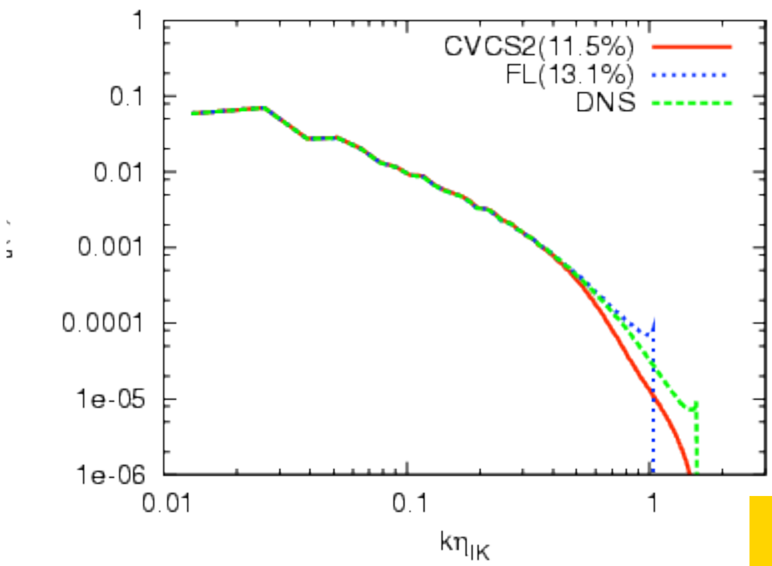
11.5% N

Skewness of the velocity gradients

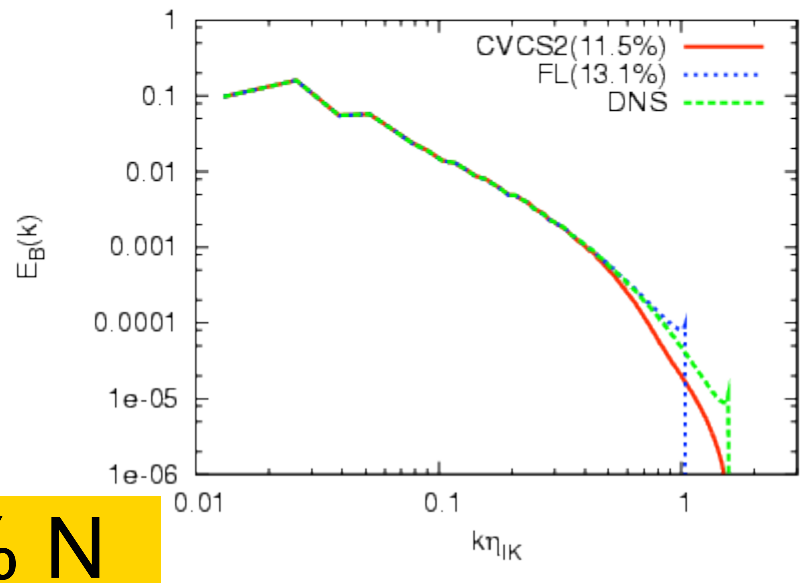


CVCS is in reasonable agreement with DNS but the Linear Fourier for the same compression rate differs by more than 20% with respect to DNS.

Kinetic energy spectrum

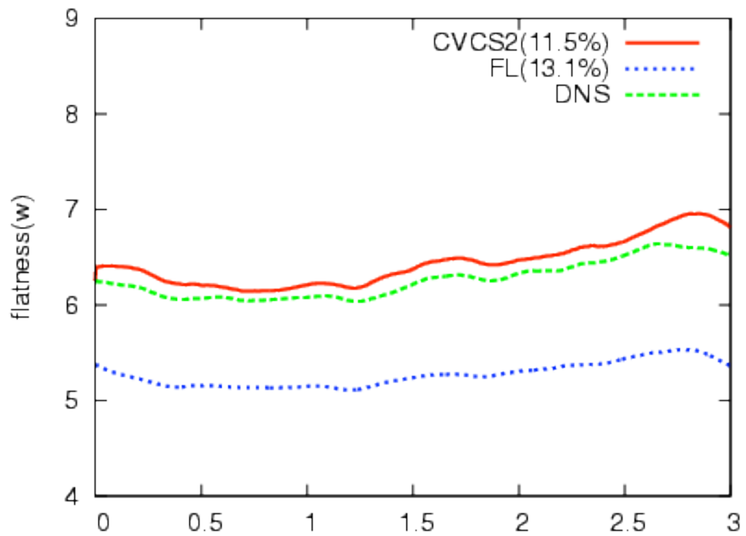


Magnetic energy spectrum

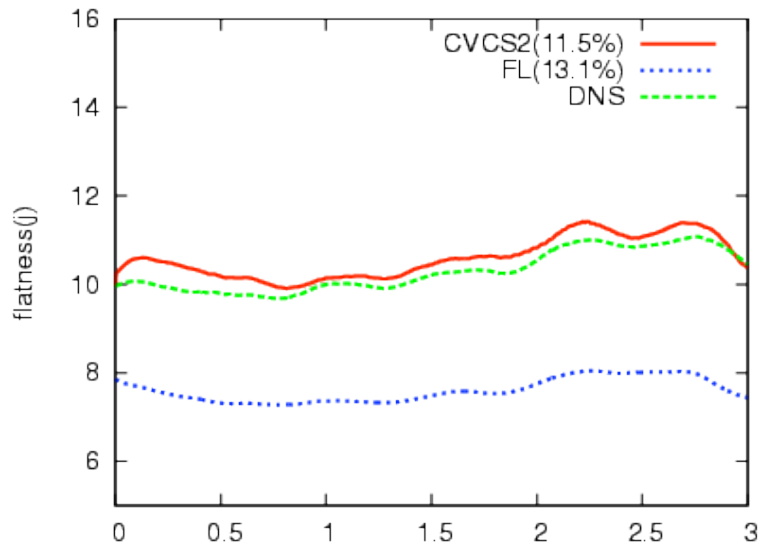


11.5% N

Flatness of vorticity



Flatness of current density



Conclusion

- We introduced **CVCE** method for extracting coherent structures out of 3D homogeneous MHD turbulence,
- About **3.2% N** wavelet coefficients are **sufficient** to represent the coherent vorticity sheets and the coherent current sheets,
- These **coherent structures** are responsible for the **flow intermittency**,
- The **statistics** of the **coherent** velocity and coherent magnetic fields are **similar** to those of the **total** velocity and total magnetic fields, respectively,
- The tests of Coherent Vorticity and Current Simulation (**CVCS**) and their comparison with Fourier/LES are **promising**.

To download papers and codes
http://wavelets.ens.fr