Scaling laws for dynamos in rotating spheres: from planets to stars

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Overview

(1) Definition of dynamo model problem
(2) Overview on proposed planetary magnetic field scaling laws
(3) Theory: Scaling laws for flow velocity
(4) Theory: Scaling laws for magnetic field
(5) Comparison with dynamo model results
(6) Selection of magnetic field geometry
(7) Nusselt number scaling laws
(8) Comparison with planetary field strength
(9) Comparison with rapidly rotating stars

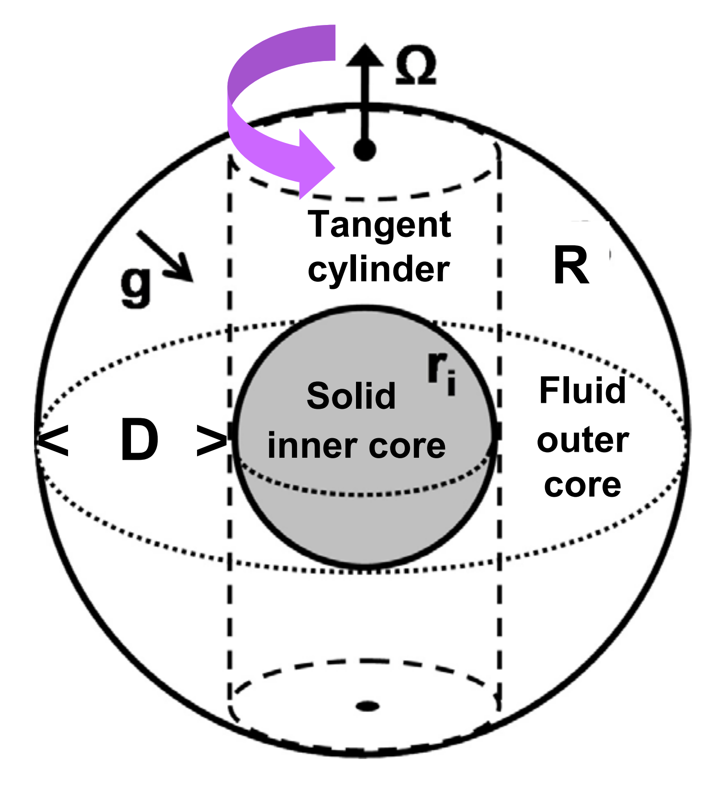
Planetary dynamos

Convection-driven magnetohydrodynamic flow in rotating and electrically conducting spherical shell

Direct numerical simulation of MHD equations possible

But not at realistic values of some key parameters

$\Rightarrow$ Find scaling laws that can possibly be extrapolated to planetary values



Boussinesq equations

$$\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u}\right) + 2\vec{e}_z \times \vec{u} + \vec{\nabla} P = \boxed{E}\, \nabla^2 \vec{u} + \boxed{Ra^*}\, \frac{\vec{r}}{r_o} T + (\vec{\nabla} \times \vec{B}) \times \vec{B}$$

Inertia **Coriolis** **Viscosity** **Buoyancy** **Lorentz**

$$\frac{\partial \mathrm{T}}{\partial t} + \vec{\mathrm{u}} \cdot \vec{\nabla} \mathrm{T} = \boxed{\frac{\mathrm{E}}{\mathrm{Pr}}}\, \nabla^2 \mathrm{T}$$

Advection **Diffusion**

$$\frac{\partial \vec{\mathrm{B}}}{\partial t} + \vec{\mathrm{u}} \cdot \vec{\nabla} \vec{\mathrm{B}} = \vec{\mathrm{B}} \cdot \vec{\nabla} \vec{\mathrm{u}} + \boxed{\frac{\mathrm{E}}{\mathrm{Pm}}}\, \nabla^2 \vec{\mathrm{B}}$$

Advection **Induction** **Diffusion**

E: Ekman number

Ra*: Rayleigh number

Pr: Prandtl number

Pm: magnet. Prandtl number

Control parameters

Definition	Name	Force balance	Earth value	Model values
$Ra^* = \alpha g \Delta T / \Omega^2 D$	Rayleigh number	Buoyancy / Rotational forces	5000 x critical ?	< 1000 x critical
$E = \nu / \Omega D^2$	Ekman number	Viscosity / Coriolis force	10^{-14}	$\geq 10^{-6}$
$Pr = \nu / \kappa$	Prandtl number	Viscosity / Thermal diffusion	0.1 - 1	0.1 – 10
$Pm = \nu / \eta$	Magnetic Prandtl #	Viscosity / Magnetic diffus.	10^{-6}	0.06 - 20

Note: If convection is driven by an imposed heat flow q rather than by a fixed ΔT, Ra* is replaced by a flux Rayleigh number $Ra_q^* = \alpha g q / (\rho c_p D^2 \Omega^3) = q / (\rho H_T D^2 \Omega^3)$.

α: therm. expansivity, g: gravity, D: shell thickness, $H_T = \rho c_p / \alpha$: temperature scale height

Diagnostic numbers

	Name	Ratio of	Earth	Model
$Re = UD/\nu$	Reynolds number	Nonlinear inertia / Viscosity	10^8	10 - 2000
$Rm = UD/\eta$	Magnetic Reynold#	Advection / Magnet. diffus.	10^3	40 - 3000
$Ro = U/\Omega D$	Rossby number	Nonlinear inertia / Coriolis	5×10^{-6}	10^{-4} - 1
$Nu = q/q_{con}$	Nusselt number	Total heat flow / Conductive heat	? (>>1)	2 - 30
$\Lambda=\sigma B^2/2\rho\Omega$	Elsasser number	Lorentz force / Coriolis force	1 - 10	.03 - 100

Scaling laws

... describe for a dynamical system the systematic dependence of diagnostic numbers on the control parameters.

Example: In (non-rotating) highly turbulent convection, the Nusselt number varies with Rayleigh number as

$$Nu \sim Ra^{2/7}$$

Planetary field scaling laws: Practical questions

What controls the strength of a magnetic field generated in a planetary dynamo ?

Candidates: Rotation rate, electrical conductivity, energy flux, density, viscosity, ...

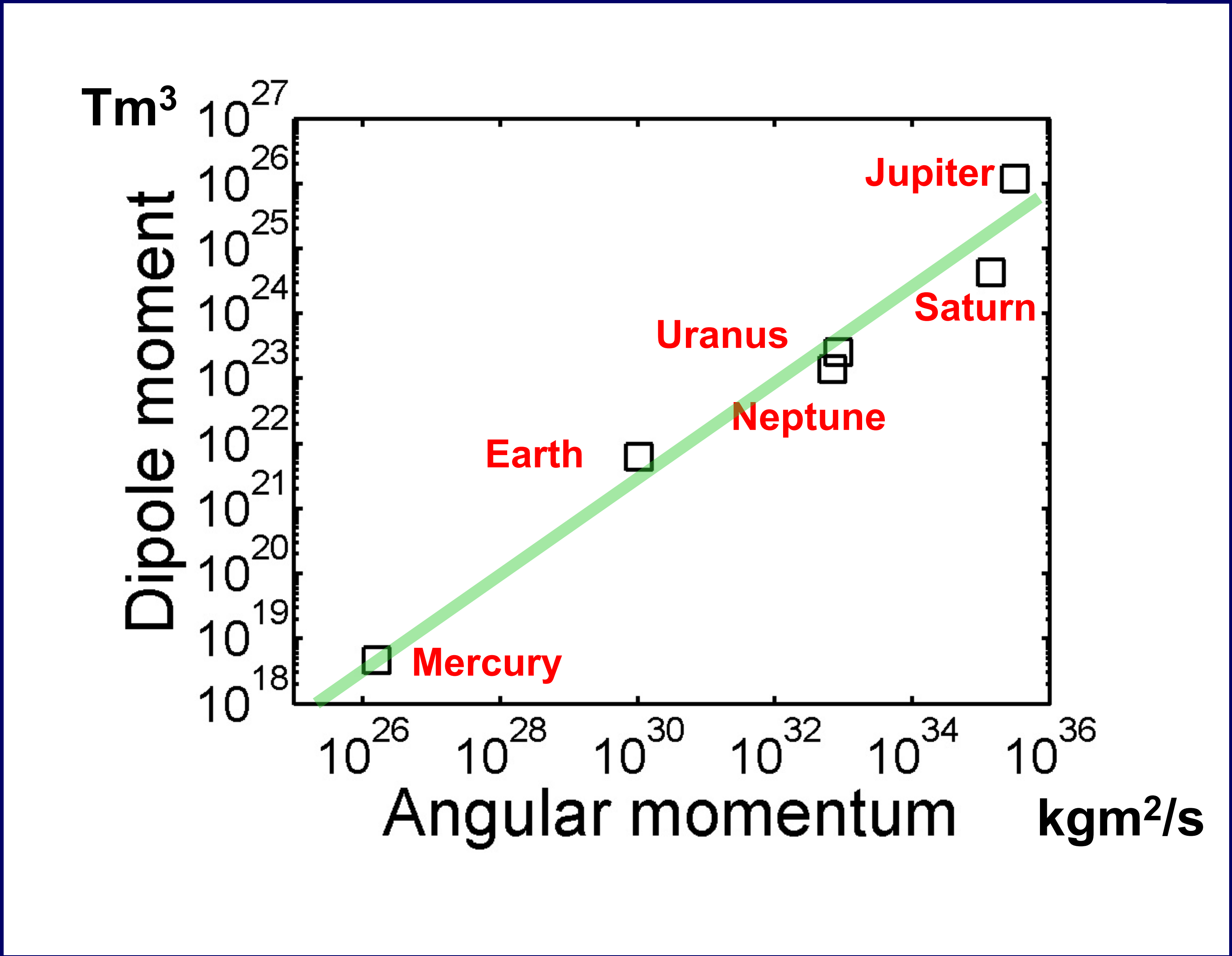
Do the same rules apply to planets and stars ?

Proposed planetary scaling laws

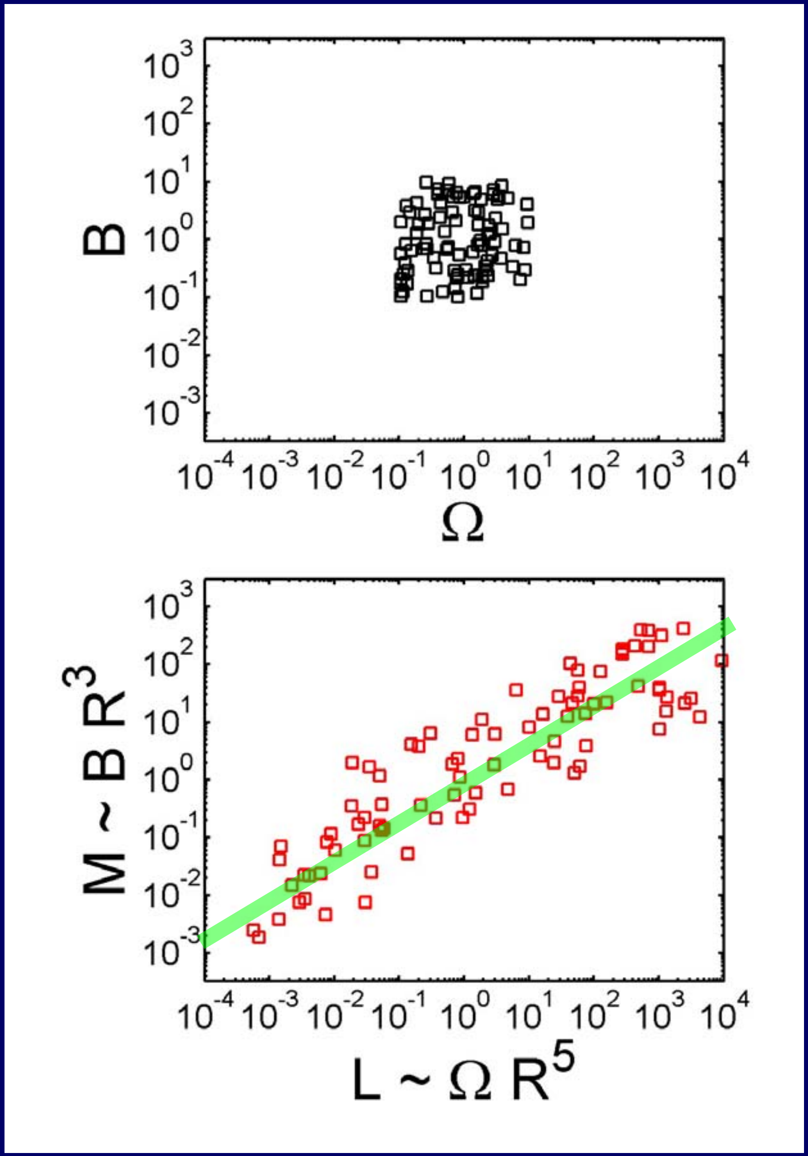
★ $BR^3 \sim (\rho\ \Omega\ R^5)^\alpha$	„Magnetic Bode law“ (Russell, 1978)
$B^2 \sim \rho\ \Omega^2\ R^2$	Busse (1976)
★ $B^2 \sim \rho\ \Omega\ \sigma^{-1}$	„Elsasser # rule“ (Stevenson, 1979)
$B^2 \sim \rho\ R^3\ q\ \sigma$	Stevenson (1984) *(for small q)*
$B^2 \sim \rho\ \Omega\ R^{5/3}\ q^{1/3}$	Curtis & Ness (1986)
$B^2 \sim \rho\ \Omega^{3/2}\ R\ \sigma^{-1/2}$	Mizutani et al. (1992)
$B^2 \sim \rho\ \Omega^2\ R$	Sano (1993)
$B^2 \sim \rho\ \Omega^{1/2}\ R^{3/2}\ q^{1/2}$	Starchenko & Jones (2002)
★ $B^2 \sim \rho\ R^{4/3}\ q^{2/3}$	Christensen & Aubert (2006)

σ: conductivity, ρ: density, Ω: rotation rate, R: core radius, q: convected heat flux

The „magnetic Bode law“



The „Bode law fallacy“



- Generate random distributions of variables Ω („rotation rate“) and B („field strength“) in range 0.1 – 10
- Generate random distribution of variable R („radius“) 1/6 < R < 6
- Introduce new variables $L=\Omega R^5$ („angul. momentum“) $M=BR^3$ („dipole moment“)
- There is a correlation between M and L

Cain et al., J. Geophys. Res. 1995

Velocity scaling: Mixing length

Balance in vorticity equation (non-magnetic):

$$\rho \ (u \bullet \nabla) \, u \quad \sim \quad \rho \alpha g \delta T \, e_r$$

Assume mixing length ℓ and balance inertia ~ buoyancy

$$U^2/\ell \quad \sim \quad \alpha g \delta T$$

Relate temp. fluctuation to convected heat flux: $q_c \sim \rho c_p U_r \delta T$

Introduce temperature scale height: $H_T^{-1} = \alpha g / c_p$

$$U \sim [q_c \, \ell \, / \, \rho \, H_T]^{1/3}$$

In astrophysics the mixing length is usually set to $\ell \sim H_\rho$.

In planets $H_\rho > D$ and it is appropriate to set $\ell \sim D$.

α: thermal exp. coeff., g: gravity, ℓ: mixing length, δT: fluctuating T, D: global length scale,

Velocity scaling: MAC-balance

$$\mathbf{j} \times \mathbf{B} \quad \sim \quad \rho\alpha g\delta T\mathbf{e}_r \quad \sim \quad 2\rho\Omega\, \mathbf{e}_z \times \mathbf{u}$$

Magnetic Archimedean Coriolis

Assuming that all three forces are of the same order, any two must balance. Considering the balance between buoyancy and Coriolis force results in

$$2\,\Omega\, U \quad \sim \quad \alpha g \delta T$$

and in terms of the convected energy flux and H_T in

$$U \sim [\, q_c / (\rho\, \Omega\, H_T)\,]^{1/2}$$

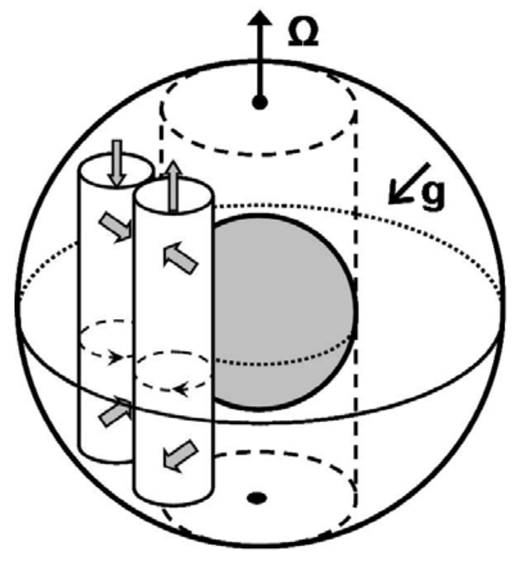
(Starchenko & Jones, 2002)

Velocity scaling: CIA-balance

$$2\rho\Omega\, \partial u/\partial z \;\sim\; \rho\, \nabla\times([\nabla\times u]\times u) \;\sim\; \rho\alpha g\, \nabla\times T e_r$$

Coriolis   Inertia   Archimedean

Assume that the (small) length scale ℓ applies to derivatives in inertia and buoyancy terms. Because of quasi-geostrophic flow structure, the length scale L associated with $\partial/\partial z$ in Coriolis term is large. Triple balance allows to determine U and ℓ. Replacing δT by q_c, the rule for U is:



$$U \;\sim\; [q_c \,/\, (\rho H_T)]^{2/5}\, (L/\Omega)^{1/5}$$

For planets L = D, but with strong density stratification $L = H_\rho$

(Aubert et al., 2001)

Velocity scaling: Non-dimensional

Divide scaling laws by DΩ ⇓ Rossby number

$$\mathbf{Ro = U/(\Omega D)}$$

Terms on RHS transform to flux Rayleigh number

$$Ra_q^* = \alpha g q_c / (\rho c_p D^2 \Omega^3) = q_c / (\rho H_T D^2 \Omega^3)$$

$Ra_q^* = Ra^*(Nu-1)E/Pr$ (Ra* is control parameter).

Ra_q^* is proportional to the (non-dim.) power generated by buoyancy forces. The three scaling laws become:

$$Ro \sim (Ra_q^*)^\gamma$$

where $\gamma=1/3$ (mixing length) , $\gamma=½$ (MAC) or $\gamma=2/5$ (CIA)

Magnetic field: Elsasser number

Balance Coriolis – Lorentz (Magnetostrophic)

$$2\rho\, \Omega \times u \quad \sim \quad j \times B$$

Generalized Ohm's law: $j = \sigma\, (E + u \times B)$

Ignore electric field

$$J \sim \sigma U B \qquad \Rightarrow \qquad 2\rho\Omega U \sim \sigma U B^2$$

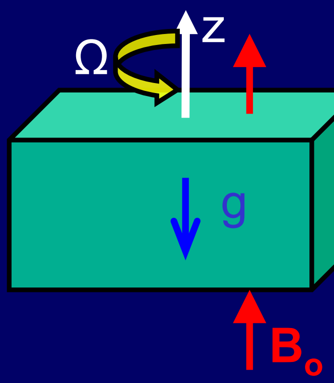
Elsasser number

$$\Lambda \quad = \quad \sigma B^2/(2\rho\Omega) \quad \sim \quad 1$$

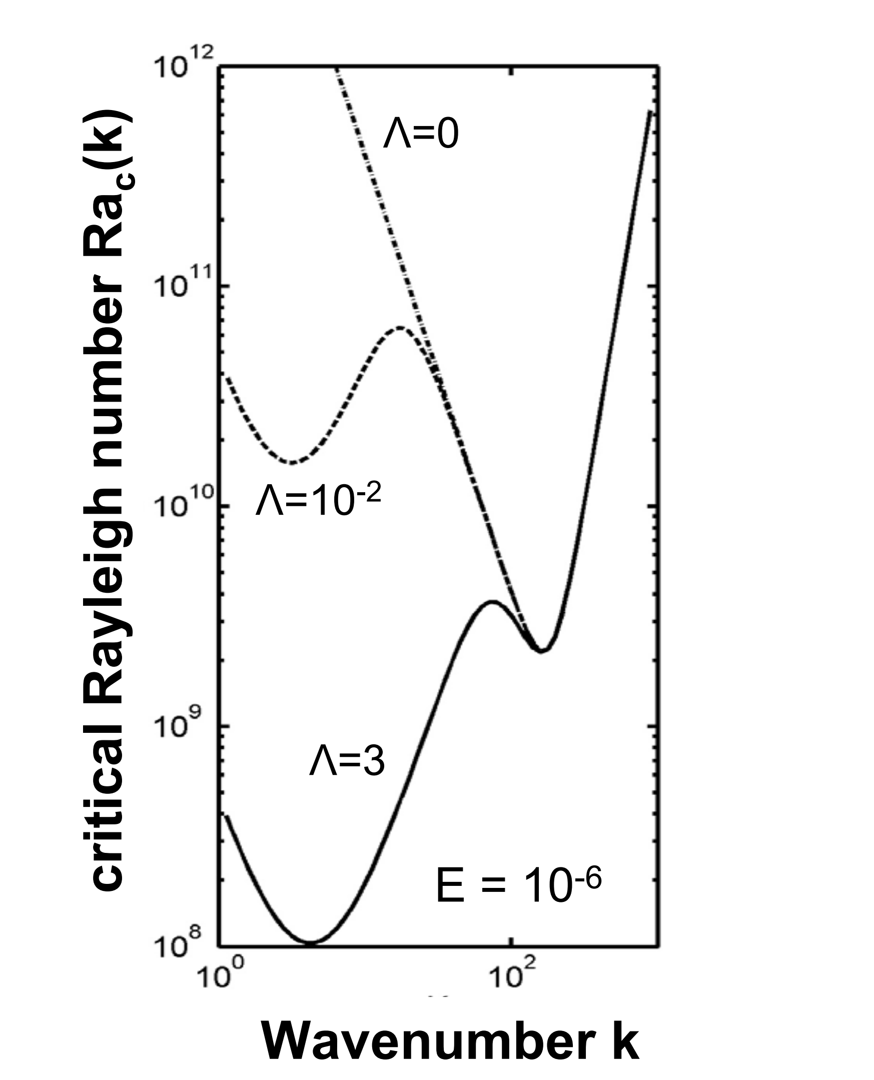
(Stevenson, 1979)

j: current density, E: electric field, capital letters indicate characteristic value

Onset of rotating convection with imposed field



- Rotation impedes convection
- Magnetic field impedes convection
- Both combined are less retarding when $\Lambda \sim 1$



Chandrasekhar, 1961.

Magnetostrophic balance – different approach

$$2\rho\, \mathbf{\Omega} \times \mathbf{u} \quad \sim \quad \mathbf{j} \times \mathbf{B}$$

$$\mathbf{j} = \mu_o^{-1}\, \nabla \times \mathbf{B} \qquad\qquad J \sim \mu_o^{-1}\, B/\ell_B$$

$$B^2/2\mu_o \quad \sim \quad \rho\, \Omega\, U\, \ell_B$$

(1) Use your preferred scaling law for U
(2) Make some assumption about ℓ_B

Leads to scaling laws proposed by Busse(1976), Curtis & Ness (1986), Mizutani et al. (1992), Sano (1993), Starchenko & Jones (2002)

Power-controlled magnetic field strength

Power generated by buoyancy per unit volume:

$$P \sim \rho g \alpha U_r \delta T \sim q_c / H_T$$

Convected heat flux $q_c = \rho c_p U_r \delta T$.

Temperature scale height $H_T^{-1} = \alpha g / c_p$

Ohmic dissipation per unit volume:

$$D_{ohm} \sim j^2/\sigma \sim \eta B^2 / (\mu_o \ell_B^2)$$

η: magnetic diffusivity, ℓ_B: magnetic field length scale. Fraction f_{ohm} of total power dissipated ohmicly (rather than viscously)

$$B^2/2\mu_o \sim f_{ohm} \ell_B^2/\eta \; (q_c/H_T)$$

Power-controlled field strength

$$E_{mag} = B^2/2\mu_o \sim f_{ohm}\, \ell_B{}^2/\eta \ (q_c/H_T)$$

(1) $\ell_B{}^2/\eta = f_1\,(U/D, \Omega)$ assume $\ell_B{}^2/\eta \sim (U/D)^{-1}$

equivalent to $\ell_B/D \sim Rm^{-1/2}$

(2) $Ro = U/D\Omega \sim (Ra_q{}^*)^{\gamma}$

$\gamma = 1/3$ (Mixing l.) $E_{mag} \sim f_{ohm}\, \rho^{1/3}\, (q_c\, D/H_T)^{2/3}$

$\gamma = 1/2$ (MAC) $E_{mag} \sim f_{ohm}\, D\, (q_c \rho \Omega/H_T)^{1/2}$

$\gamma = 2/5$ (CIA) $E_{mag} \sim f_{ohm}\, \rho^{2/5} \Omega^{1/5} D^{4/5}\, (q_c/H_T)^{3/5}$

Non-dimensional scaling laws

Many different ways to non-dimensionalize possible.
Here use „rotational scaling“:

$E^m{}_\Omega := E_{mag} / (\rho\Omega^2 D^2)$ $\qquad\qquad$ $q_\Omega := q_c / (\rho\Omega^3 D^3)$

$$E^m{}_\Omega \sim f_{ohm} \, (q_\Omega \, D/H_T)^p \sim f_{ohm} \, (Ra_q{}^*)^p$$

where $\quad p = 2/3 \quad$ mixing length balance ($\gamma=1/3$)

$p = 1/2 \quad$ MAC balance ($\gamma=1/2$)

$p = 3/5 \quad$ CIA balance ($\gamma=2/5$)

Thermodynamic efficiency

$$E_{mag} = B^2/2\mu_o \sim f_{ohm}\, \rho^{1/3} (q_c D/H_T)^{2/3}$$

So far, balance power generation / ohmic dissipation treated locally. Global balance more likely in dynamos with large-scale field. Complication: q_c, ρ and H_T vary with radius. Assumption: volume average is meaningful $\Rightarrow$ efficiency factor F

$$F^{2/3} = \frac{1}{V}\int_{r_i}^{R} \left(\frac{q_c(r)}{q_o}\frac{L(r)}{H_T(r)}\right)^{2/3} \left(\frac{\rho(r)}{\bar{\rho}}\right)^{1/3} 4\pi r^2 dr$$

Dim: $E_{mag} = c\, f_{ohm}\, \rho_o^{1/3} (Fq_o)^{2/3}$

Non-dim: $E^m = c\, f_{ohm} (Fq)^{2/3}$

F can be calculated for a given structural model of planet/star.

q_o, ρ_o: reference values, c: constant of proportionality.

Test of scaling laws

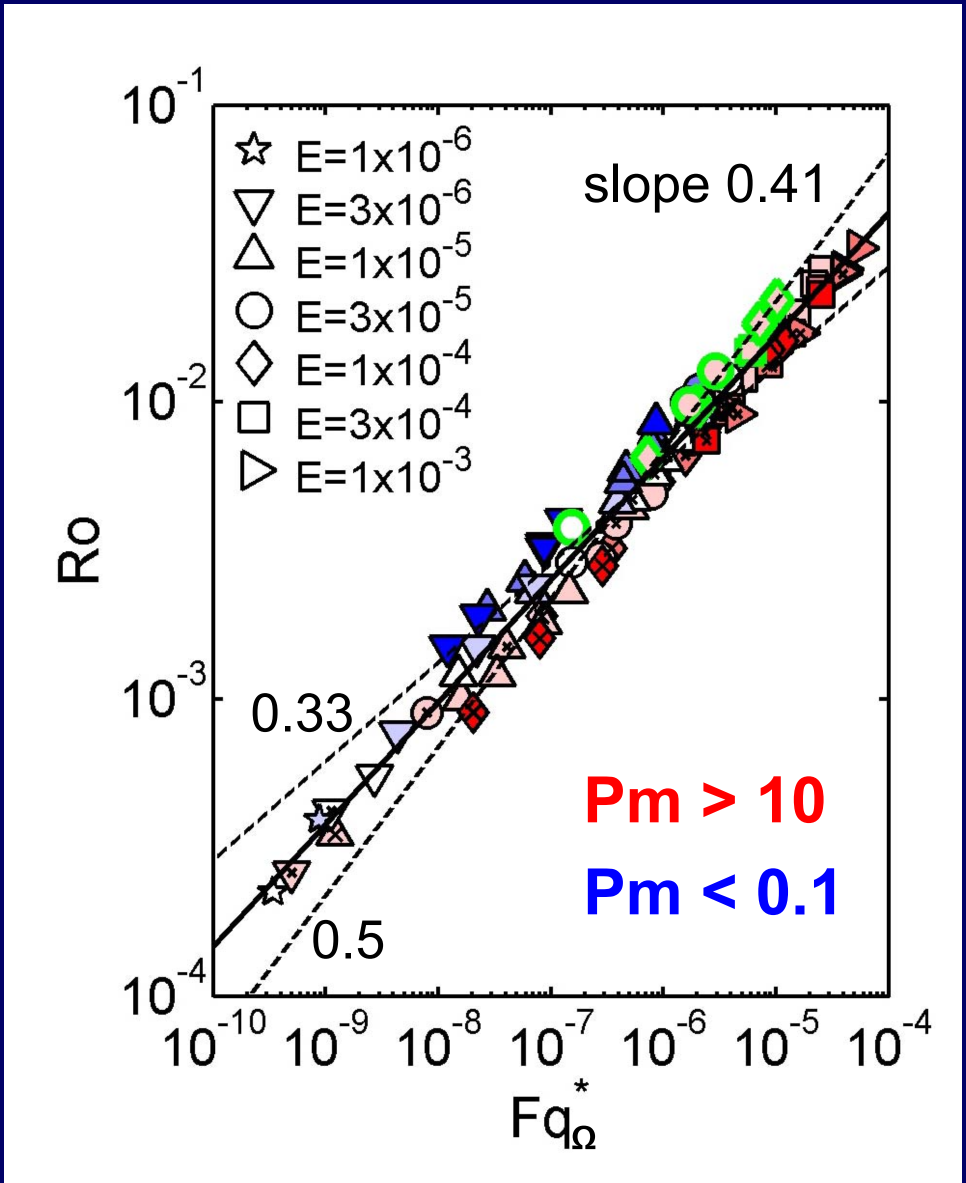
A large number of numerical dynamo simulations is now available, which cover a sufficiently wide range in parameter space, to test scaling laws.

$10^{-6} \leq E \leq 10^{-3}$ $0.06 \leq Pm \leq 15$ $0.1 \leq Pr \leq 10$

Select dynamo models with a dipole-dominated magnetic field.

Use only cases with significantly supercritical convection ($Nu > 2$) to test scaling laws.

Rossby number scaling



Symbol shape: different Ekman number (viscosity vs. rotation)

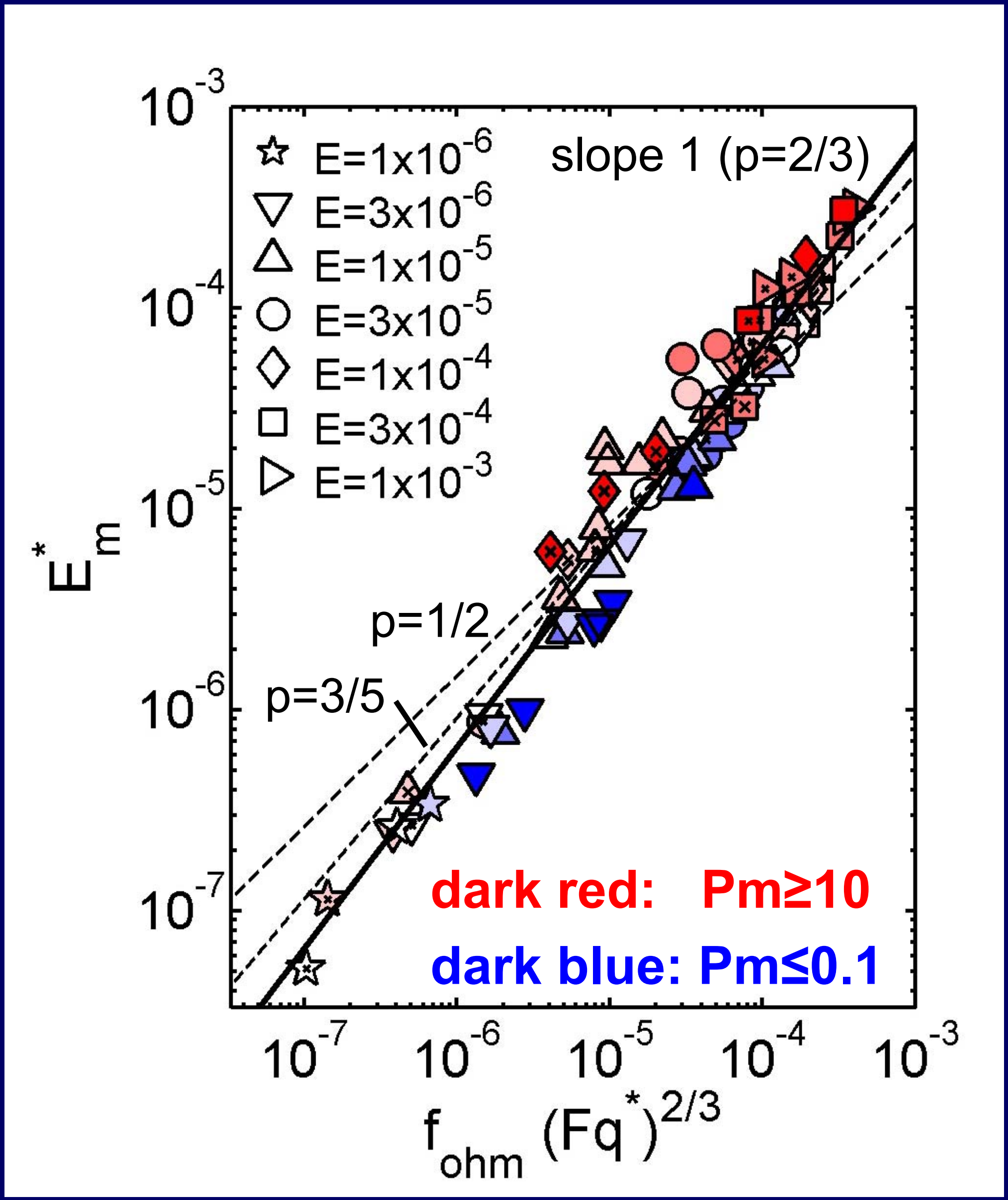
Color: different magn. Prandtl # (viscosity / magnetic diffusivity)

Green-rimmed symbols for fixed heat flux

Best agreement with CIA-scaling theory

Christensen & Aubert, 2006 (updated)

Magnetic energy scaling



Symbol shape: different Ekman number (viscosity vs. rotation)

Color: different magn. Prandtl # (viscosity / magnetic diffusivity)

f_{ohm} recorded in each model run (0.3 – 0.8)

Best agreement with mixing length theory

Christensen & Aubert (2006),
Christensen et al. (2009)
Takahashi et al. (2008)

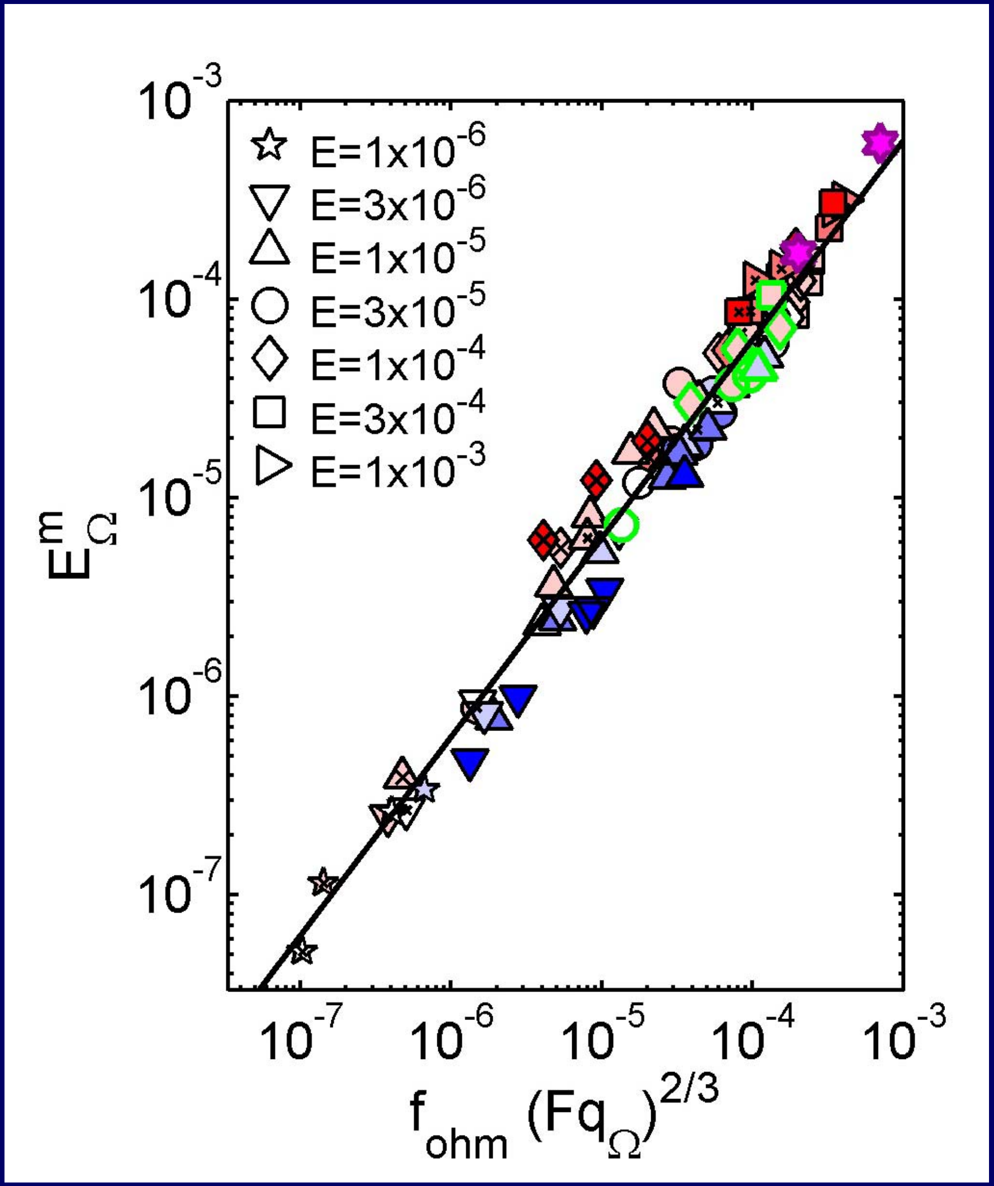
Interim conclusion

Results for different Ekman number, Prandtl number and magnetic Prandtl number nearly collapsed on single line $\Rightarrow$ viscosity, diffusivities and rotation play no role.

A weak residual dependence on the magnetic Prandtl number may exist.

Conflict between best fitting exponents for Ro and for E_{mag}.

Scaling versus dynamo models



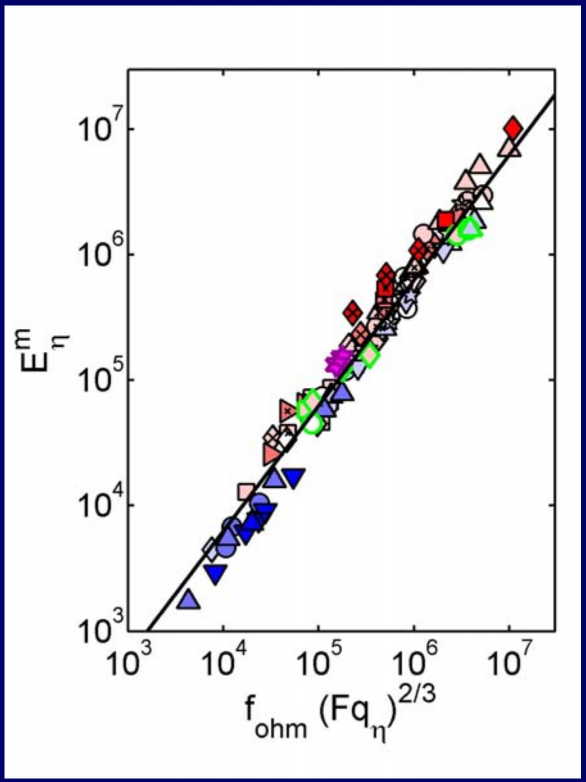
Test of recipe for calculating F-factor

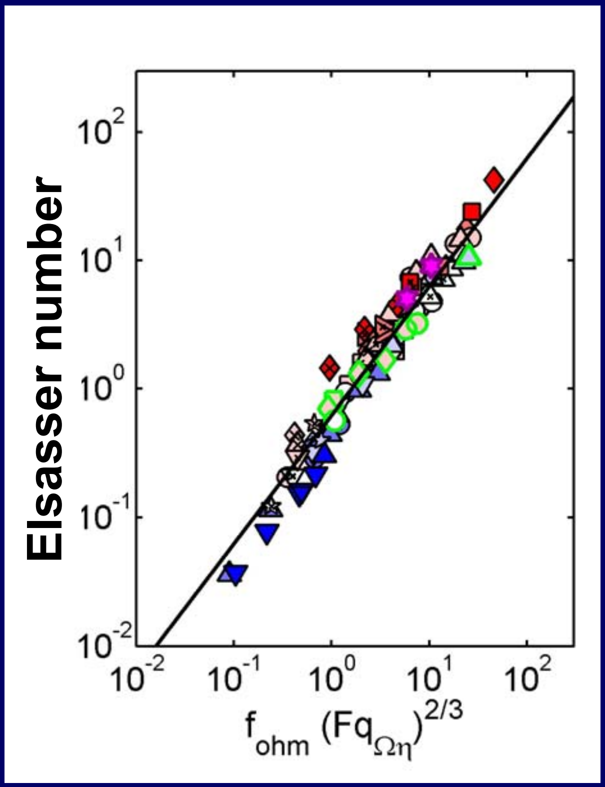
Green: „Compositional convection“ – zero flux on outer boundary

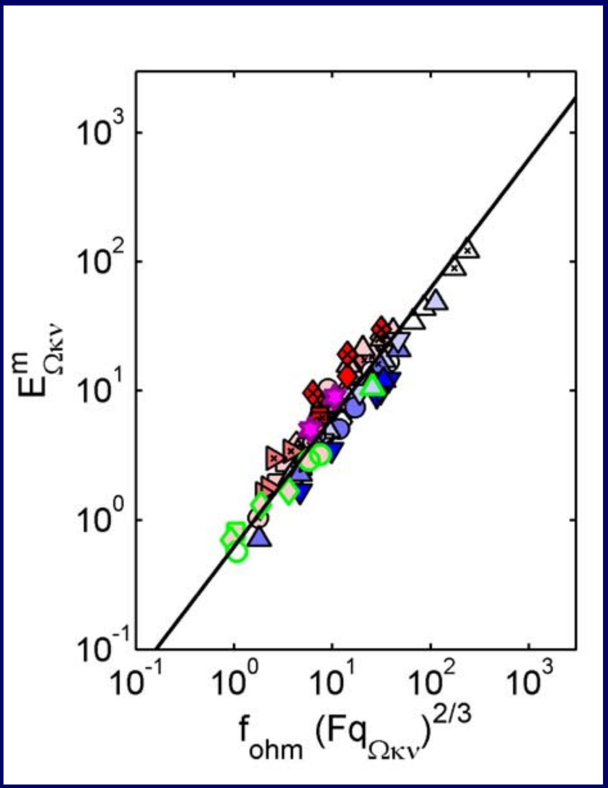
Pink: Moderately com-pressible dynamo models

(Dobler et al., ApJ, 2006)

Different non-dimensionalizations







$q' = q_o R^3/\rho\eta^3$

$E^m = B^2R^2/2\mu_o\rho\eta^2$

magnetic scaling

$q' = q_o/\rho(\Omega\eta)^{3/2}$

$E^m = B^2/2\mu_o\rho\Omega\eta$

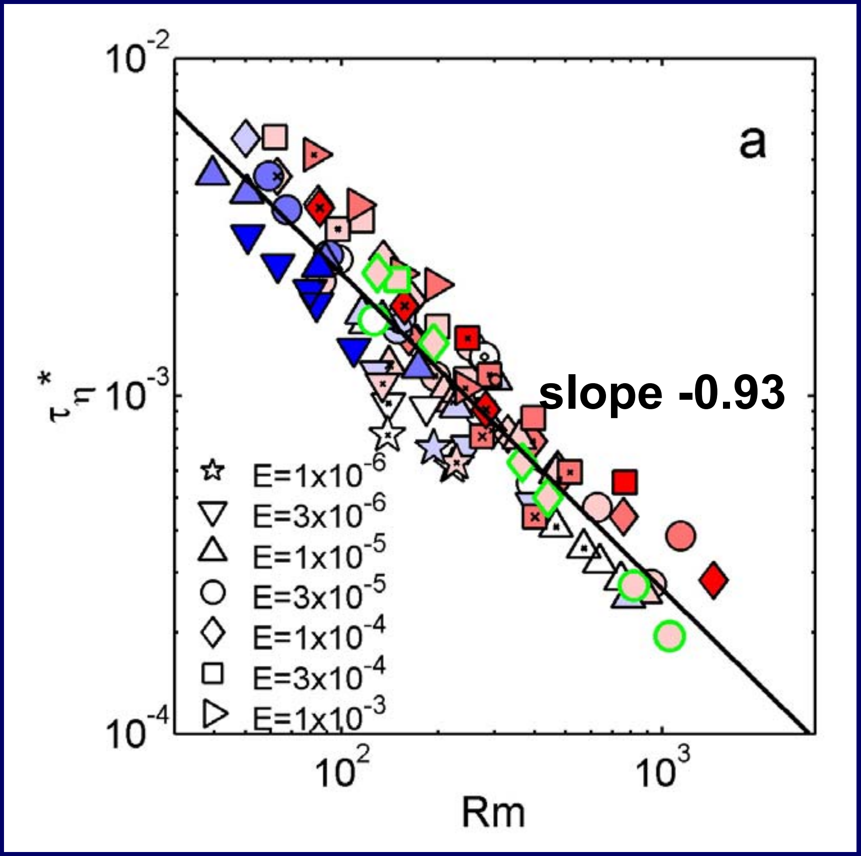
magneto-rotational scaling

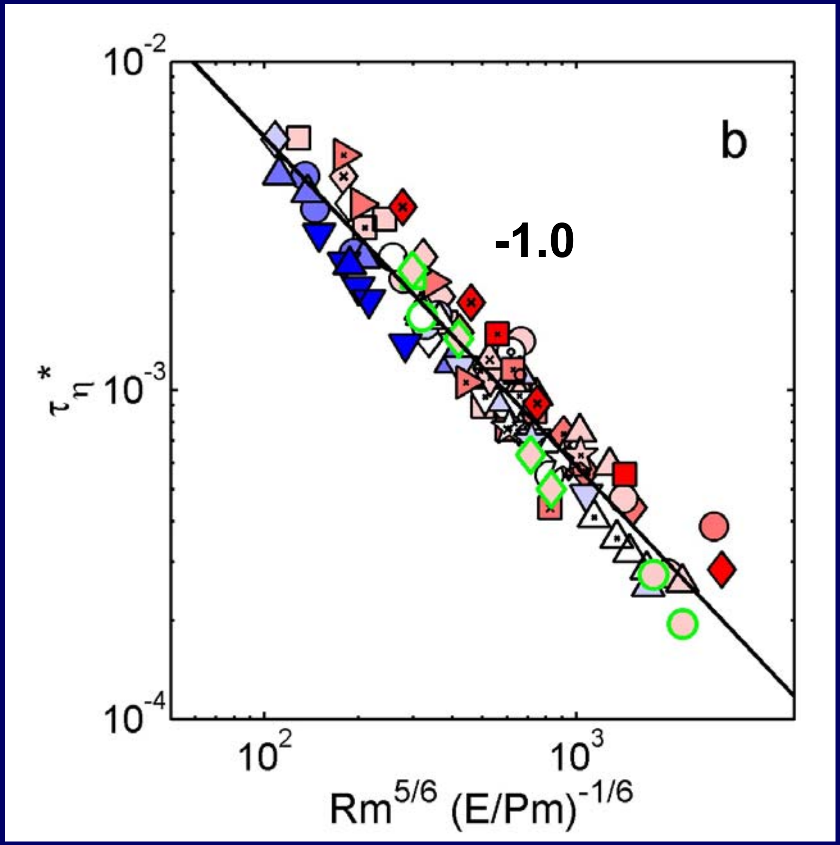
$q' = q_o R^{1/3}/\rho\Omega^{4/3}\nu^{-1/3}\kappa^2$

$E^m = B^2R^{2/9}/2\mu_o\rho\Omega^{8/9}\nu^{-2/9}\kappa^{4/3}$

critical Rayleigh number scaling

Ohmic dissipation time





Ohmic dissipation time $\tau_\eta = E_{mag}/D_{ohm} \sim \ell_B^2/\eta$

Non-dimensional: $\tau_\eta^* = \tau_\eta\,(\eta/D^2) \sim (\ell_B/D)^2$

see also Christensen & Tilgner (2004)

Ohmic dissipation time

Dependence of τ_η* on „magnetic Ekman number“
E_η = E/Pm implies (weak) dependence of τ_η on Ω

$$\tau_\eta \sim \ell_B{}^2/\eta \sim (D/U)^{5/6}\,\Omega^{-1/6}$$

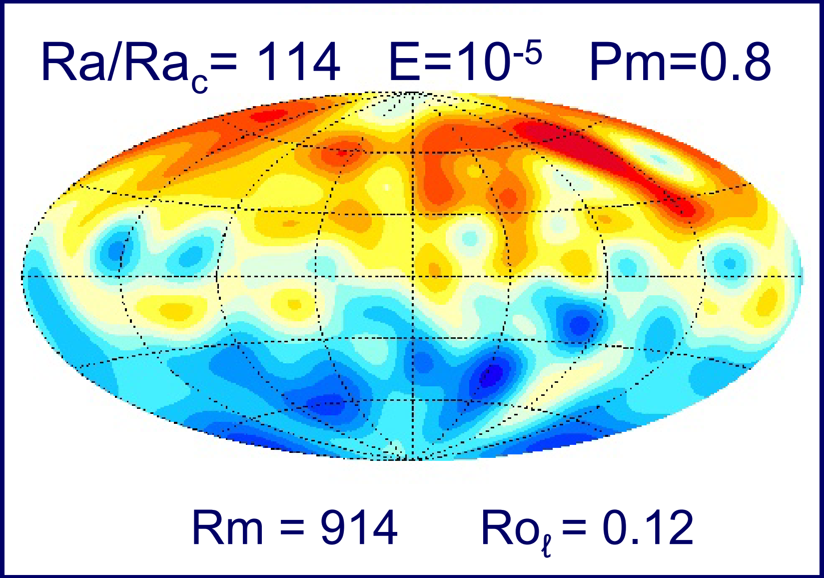
This, combined with a 2/5-power law for the Rossby number, gives exactly the 1/3-power law for the magnetic energy density.

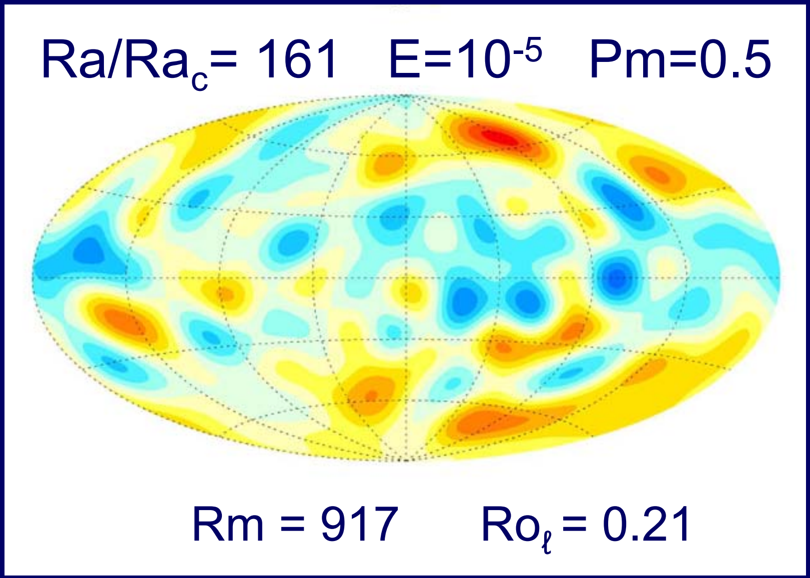
What is the role of rotation ?

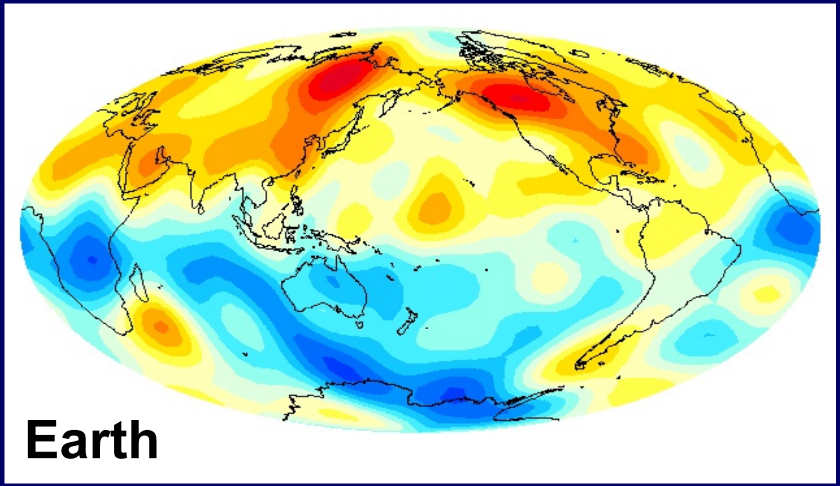
So far, we considered only rapidly rotating dynamos. What changes when inertial forces can compete with the Coriolis forces ?

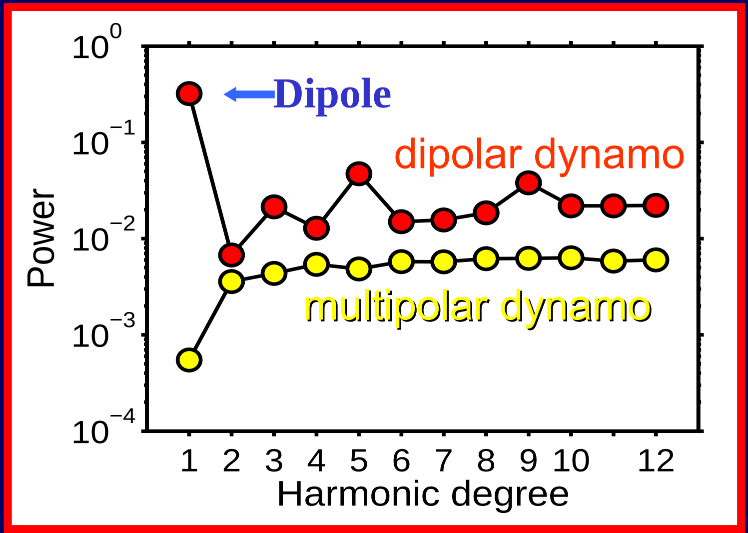
$\Rightarrow$ Field morphology changes

Magnetic field morphology

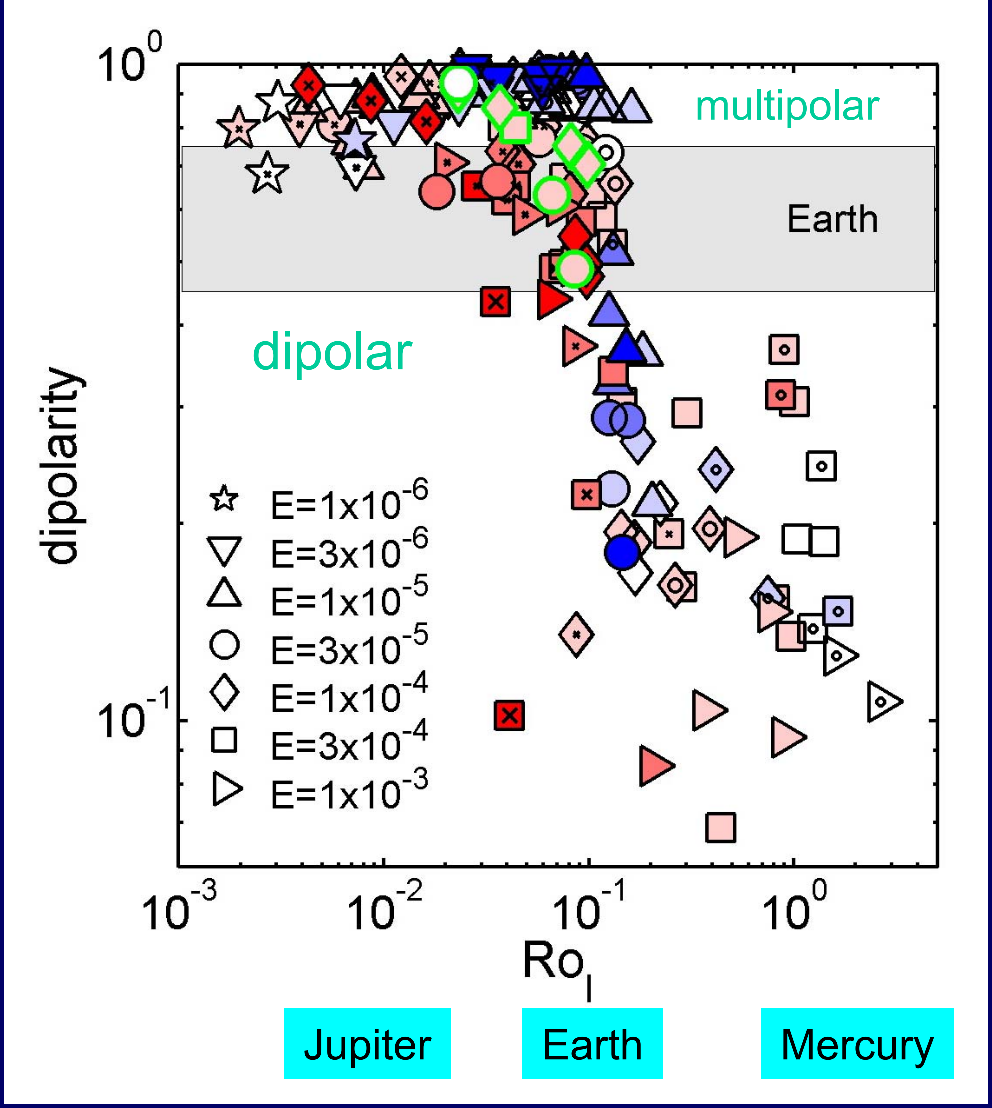








The local Rossby number



Inertial vs. Coriolis force:

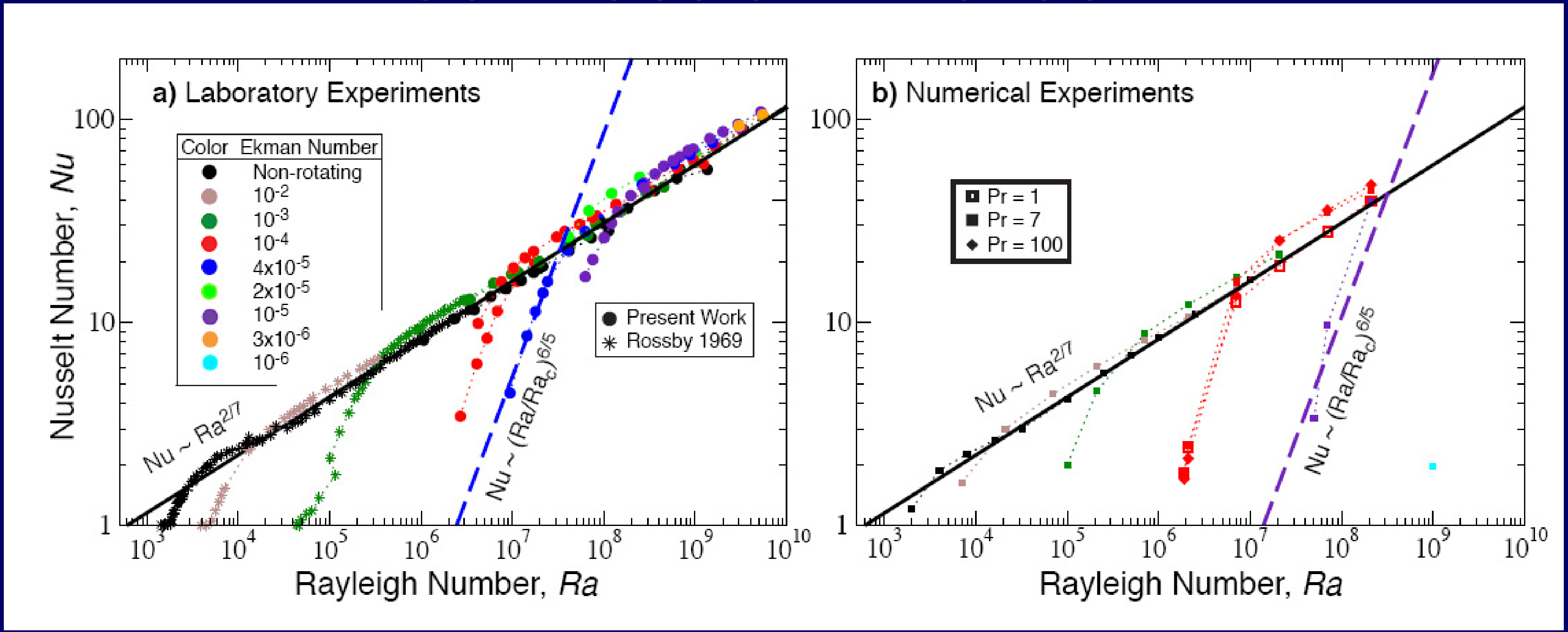
Rossby number Ro_ℓ calculated with mean length scale ℓ in the kinetic energy spectrum

$$Ro_\ell = U/\Omega\ell$$

Regime boundary at $Ro_\ell \approx 0.12$

Christensen & Aubert (2006); Olson & Christensen (2006) – updated.

Nusselt number scaling: rotating convection in a box



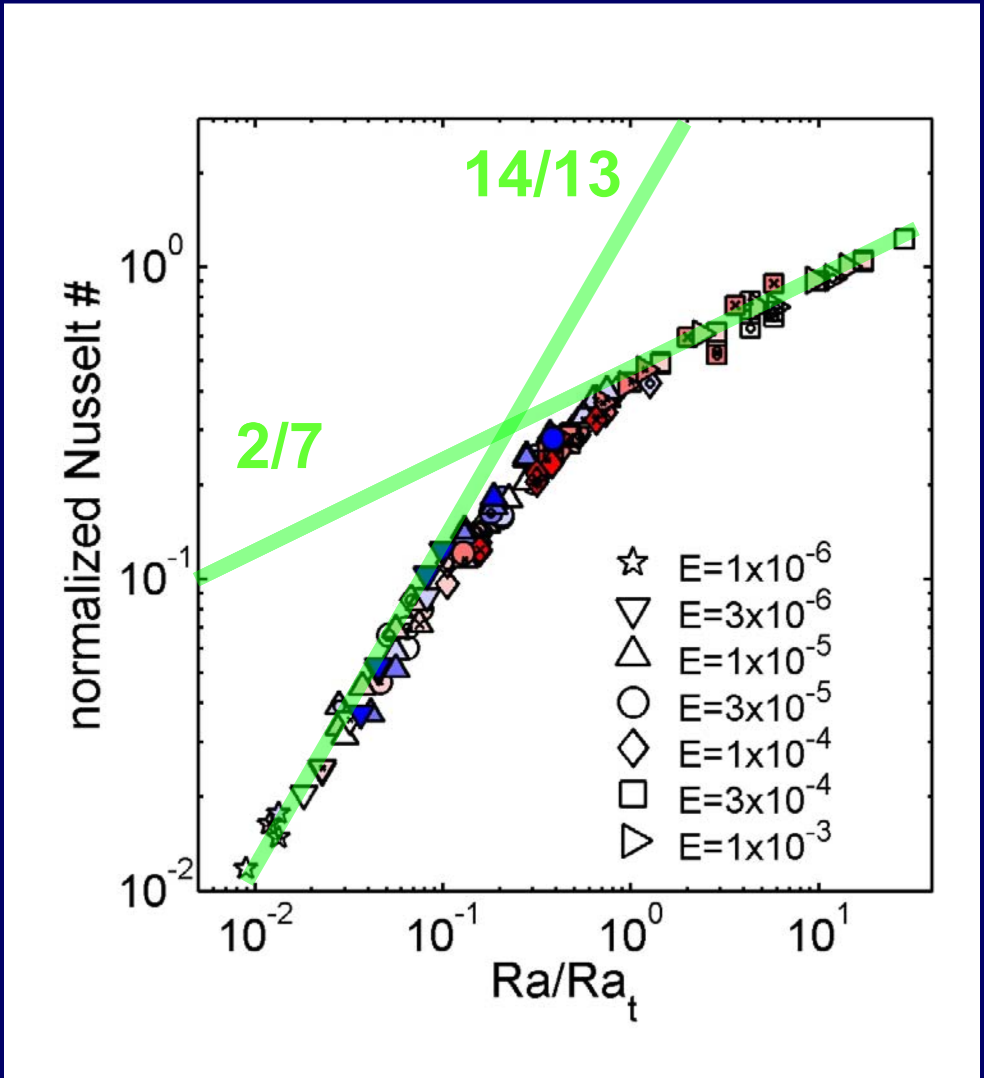
Rotating convection in cartesian geometry:

$Nu \sim Ra^{6/5}$ for $Ra_{crit} < Ra < Ra_t$ $Nu \sim Ra^{2/7}$ for $Ra > Ra_t$

at $Ra=Ra_t$ thickness of thermal boundary layer ≈ Ekman layer

(King et al., Nature, 2009)

Heat transfer scaling: dynamos



Transition Rayleigh #

$$Ra_t \sim E^{-7/4}$$

Normalized Nusselt #

$$Nu' = Nu / Ra_t^{2/7}$$

Transition in Nusselt number nearly coincident with dipole – multipole transition.

$Ra_t/Ra_{crit} \approx 100$ at $E = 10^{-5}$

$Ra_t/Ra_{crit} \approx 10^5$ at $E = 10^{-14}$

Application to celestial dynamos

Use structural models to evaluate

$$F^{2/3} = \frac{1}{V}\int_{r_i}^{R}\left(\frac{q_c(r)}{q_o}\frac{L(r)}{H_T(r)}\right)^{2/3}\left(\frac{\rho(r)}{\bar{\rho}}\right)^{1/3} 4\pi r^2 dr$$

setting the length scale $L = \min(D, H_\rho)$

Jupiter	F = 1.1
Earth's core	F = 0.27 – 0.52
Stars	F = 0.78 – 1.14

<...>: volume average, D: dynamo layer thickness, H_ρ: density scale height

Christensen et al. (2009)

From surface to interior field

Observations tell us about the field at the surface of the dynamo (or only its long-wavelength part).

The scaling law makes a prediction about the mean internal field.

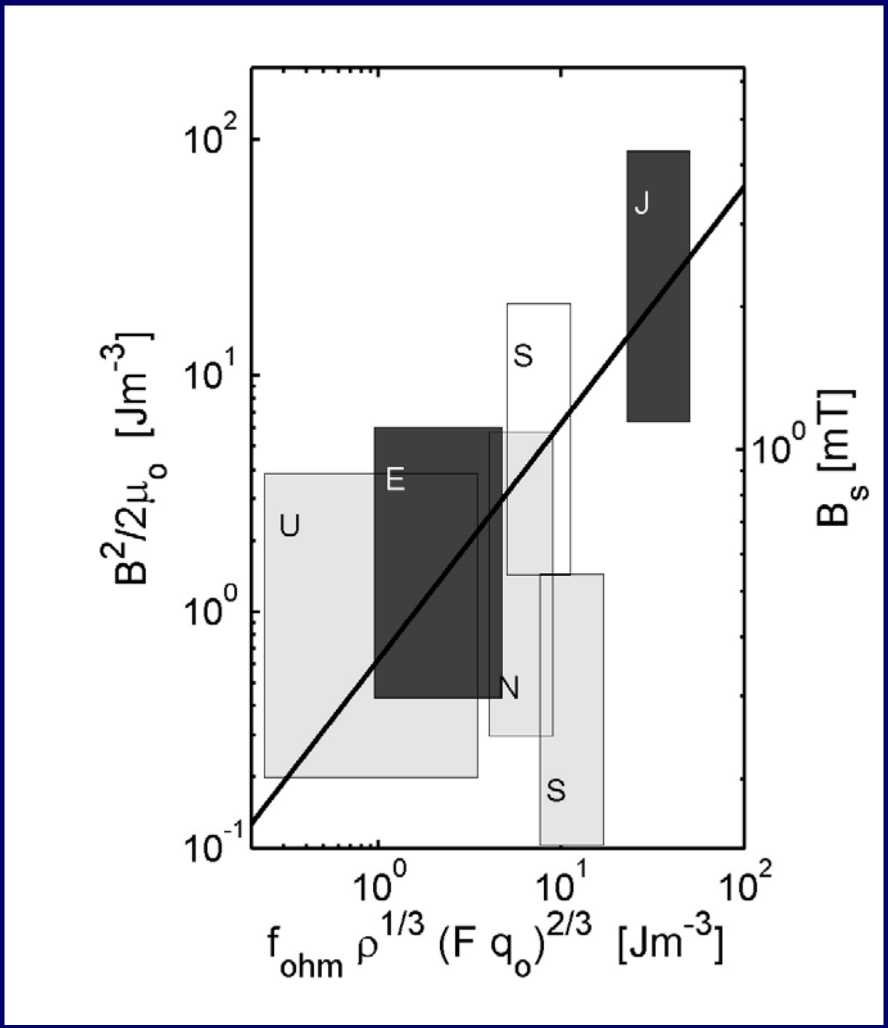
How are they related ?

$\Rightarrow$ Use results from dynamo models for ratios:

$B_{interior} / B_{surface} \approx 2.5 – 5.5$ (3.5 nominally)

$B_{interior} / B_{dipole} \approx 4 - 20$ (7 nominally)

Field strength scaling of planets



- Mercury is slow rotator & predicted to be in multi-polar regime. Energy flux unknown.
- In Saturn and in Mercury the dynamo may operate below a stably stratified conducting layer of unknown thickness
- Uranus‘ and Neptune‘s dynamos may operate in thin convecting shell above stable fluid region
- Perhaps only Earth and Jupiter have „simple“ dynamos (i.e. in a deep convecting shell without shielding stable layer)

Application of field strength scaling law to stars

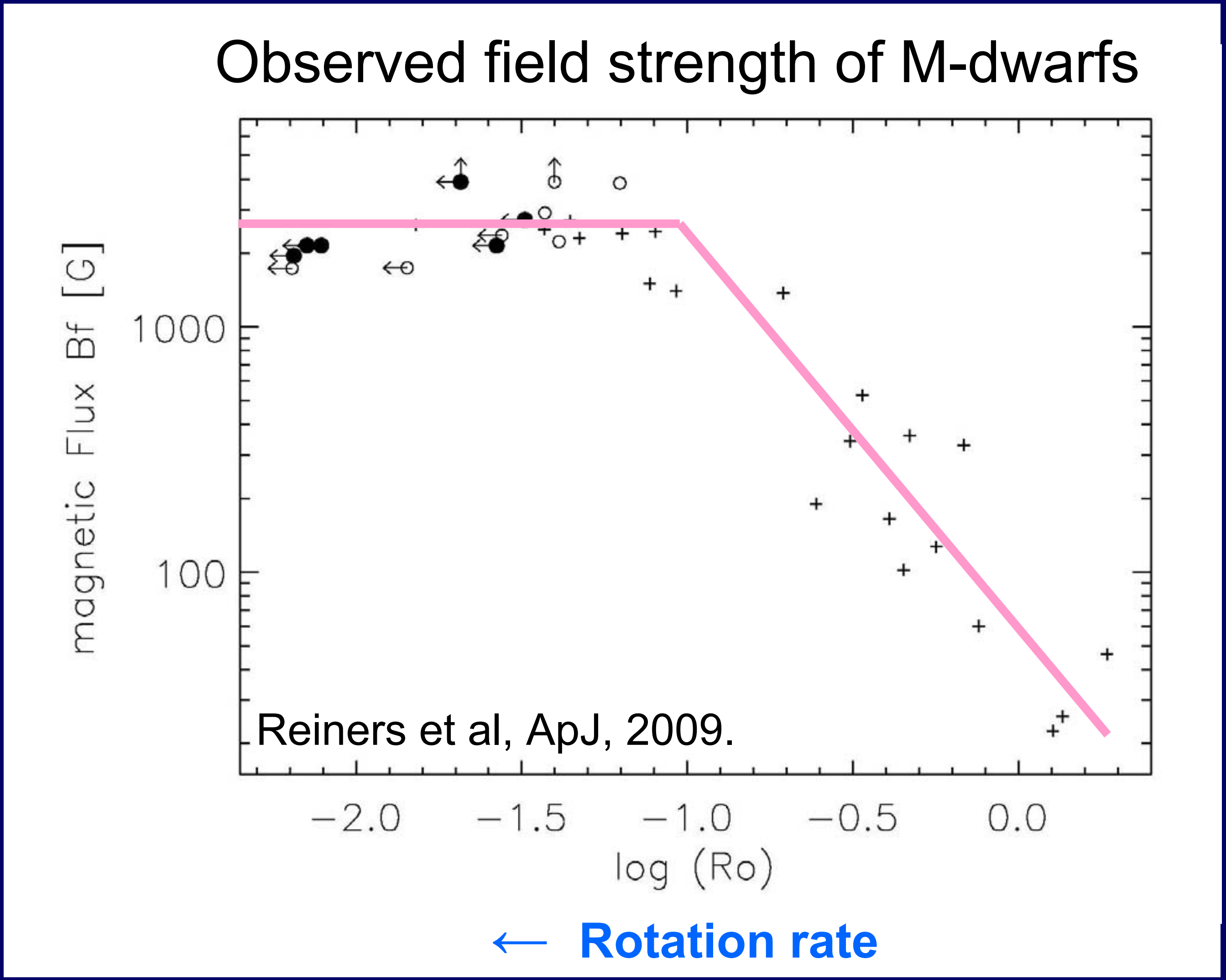
Problems with sun:

- Slow rotation
- Large scale field not dominant
- Tachocline may lead to different type of dynamo

Fully convective, rapidly rotators more suitable

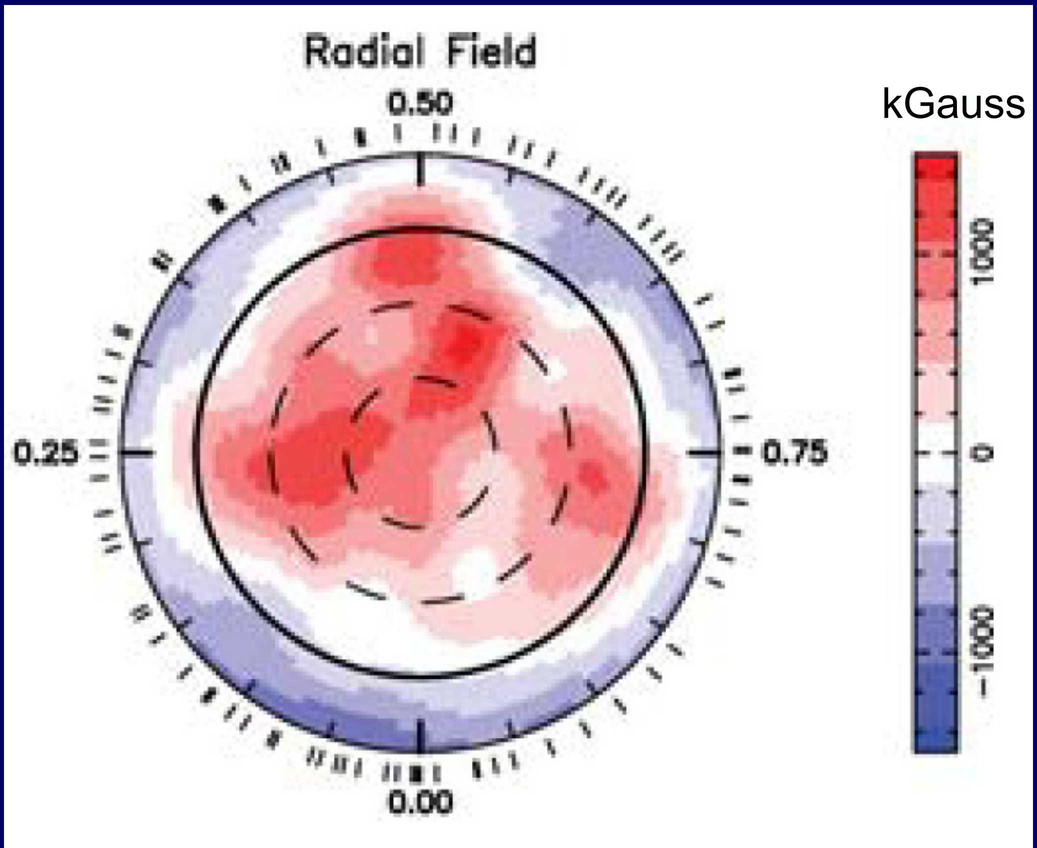
- Low-mass M-dwarfs ($M < 0.35\ M_{sun}$)
- T-Tauri stars (Pre-main sequence, contracting)

M-dwarfs: surface field vs. rotation



Field strength increases with rotation rate but saturates at Ro < 0.1

Stellar field morphology



Slowly rotating solar-type stars: small-scale fields

Rapidly rotating low-mass stars: significant large scale field component

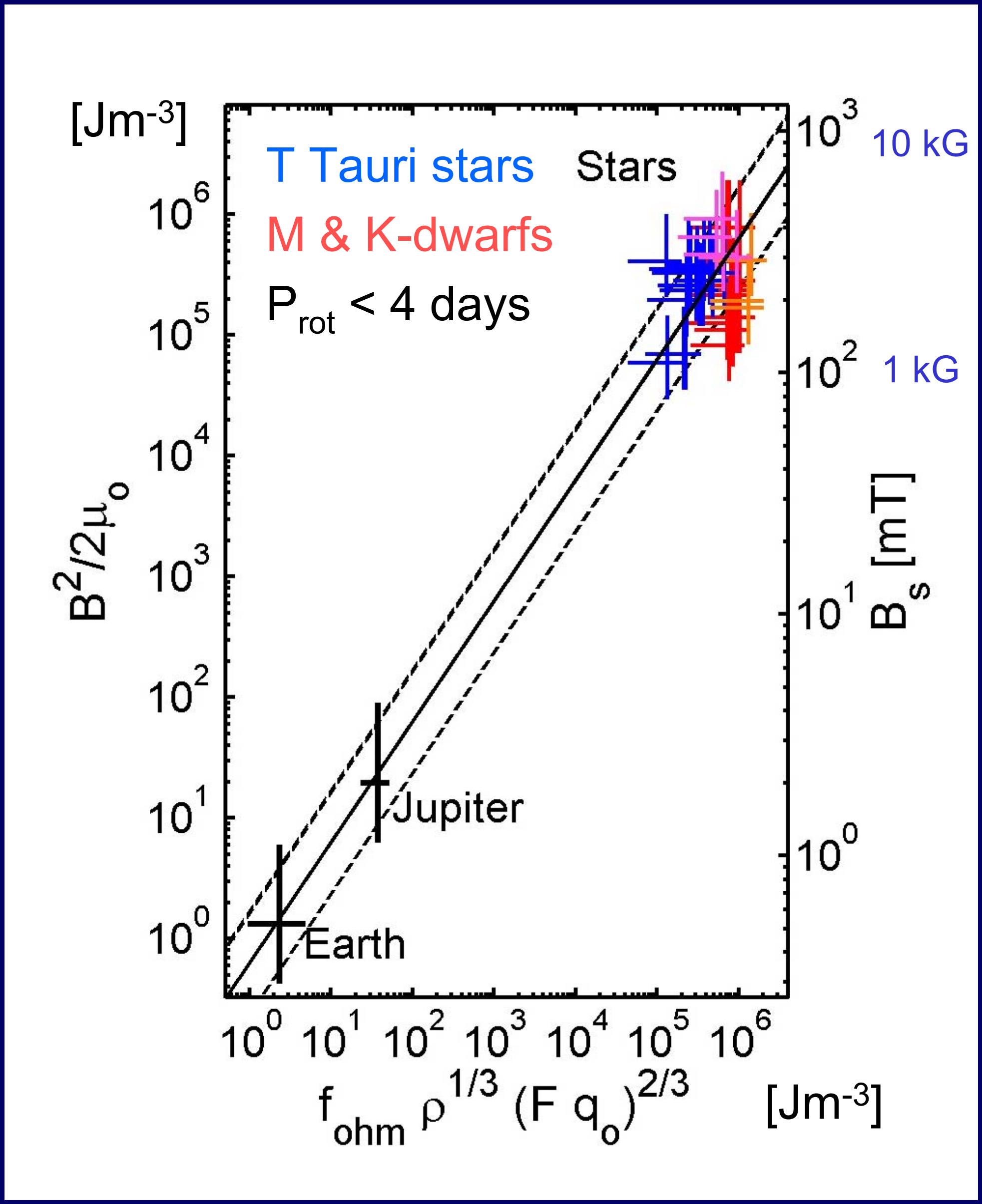
V374 Peg

M = 0.28 M_{sun} Rotation period 0.45 d

Field mapped by Zeeman Doppler tomography

Donati et al. (2006,2008)

From planets to stars

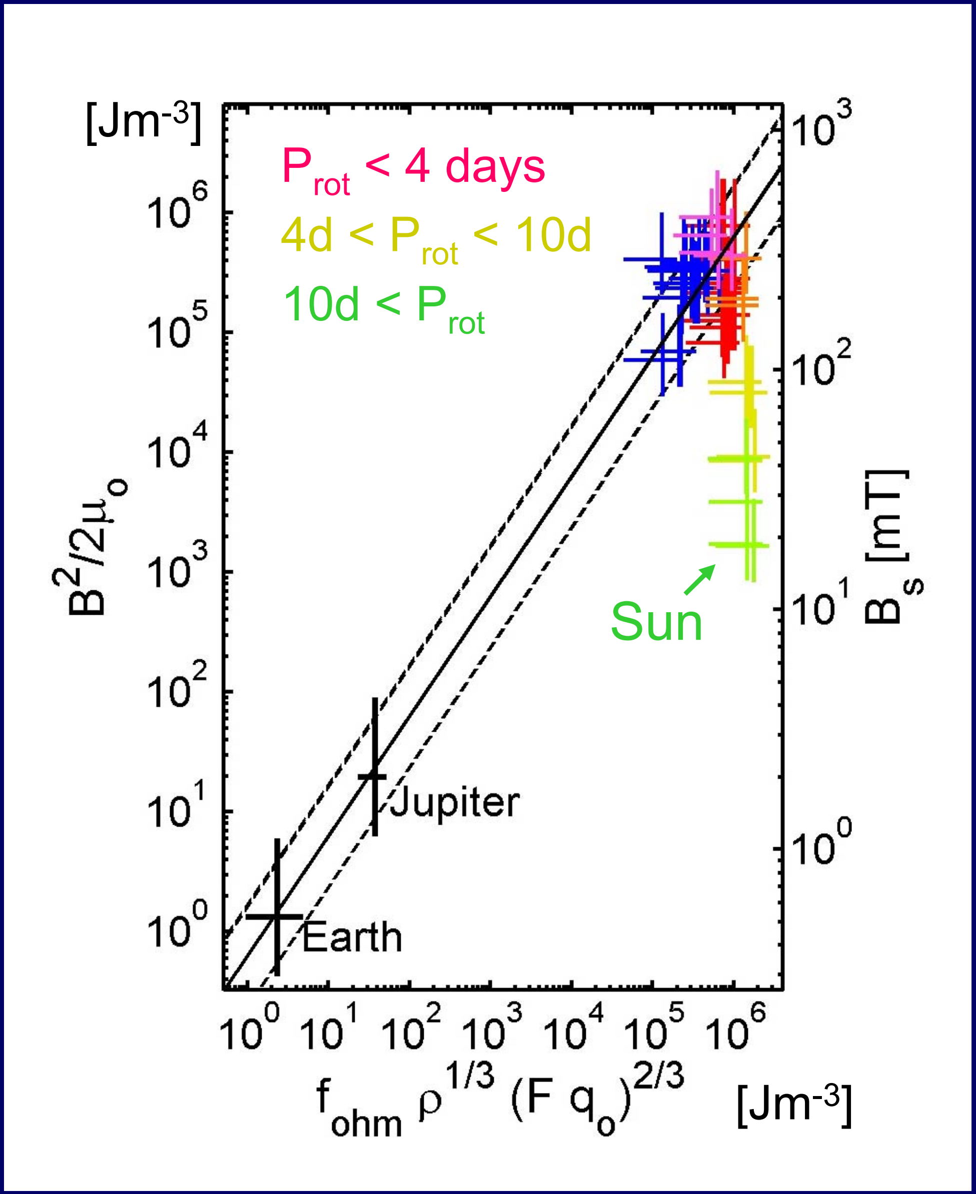


The observed field of rapidly rotating low-mass stars agrees with the prediction as well as that of Jupiter and Earth

$\Rightarrow$ confirmation for scaling law

$\Rightarrow$ dynamos in planets and (some) stars may be similar

Add slowly rotating stars



Slowly rotating K- and G-stars fall below the prediction.

Rapid rotation seems the essential prerequisite for the applicability of the scaling law.

Christensen et al. (2009)

Summary

- In dipole-dominant natural dynamos with rapid rotation magnetic field strength is controlled by the energy flux and is independent of rotation rate, conductivity and viscosity
- For slow rotation ($Ro_{local} > 0.1$) the field is weaker and multipolar
- The predictions of the scaling law for B agree with Earth's and Jupiter's field strength and with the field strength of rapidly rotating low-mass stars
- In several solar system planets, thin convective shells or stably stratified layers could make a comparison difficult
- Strong and possibly observable magnetic fields are predicted for rapidly rotating brown dwarfs and massive extrasolar planets

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