Lecture 5: Line Stretching

Statistics of line elements, Lyapunov exponents, F(p) and G(h), the Kraichnan-Kazantsev model, Kraichnan's Gaussian bloblet

Line element stretching: general results

J. Fluid Mech. (1984), *vol.* 144, *pp.* 1–11
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Kinematic dynamo problem in a linear velocity field

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Turbulent stretching of line and surface elements

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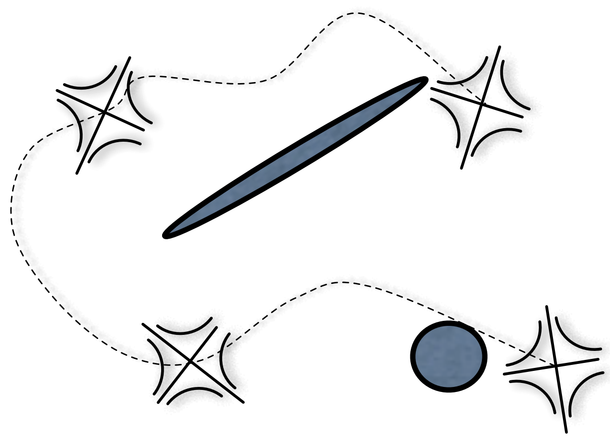
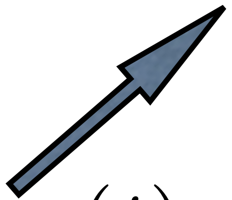
The “local stretching model”

- Focus on small scales, and elaborate Townsend’s hot-spot model.



- Solve the line-element equation:

$$\partial_t \boldsymbol{\xi} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$



No molecular diffusion (yet)

- In this Lagrangian frame we have a stochastic differential equation:

$$\dot{\boldsymbol{\xi}} = \boldsymbol{W}(t) \boldsymbol{\xi}$$

- We desire the statistical properties of line-element lengths.

Notation: $\ell(t) = |\boldsymbol{\xi}(t)|$

$$h(t) \equiv \frac{1}{t} \ln \left(\frac{\ell(t)}{\ell_0} \right)$$

Definition of the Lyapunov exponent

☛ For the moment, we use the definition:

$$\gamma_{\rm Lpv} \equiv \lim_{t\to\infty} \frac{1}{t} \left\langle \ln \left(\frac{\ell(t)}{\ell_0} \right) \right\rangle$$

☛ Using the golden rule for multiplicative processes:

$$\boxed{\ell_{\rm mp} = \ell_0 {\rm e}^{\gamma_{\rm Lpv} t}}$$

☛ According to Batchelor, all elements would stretch at this rate. This is not exactly true - we need a more complete characterization of stretching statistics.

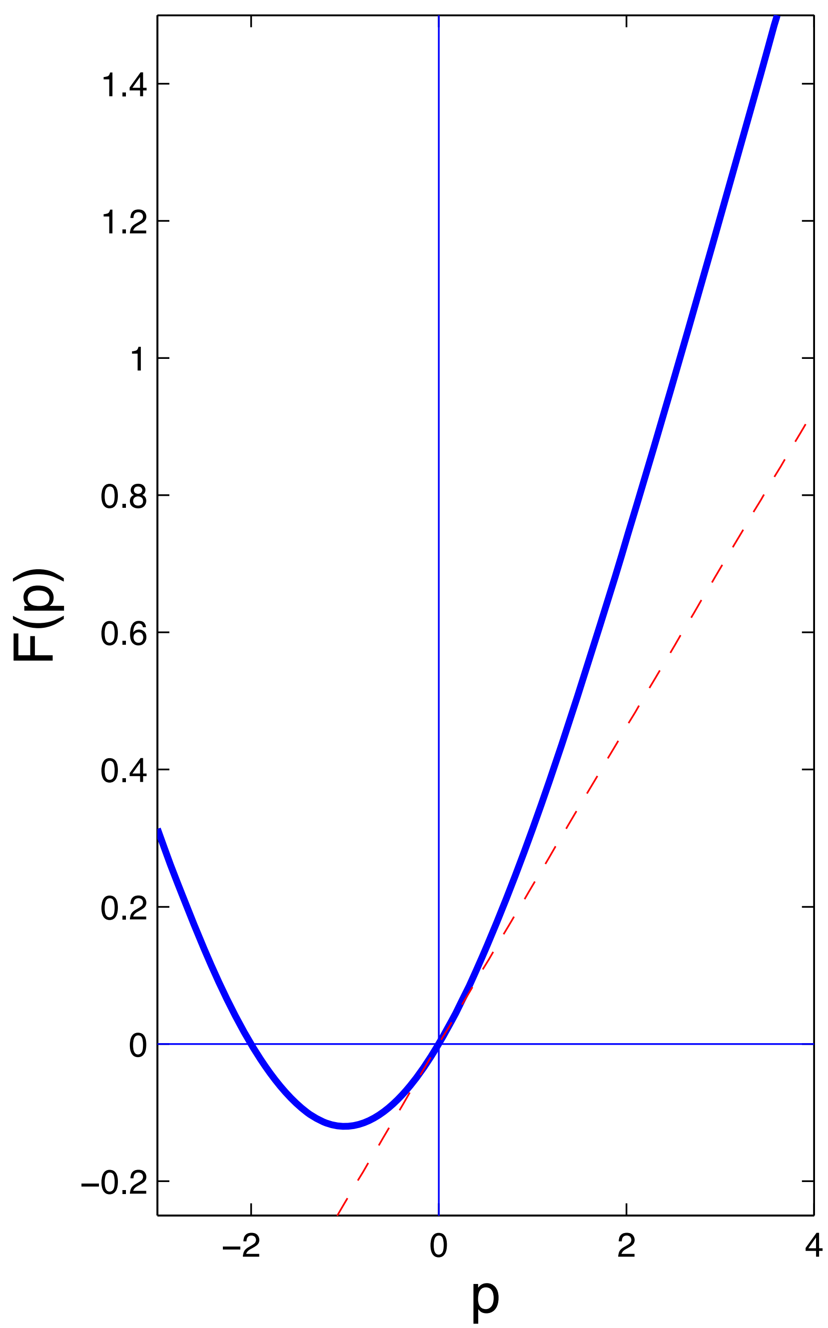
The big picture for line-element stretching

☛ $$F(p) \equiv \lim_{t\to\infty} \frac{1}{t} \log \left\langle \left(\frac{\ell}{\ell_0}\right)^p \right\rangle$$

or $$\langle \mathrm{e}^{ph} \rangle = \mathrm{e}^{tF(p)}\,, \qquad \text{as } t \to \infty$$

F(p) is the CGF.

☛ $$\gamma_{\mathrm{Lpv}} = \lim_{p\to 0} \frac{F(p)}{p}$$



☛ F(p) is convex, and

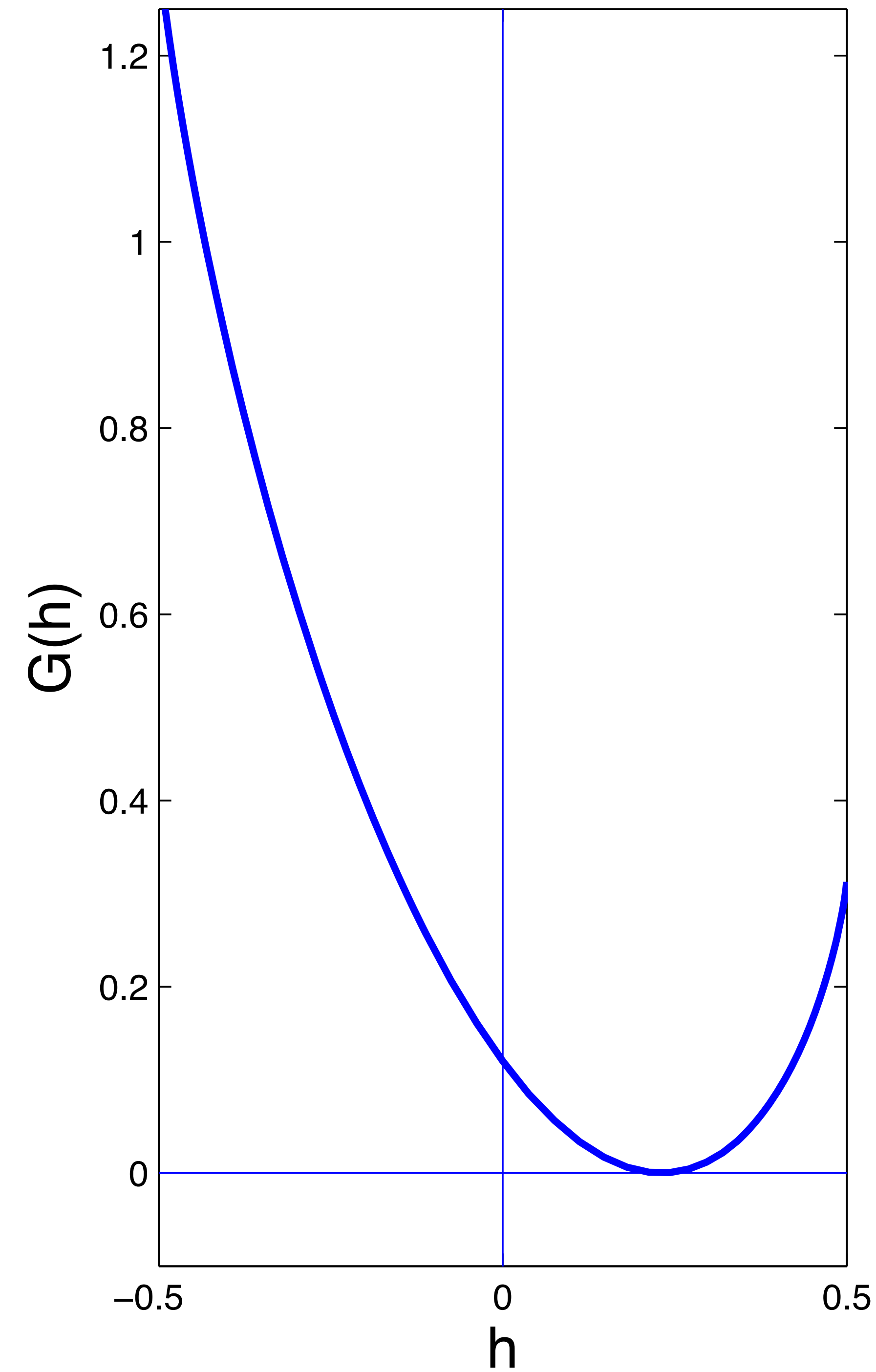
$$F(0) = 0\,, \quad F'(0) > 0\,, \quad F(-d) = 0\,,$$ and $$\underbrace{F(p) = F(-2-p)}_{d=2}$$

The other half of the big picture

☛ Large deviation theory gives

$$\mathrm{pdf}(h) \approx \sqrt{\frac{tG''(h)}{2\pi}}\mathrm{e}^{-tG(h)}$$

and $$G\left(\gamma_{\mathrm{Lpv}}\right) = 0$$



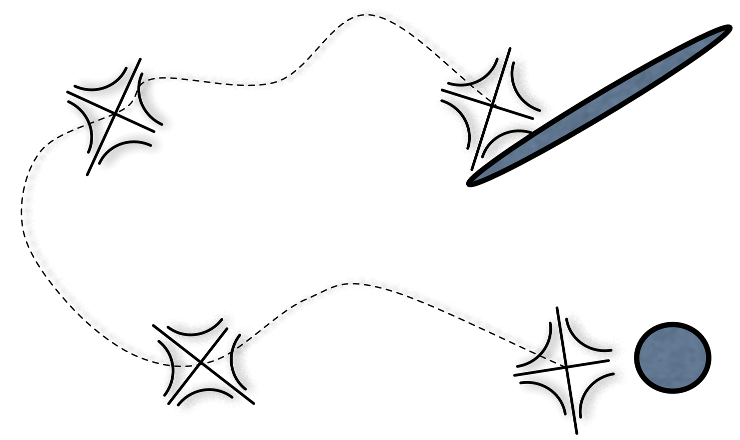
☛ F(p) and G(h) contain equivalent information:

$$G(h) = \sup_{\forall p}\left(ph - F(p)\right) \qquad F(p) = \sup_{\forall h}\left(ph - G(h)\right)$$

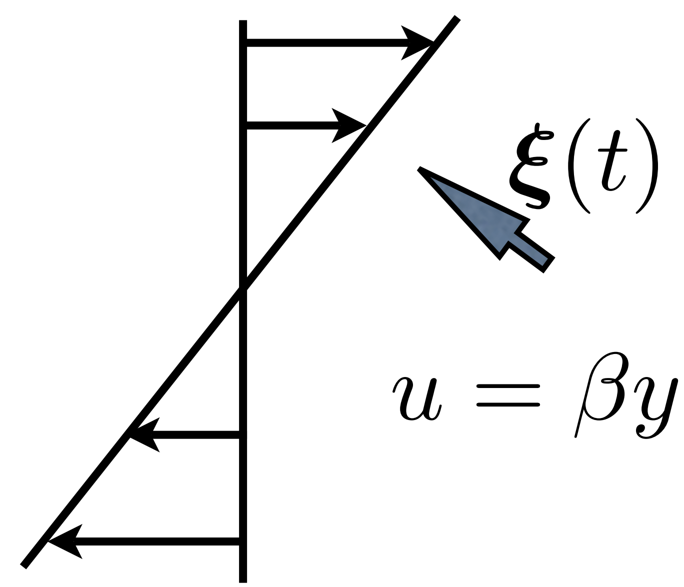
An example illustrating the main features of the big picture

Renewing Couette flow

☛ The illustration at right is misleading:
hyperbolic points are not necessary for exponential stretching.

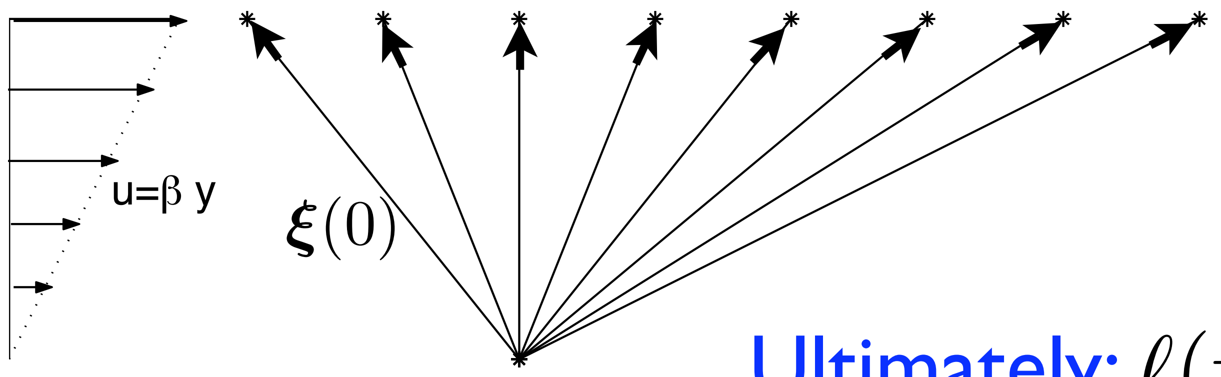


☛ Even Couette flow can produce exponential stretching, provided there is "random realignment". (Think of our renewing wave models.)



Couette flow $\dot{\boldsymbol{\xi}} = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \boldsymbol{\xi}$

☛ Recall elementary Couette flow - a material line moves like this:



Ultimately: $\ell(t) \equiv |\boldsymbol{\xi}(t)| \propto \ell_0 \times \beta t$

☛ The solution of the line-element equation is:

$$\boldsymbol{\xi}(t) = \ell_0 \begin{pmatrix} \cos\theta + \beta t \sin\theta \\ \sin\theta \end{pmatrix}$$

and $\ell^2(t) = \left[1 + \beta t \sin 2\theta + \beta^2 t^2 \sin^2\theta\right] \ell_0^2$

Renewing Couette flow

☛ At the end of each epoch,

$$[0 \leq t < \tau] \quad [\tau \leq t < 2\tau] \quad [2\tau \leq t < 3\tau]$$

randomly rotate the direction of the Couette flow.

☛ At the end of the n'th epoch, $\ell(n\tau) = \prod_{k=1}^{n} \underbrace{\sqrt{1 + \beta\tau \sin 2\theta_k + \beta^2\tau^2 \sin^2\theta_k}}_{\equiv m(\theta_k)} \ell_0$

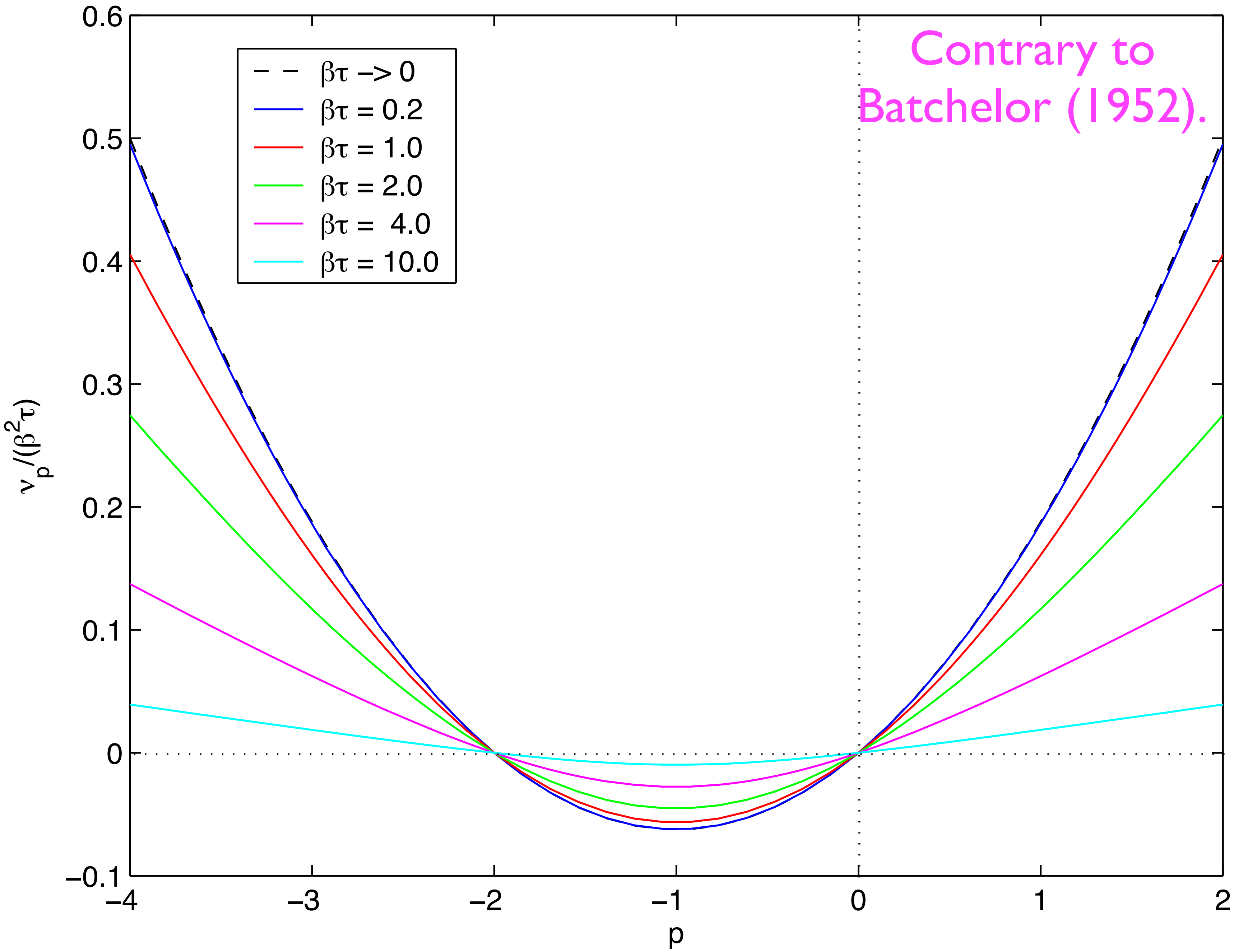
☛ Since this is a random product: $\ell_{\mathrm{mp}} = \mathrm{e}^{\langle \ln \ell \rangle} = \mathrm{e}^{n\langle m(\theta)\rangle} = \mathrm{e}^{t\gamma_{\mathrm{Lpv}}}$

$$\gamma_{\mathrm{Lpv}} = \frac{1}{2\tau} \oint \ln\left(1 + \beta\tau \sin 2\theta + \beta^2\tau^2 \sin^2\theta\right) \frac{\mathrm{d}\theta}{2\pi}$$

or $\quad \gamma_{\mathrm{Lpv}} = \frac{1}{2\tau} \ln\left(1 + \frac{\beta^2\tau^2}{4}\right)$

Homework: the renewing sinusoid

The CGF of Renewing Couette flow



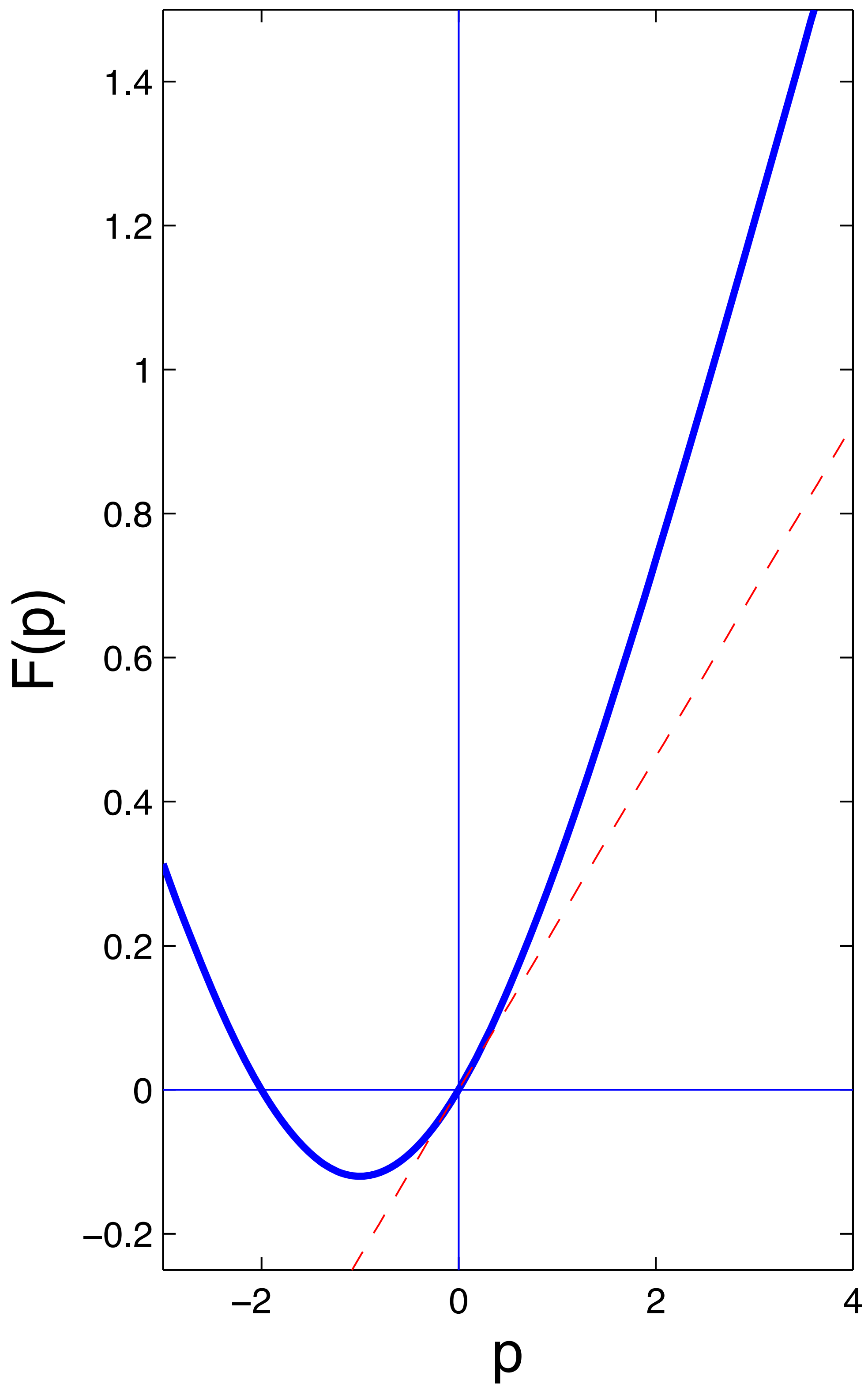
$$F(p) = \frac{1}{\tau} \ln \left[\oint \left(1 + \beta\tau \sin 2\theta + \beta^2 \tau^2 \sin^2 \theta\right)^{p/2} \frac{\mathrm{d}\theta}{2\pi} \right]$$

(Legendre functions)

Properties of F(p): some “proofs”

$$F(p) \equiv \lim_{t\to\infty} \frac{1}{t} \log \left\langle \left(\frac{\ell}{\ell_0} \right)^p \right\rangle$$

F(p) is convex



$$F(p) \equiv \lim_{t\to\infty} \frac{1}{t} \log \left\langle \left(\frac{\ell}{\ell_0} \right)^p \right\rangle$$

☛ It is evident that F(0)=0.

☛ Use Cauchy-Schwarz:

$$\left\langle \ell^{\frac{1}{2}(p+q)} \right\rangle < \left(\langle \ell^p \rangle \langle \ell^q \rangle \right)^{\frac{1}{2}}$$

$$\Rightarrow \quad F\left(\frac{p+q}{2} \right) < \tfrac{1}{2} \left(F(p) + F(q) \right)$$

In isotropic flow $\gamma_{\mathrm{Lpv}} = F'(0) > 0$

☛ Since the line-stretching equation is linear:

$$\boldsymbol{\xi}(t) = \boldsymbol{J}(t)\boldsymbol{\xi}(0) \quad \Rightarrow \quad \ln\left(\frac{\ell}{\ell_0}\right) = \tfrac{1}{2}\ln\left(\boldsymbol{e}^\top \boldsymbol{J}^\top \boldsymbol{J}\boldsymbol{e}\right), \qquad |\boldsymbol{e}|^2 = 1$$

and $\det \boldsymbol{J} = 1$ where $\boldsymbol{\xi}(0) = \ell_0 \boldsymbol{e}$

☛ Three things we know about: $\boldsymbol{J}^\top \boldsymbol{J}$

☛ Use Jensen's inequality log(avg) > avg(log):

$$\begin{aligned}
\left\langle \ln\left(\frac{\ell}{\ell_0}\right)\right\rangle_{\boldsymbol{e}} &= \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\ln\left(\lambda_1 e_1^2 + \lambda_2 e_2^2 + \lambda_3 \mathrm{e}_3^2\right)\mathrm{d}S \\
&\geq \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\left(e_1^2 \ln\lambda_1 + e_2^2 \ln\lambda_2 + e_3^3 \ln\lambda_3\right)\mathrm{d}S \\
&= \frac{1}{6}\ln(\lambda_1\lambda_2\lambda_3) = 0
\end{aligned}$$

In isotropic flow $\gamma_{\rm Lpv} = F'(0) > 0$

☛ Since the line-stretching equation is linear:

$$\boldsymbol{\xi}(t) = \boldsymbol{J}(t)\boldsymbol{\xi}(0) \quad \Rightarrow \quad \ln\left(\frac{\ell}{\ell_0}\right) = \tfrac{1}{2}\ln\left(\boldsymbol{e}^\intercal \boldsymbol{J}^\intercal \boldsymbol{J}\boldsymbol{e}\right), \qquad |\boldsymbol{e}|^2 = 1$$

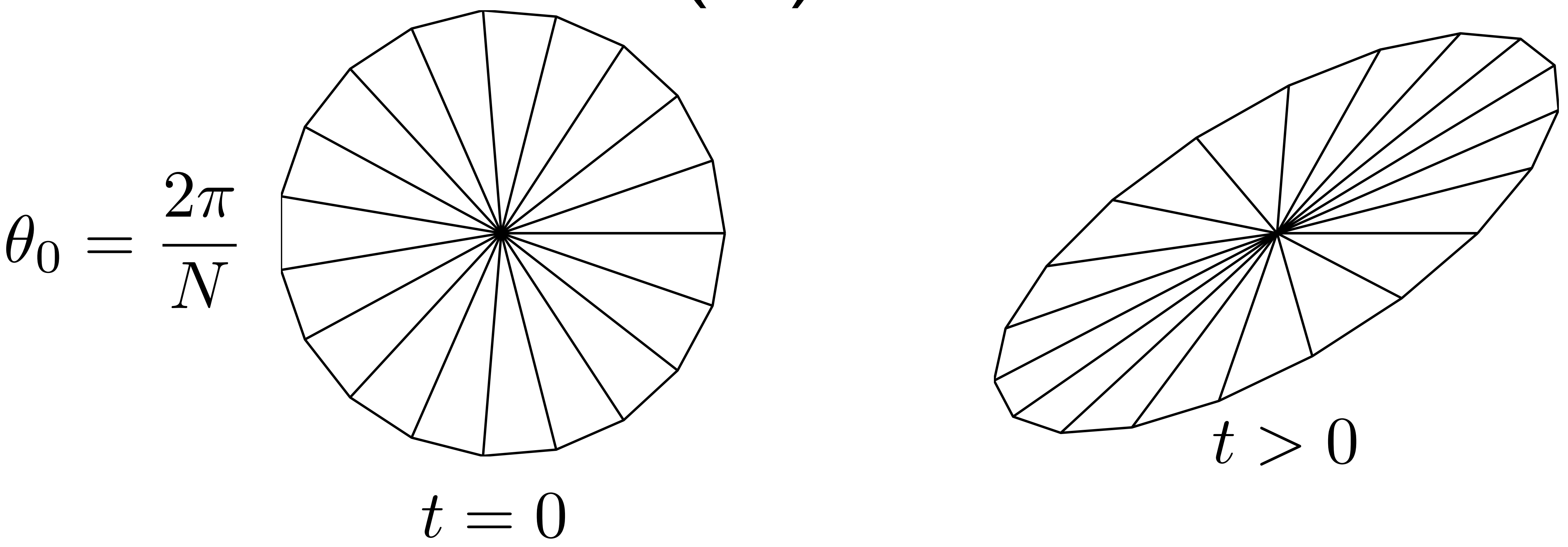
and $\det \boldsymbol{J} = 1$ where $\boldsymbol{\xi}(0) = \ell_0 \boldsymbol{e}$

☛ Three things we know about: $\boldsymbol{J}^\intercal \boldsymbol{J}$ symmetric, det =1, and eigenvalues are positive.

☛ Use Jensen's inequality log(avg) > avg(log):

$$\begin{aligned}
\langle \ln\left(\frac{\ell}{\ell_0}\right)\rangle_{\boldsymbol{e}} &= \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\ln\left(\lambda_1 e_1^2 + \lambda_2 e_2^2 + \lambda_3 \mathrm{e}_3^2\right) \mathrm{d}S \\
&\geq \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\left(e_1^2 \ln\lambda_1 + e_2^2\ln\lambda_2 + e_3^3 \ln\lambda_3\right)\,\mathrm{d}S \\
&= \frac{1}{6}\ln(\lambda_1\lambda_2\lambda_3) = 0
\end{aligned}$$

$F(-d) = 0$



☛ Incompressibility (conservation of area in *d*=2) is the key:

$$A = \ell_0^2 \theta_0 = \ell_n^2(t)\theta_n(t) \quad n = 1\,, 2\,, \cdots N$$

$$\Rightarrow \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\ell_0}{\ell_n(t)} \right)^2 = 1$$

The Kraichnan-Kazantsev model

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Convection of a passive scalar by a quasi-uniform random straining field

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K^2 flows

☛ Once again, we use a renewal model:

$$[0 \leq t < \tau] \quad [\tau \leq t < 2\tau] \quad [2\tau \leq t < 3\tau]$$ etc

☛ But now: $\tau \to 0\,, \quad \text{with} \quad \boldsymbol{u}_i \propto \tau^{-1/2}$

☛ Thus $\langle u_i(\boldsymbol{x}_1, t_1) u_j(\boldsymbol{x}_2, t_2)\rangle = 2\,\mathcal{U}_{ij}(\boldsymbol{x}_2 - \boldsymbol{x}_1)\delta_\tau(t_2 - t_1)$

☛ Although the kinetic energy of a K^2-flow is infinite, we are not deterred.

☛ Homework/discussion: calculate the velocity correlation function for our favorite example, the renewing sinusoid.

The simplest example: Eddy diffusivity

- Start with: $c_t + u_i c_{,i} = \kappa \nabla^2 c\,,$

- We only have to consider the first epoch:

$$c(\boldsymbol{x},\tau) = c(\boldsymbol{x},0) - \underbrace{\tau u_i c_{,i}(\boldsymbol{x},0)}_{O(\tau^{1/2})} + \tfrac{1}{2}\underbrace{\tau^2 u_j \left[u_i c_{,i}(\boldsymbol{x},0)\right]_{,j}}_{O(\tau)} + \underbrace{O\left(\tau^3 u^3\right)}_{O(\tau^{3/2})}\,.$$

or

$$\frac{\Delta c}{\tau} = -u_i c_{,i}(\boldsymbol{x},0) + \tfrac{1}{2}\tau u_j u_{i,j} c_{,i}(\boldsymbol{x},0) + \tfrac{1}{2}\tau u_j u_i c_{,ij}(\boldsymbol{x},0)\,.$$

- Ensemble average, and take the limit: $\langle c\rangle_t = \kappa \nabla^2 \langle c\rangle + \mathcal{U}_{ij}(0)\langle c\rangle_{,ij}\,.$

(Assuming spatial homogeneity.)

- Recall $\langle u_i(\boldsymbol{x}_1,t_1)u_j(\boldsymbol{x}_2,t_2)\rangle = 2\,\mathcal{U}_{ij}(\boldsymbol{x}_2-\boldsymbol{x}_1)\delta_\tau(t_2-t_1)$

The second simplest example: line element FP eqn.

- The line-element equation: $\partial_t \xi_i = \underset{O\left(\tau^{-1/2}\right)}{W_{ip}}\xi_p$

- Zero mean $\langle W_{ip} \rangle = 0$, and spatially homogeneous $\langle W_{ip} W_{qi} \rangle = 0$

- The Liouville equation: $\partial_t \tilde{P} + \left(W_{ip} \xi_p \tilde{P} \right)_{,i} = 0$

unaveraged density of line elements

- Advance through the first epoch:

$$\frac{\tilde{P}(\boldsymbol{\xi},\tau) - \tilde{P}(\boldsymbol{\xi},0)}{\tau} = \underbrace{\partial_t \tilde{P}(\boldsymbol{\xi},0)}_{O(\tau^{-1/2})} + \frac{1}{2}\underbrace{\tau \partial_t^2 \tilde{P}(\boldsymbol{\xi},0)}_{O(\tau^0)} + O(\tau^{1/2})$$

- Ensemble average: $\partial_t P = \Gamma_{ijpq} \xi_p \xi_q P_{,ij}$ where $\Gamma_{ijpq} \equiv \frac{1}{2}\tau \langle W_{ip} W_{jq} \rangle$

The isotropic case

☛ With isotropic statistics:

$$\Gamma_{ijpq} = \Gamma\left[(d+1)\delta_{ij}\delta_{pq} - \delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jq}\right]$$

and the solution is isotropic

$$P(\boldsymbol{\xi}) \propto P(\ell) \quad \text{with} \quad \ell \equiv |\boldsymbol{\xi}|$$

☛ After some algebra:

$$\boxed{P_t + \gamma P_q = \frac{\gamma}{d} P_{qq}}$$

where $\gamma \equiv d(d-1)\Gamma = \Gamma_{ijij}$ and $q \equiv \ln \dfrac{\ell}{\ell_0}$

☛ We don't need large deviation theory, because everything is Gaussian

☛ There is an equivalent SDE:

$$\dot{\ell} = s(t)\ell\,,$$

with

$$s(t) = \gamma + s'(t)\,, \qquad \langle s'(t_1)s'(t_2)\rangle = \frac{2\gamma}{d}\delta(t_2 - t_1)\,.$$

Decay of Kraichnan's bloblet

☛ The model: $$c_t + \left[\gamma + s'(t)\right] x c_x - \left[\gamma + s'(t)\right] y c_y = \kappa \nabla^2 c$$

where $\langle s'(t_1) s'(t_2) \rangle = \gamma \delta(t_1 - t_2)$ in $d = 2$

☛ How fast does the blob decay, on average?

☛ The solution: $$c = \frac{\alpha(t)\beta(t)}{2\pi} \exp\left[-\tfrac{1}{2}\alpha^2(t)x^2 - \tfrac{1}{2}\beta^2(t)y^2\right]$$

and
$$\dot{\alpha} = -(\gamma + s')\alpha - \kappa\alpha^3\,,$$
$$\dot{\beta} = (\gamma + s')\beta - \kappa\beta^3\,.$$

☛ We'd like to calculate: $$\left\langle \iint c^{n+1}(x, y, t)\, \mathrm{d}x\mathrm{d}y \right\rangle \propto \langle \alpha^n \beta^n \rangle$$

Solution of the bloblet model

☛ Change variables: $\alpha = \ell^{-1}\mathrm{e}^{-p}\,, \qquad \beta = \ell^{-1}\mathrm{e}^{-q}$ where $\ell \equiv \sqrt{\kappa/\gamma}$.

☛ The noise is now additive:

$$\dot{p} = \gamma(1+\mathrm{e}^{-2p}) + s'\,,$$
$$\dot{q} = -\gamma(1-\mathrm{e}^{-2q}) - s'\,.$$

☛ The corresponding FP equation is:

$$F_{t'} + \left[(1+\mathrm{e}^{-2p})F\right]_p - \left[\left(1-\mathrm{e}^{-2q}\right)F\right]_q = \tfrac{1}{2}\left(F_{pp} - 2F_{pq} + F_{qq}\right)$$

with $t' = \gamma t$

☛ Resort to an approximate solution: $F(p,q,t) \approx P(p,t)Q(q,t)$

The marginal densities

☛ For the expanding axis: $P_{t'} + \left[\left(1+\mathrm{e}^{-2p}\right) P\right]_p = \frac{1}{2} P_{pp}\,.$

or $P(p,t') \approx \dfrac{\mathcal{N}}{\sqrt{2\pi t'}} \exp\left(-\dfrac{(p-t')^2}{2t'}\right) H(p)$

☛ For the shrinking axis: $Q_{t'} - \left[\left(1-\mathrm{e}^{-2q}\right) Q\right]_q = \frac{1}{2} Q_{qq}$

and therefore
$$\begin{aligned} Q_{\mathrm{eq}}(q) &= 2\exp\left(-2q - \mathrm{e}^{-2q}\right)\,, \\ &= \frac{\mathrm{d}}{\mathrm{d}q}\exp\left(-\mathrm{e}^{-2q}\right)\,. \end{aligned}$$

☛ The approximate solution is: $F(p,q,t') \approx Q_{\mathrm{eq}}(q) \times \dfrac{\mathcal{N}}{\sqrt{2\pi t'}} \exp\left(-\dfrac{(p+t')^2}{2t'}\right) H(p)$

Calculate some statistics

☛ For the shrinking axis: $\ell^{(m+n)/2}\langle\alpha^m\beta^n\rangle = \langle \mathrm{e}^{-mp-nq}\rangle\,,$

$$\approx \ell^{(m+n)/2}\langle \mathrm{e}^{-mp}\rangle\langle \mathrm{e}^{-nq}\rangle$$

☛ Note the necessity of H(p) when m is large.

☛ The easy average is: $\langle \mathrm{e}^{-nq}\rangle = 2\int_{-\infty}^{\infty}\exp\left[-(n+2)q-\mathrm{e}^{-2q}\right]\mathrm{d}q = \Gamma\left(\frac{n+2}{2}\right)$

The tricky average

☛ We have $\langle \mathrm{e}^{-mp} \rangle = \int_0^\infty \mathrm{e}^{-\varphi_m(p,t')} \frac{\mathrm{d}p}{\sqrt{2\pi t'}}$,

$$\varphi_m(p,t') \equiv mp + \frac{(p-t')^2}{2t'} = \frac{p^2}{2t'} + (m-1)p + \frac{t'}{2}$$

☛ The max of the integrand is at: $p_* = \begin{cases} (1-m)t' \,, & \text{if } m < 1, \\ 0\,, & \text{if } m \geq 1. \end{cases}$

☛ We find: $\langle \mathrm{e}^{-mp} \rangle = \begin{cases} \exp\left[\left(\frac{1}{2}m^2 - m\right) t'\right] \,, & \text{if } m < 1, \\ \frac{1}{2}\mathrm{e}^{-t'/2} \,, & \text{if } m = 1, \\ (m-1)^{-1}(2\pi t')^{-1/2} \exp\left(-\frac{1}{2}t'\right) \,, & \text{if } m > 1. \end{cases}$

The answer

- ☛ The most basic measure of decay is: $$\left\langle \iint c^2(x,y,t)\,\mathrm{d}x\mathrm{d}y \right\rangle \propto \mathrm{e}^{-\gamma t/2}$$

- ☛ Fluctuations in the stretching rate shield some ensemble members from the mean stretching. these long-term survivors dominate the ultimate statisitcs