Lecture 1: Introduction & Overview

Stirring & mixing; differential advection; and passive scalar decay; uniform strain solution; Gaussianology.

As instructed by the organizers, I'll assume that you are totally ignorant, infinitely intelligent, and not shy to interrupt.

The passive-scalar problem

- $$c_t + \boldsymbol{u}\cdot\boldsymbol{\nabla} c = \kappa\nabla^2 c\,, \qquad\qquad \nabla\cdot\boldsymbol{u} = 0$$

  + initial and boundary conditions

  + a prescription for the velocity

- D=2 $\qquad \boldsymbol{x} = (x, y) \qquad \boldsymbol{u} = (u, v)$

- We're studying the little brother of the induction equation:

$$\boldsymbol{b}_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \kappa\nabla^2\boldsymbol{b}$$

- $c_t + \boldsymbol{u}\cdot\boldsymbol{\nabla} c = \kappa\nabla^2 c\,,$ is equivalent to the SDE

$$\dot{\boldsymbol{x}} = \boldsymbol{u}(\boldsymbol{x}, t) + \sqrt{2\kappa}\,\mathrm{d}\boldsymbol{W}_t$$

One version of the passive-scalar problem: how long does mixing take?

Eckart (1948) - “stirring” versus “mixing”

☛ $\kappa_{\rm sugar} = 4 \times 10^{-10} {\rm m}^2 {\rm s}^{-1}$

(This coffee drinker is in free fall...)

☛ With coffee-cup BCs:

$$\frac{{\rm d}}{{\rm d}t} \int c(\boldsymbol{x}, t)\, {\rm d}V = 0$$

so we subtract the mean.

With only diffusion, mixing takes a long time

- ☛ In a two-dimensional coffee cup, the second slowest mode is

$$c(\boldsymbol{x}, t) = J_0\left(3.832\frac{r}{R}\right)\exp\left(-\frac{(3.832)^2\kappa}{R^2}t\right)$$

- ☛ Note that the decay is exponential, and there is no 'intermittency':

$$\left(\int c^p\,\mathrm{d}V\right)^{1/p} \propto \mathrm{e}^{-\nu t}$$

The scalar variance integral

☛ $$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\int c^2\,\mathrm{d}V = -\kappa\int|\boldsymbol{\nabla} c|^2\,\mathrm{d}V}$$

☛ Differential advection accelerates mixing by increasing concentration gradients (stirring).

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int|\boldsymbol{\nabla} c|^2\mathrm{d}V + \int\boldsymbol{\nabla} c\boldsymbol{\cdot}\boldsymbol{e}\boldsymbol{\cdot}\boldsymbol{\nabla} c\,\mathrm{d}V = -\kappa\int(\nabla^2 c)^2\,\mathrm{d}V$$

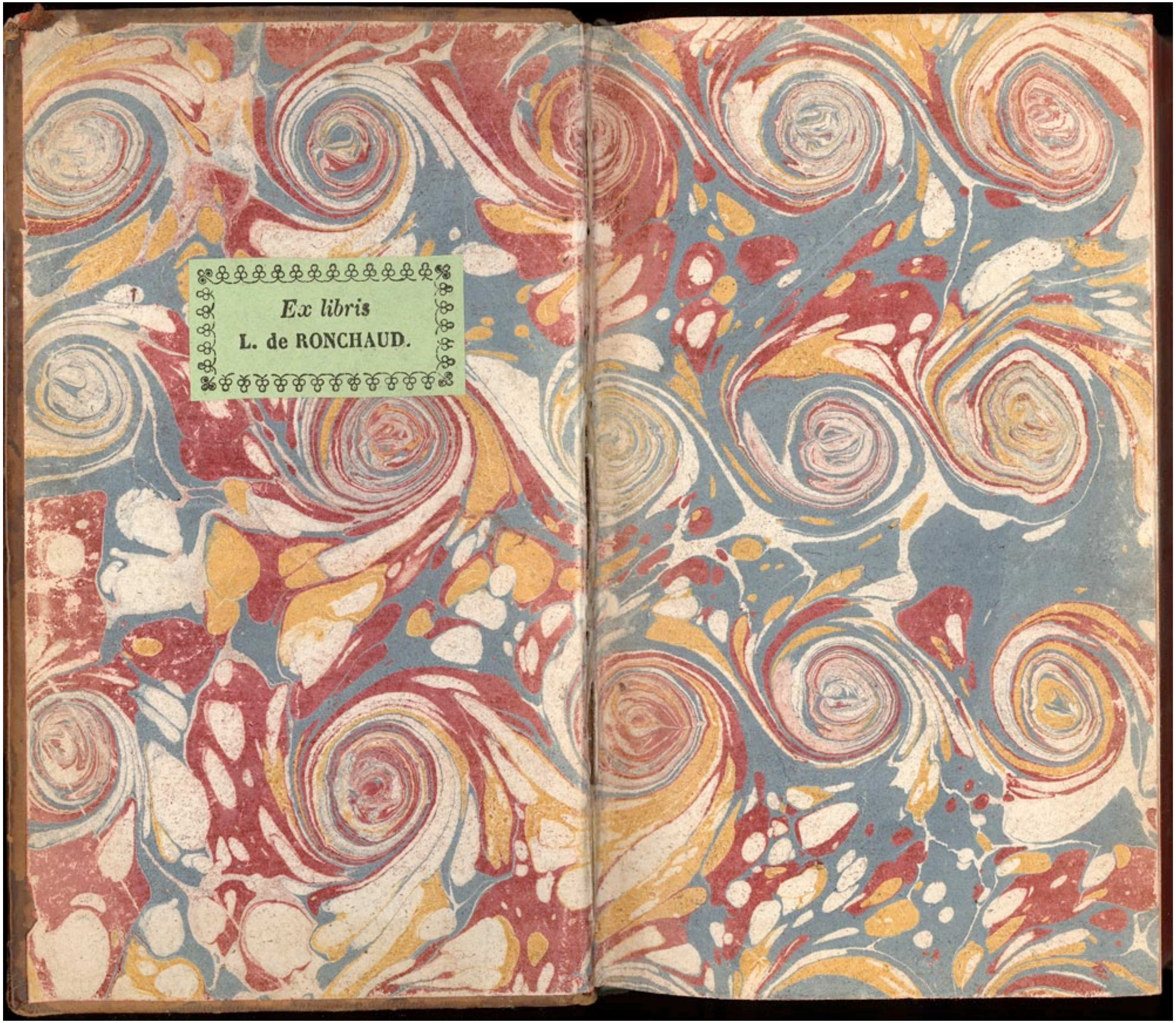
$$\boldsymbol{e}_{ij} \equiv \frac{1}{2}\left(\boldsymbol{u}_{i,j} + \boldsymbol{u}_{j,i}\right)$$

☛ Stirring can be so strong that the mixing rate is independent of molecular diffusivity.

Stirring without mixing: an artistic application

$\kappa = 0$

“Marbled end papers”



“Marbling” or “suminagashi” (floating ink)

Stirring without mixing: the Cauchy solution

$\kappa = 0$

- Lagrangian trajectories: $\dot{\boldsymbol{x}} = \boldsymbol{u}(\boldsymbol{x}(t), t) \qquad \boldsymbol{x}(0) = \boldsymbol{a}$

The solution defines the "motion map": $\mathcal{M}_t \boldsymbol{a} = \boldsymbol{x}$

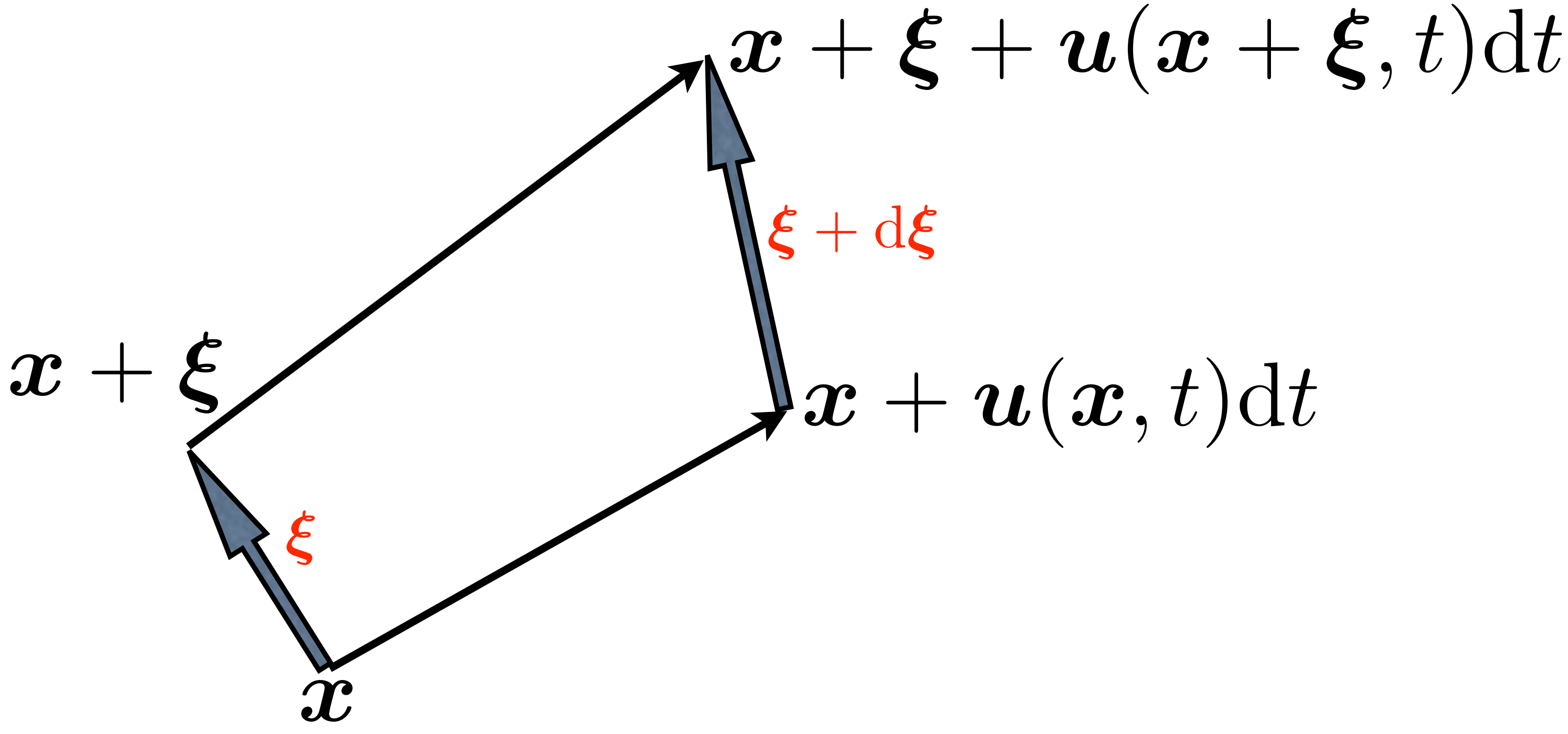
- The solution of the advection equation:

$$\partial_t c + \boldsymbol{u} \cdot \boldsymbol{\nabla} c = 0 \qquad c(\boldsymbol{x}, 0) = c_0(\boldsymbol{x})$$

is $c(\boldsymbol{x}, t) = c_0(\boldsymbol{a}(\boldsymbol{x}, t)) = c_0(\mathcal{M}_{-t} \boldsymbol{x})$

- Simple examples: unidirectional shear flow, axisymmetric vortices etc.

Material line elements



$$\partial_t \boldsymbol{\xi} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$

- The Cauchy solution:

$$\boldsymbol{\xi}(\boldsymbol{x}, t) = \boldsymbol{J}(\boldsymbol{a}, t)\boldsymbol{\xi}_0(\boldsymbol{a}) \qquad \text{where} \qquad \boldsymbol{J}_{ij}(\boldsymbol{a}, t) \equiv \frac{\partial \boldsymbol{x}_i}{\partial a_j}$$

- Simple examples: unidirectional shear flow, axisymmetric vortices.

- Homework/discussion

$$\partial_t \left(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} c\right) + \boldsymbol{u} \cdot \boldsymbol{\nabla} \left(\boldsymbol{\xi} \cdot \boldsymbol{\nabla} c\right) = 0 \qquad (\text{if } \kappa = 0)$$

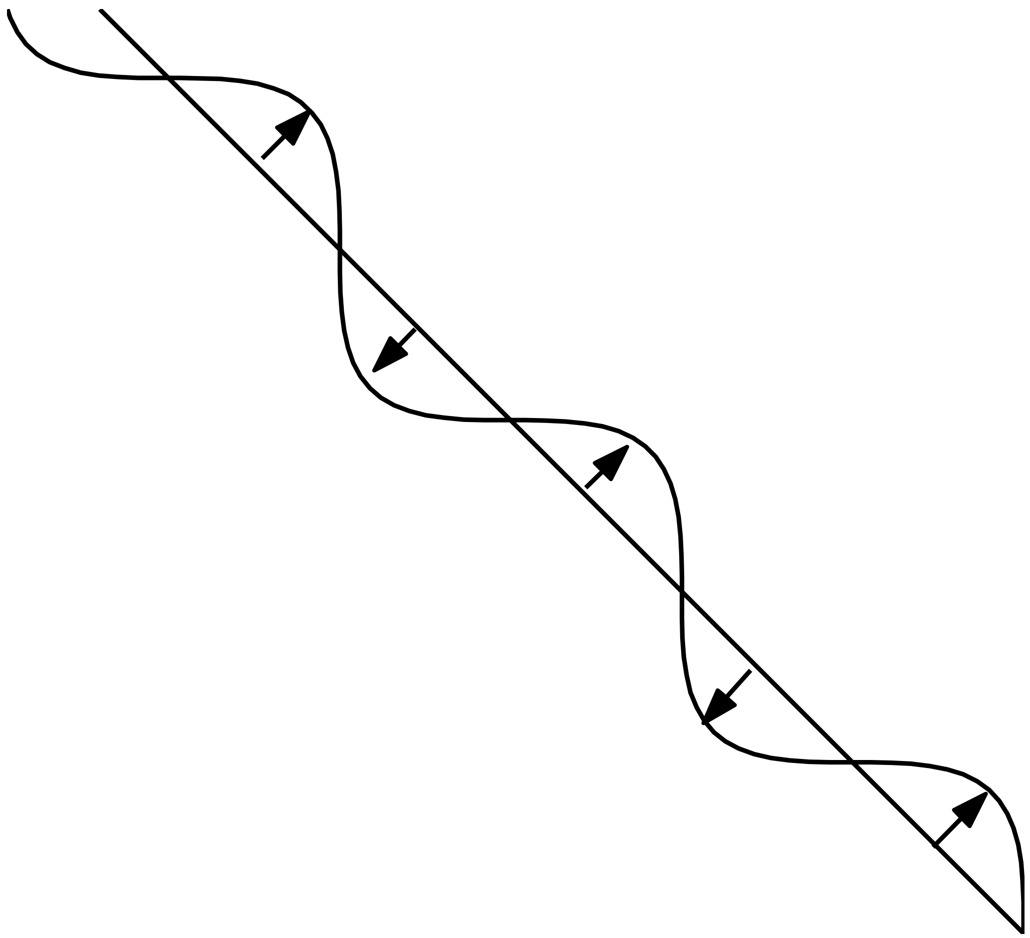
Renewing flows and random maps

Random flows produce random maps

☛ Use a renewal model,

$[0 \le t < \tau]$ $[\tau \le t < 2\tau]$ $[2\tau \le t < 3\tau]$

the first epoch the second epoch the third epoch



$$\psi_n(x, y, t) = k^{-1} U \cos[k \cos\theta_n\, x + k \sin\theta_n\, y + \varphi_n]\,,$$

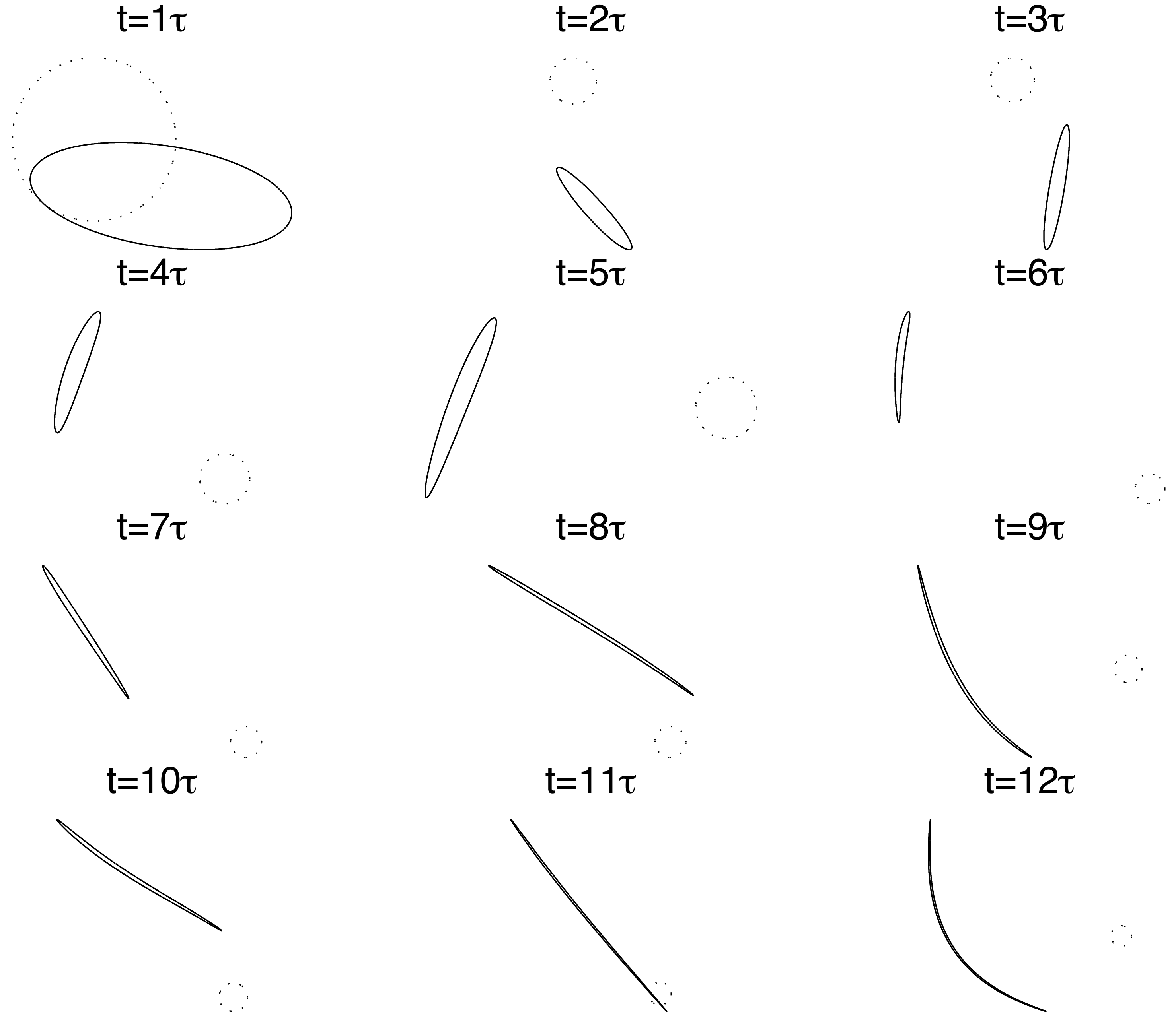
☛ The Cauchy solution is equivalent to a random map:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} + \tau U \sin\left(k c_n x + k s_n y + \varphi_n\right) \begin{pmatrix} s_n \\ -c_n \end{pmatrix}$$

$$s_n \equiv \sin\theta_n \text{ and } c_n \equiv \cos\theta_n$$

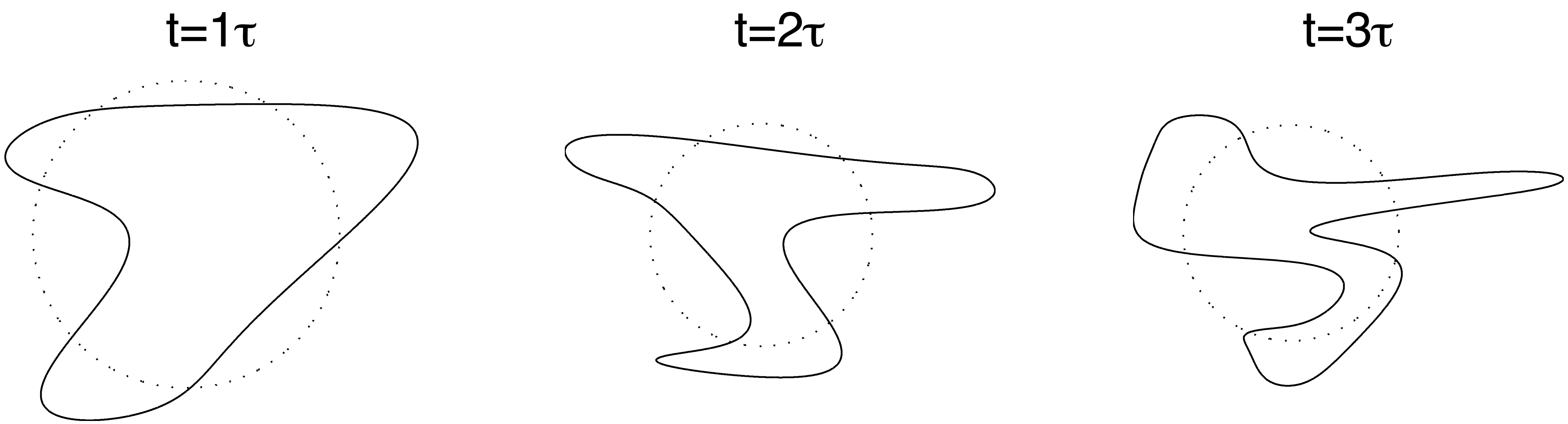
Deformation of a spot, $r_0 k \ll 1$

$$\ell \propto \exp(\gamma_{Lpv} t)$$



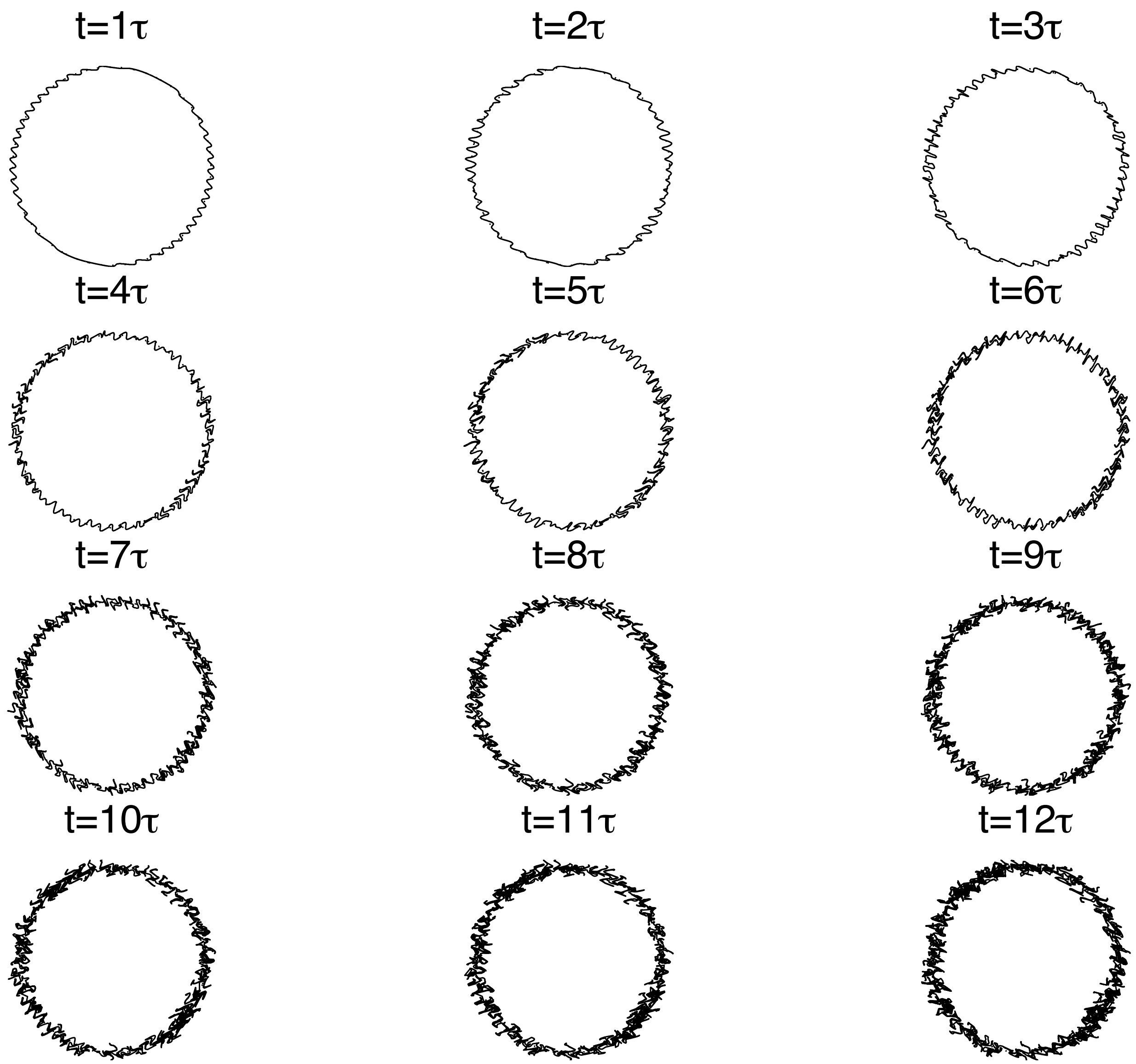
The dotted circle is the initial spot - the major axis grows exponentially with time.

Distortion of a patch, $r_0 k = 1$



The dotted circle is the initial patch

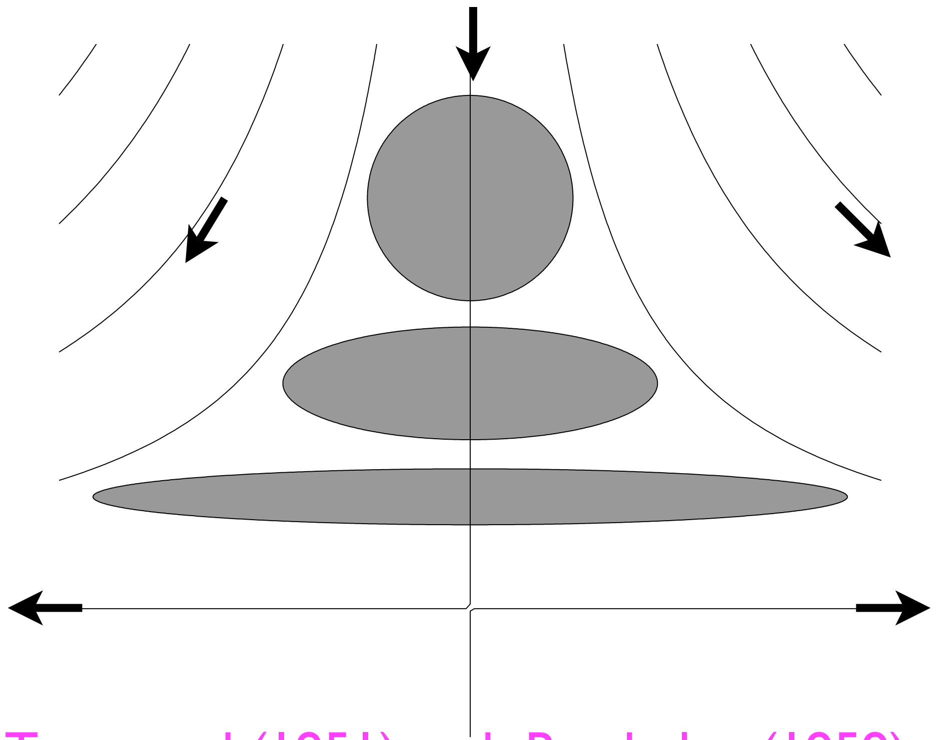
Dispersion of a big blob, $r_0 k \gg 1$



☛ This illustrates "eddy diffusivity".

Very important example:
uniform strain

Stirring and mixing by a straining flow



Townsend (1951) and Batchelor (1959)

☛ $c_t + \sigma x c_x - \sigma y c_y = \kappa \nabla^2 c$

$$\boldsymbol{e} = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix}$$ and $$\psi = -\sigma x y$$

☛ The Batchelor length is:

$$\boxed{\ell_{\mathrm{B}} = \sqrt{\frac{\kappa}{\sigma}}}$$

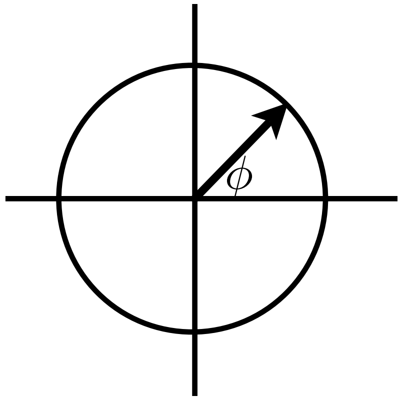
The Cauchy solution $\kappa = 0$

- Lagrangian coordinates:

$$(\dot{x}, \dot{y}) = \sigma(x, -y), \qquad \Rightarrow \qquad (x, y) = \left(\mathrm{e}^{\sigma t} a, \mathrm{e}^{-\sigma t} b\right).$$

and $$\boldsymbol{J} = \begin{pmatrix} \mathrm{e}^{\sigma t} & 0 \\ 0 & \mathrm{e}^{-\sigma t} \end{pmatrix}$$

- The initial condition:



$$\boldsymbol{\xi}(0) = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix}$$

- Most material line elements (eventually) stretch. $\ell(t) \equiv |\boldsymbol{\xi}(t)|$

$$\mathrm{prob}\left[\ell(t) \geq \ell(0)\right] = \frac{2}{\pi} \tan^{-1}\left(\sqrt{\frac{\mathrm{e}^{2\sigma t} - 1}{1 - \mathrm{e}^{-2\sigma t}}}\right)$$

The signature of incompressibility

☛ Of course area is preserved, but this doesn't seem to affect line element stretching in an obvious way.

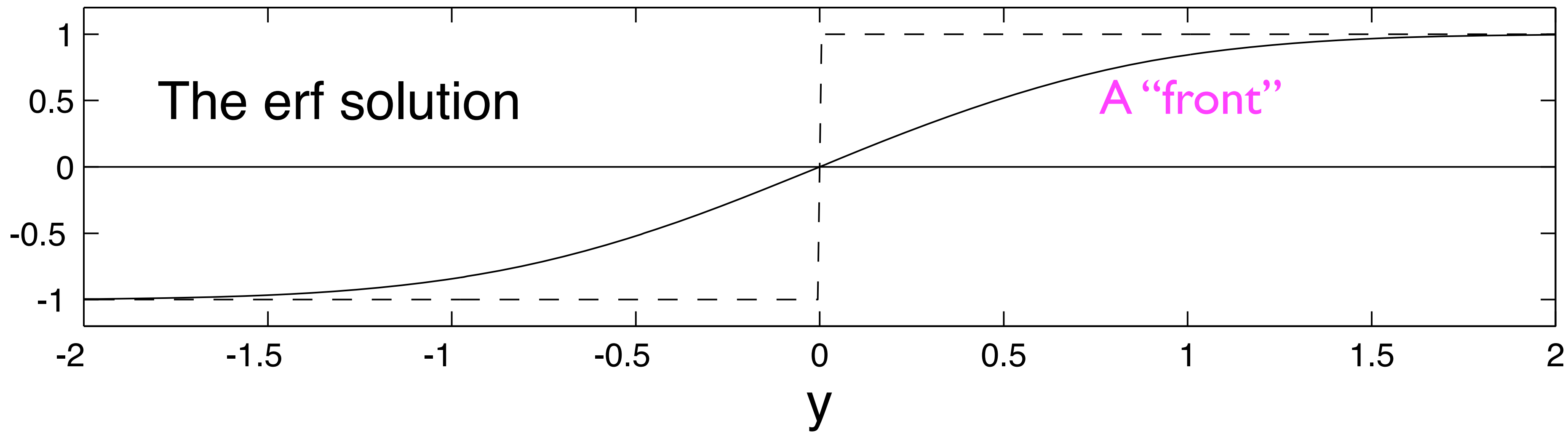
☛ But there is a non-obvious signature of incompressibility:

$$\langle \ell^{-2} \rangle = \ell_0^2 \underbrace{\int \frac{1}{\mathrm{e}^{2\sigma t}\cos^2\phi + \mathrm{e}^{-2\sigma t}\sin^2\phi} \frac{\mathrm{d}\phi}{2\pi}}_{=1}$$

☛ Homework: find the pdf of line element length.

Solution 1: a front $\kappa \neq 0$

☛ $$-\sigma y c_y = \kappa c_{yy} \qquad \Rightarrow \qquad \frac{\mathrm{d}c}{\mathrm{d}y} = A\exp\left(-\frac{y^2}{2\ell_{\mathrm{B}}^2}\right)$$



☛ The IVP, starting with a discontinuity, identifies a time scale:

$$\sqrt{\kappa t} \sim \ell_{\mathrm{B}} \qquad \Rightarrow \qquad t \sim \frac{1}{\sigma}$$

☛ OTOH, starting with a large-scale transition:

$$L\mathrm{e}^{-\sigma t} \sim \ell_{\mathrm{B}} \qquad \Rightarrow \qquad t \sim \frac{1}{\sigma}\ln\left(\frac{L}{\ell_B}\right)$$

Solution 2: super-exponential decay of a plane wave

- Lagrangian coordinates

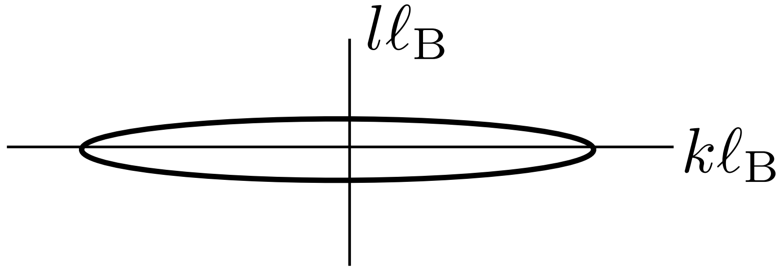
$$(\dot{x},\dot{y}) = \sigma(x,-y), \qquad \Rightarrow \qquad (x,y) = \left(\mathrm{e}^{\sigma t}a, \mathrm{e}^{-\sigma t}b\right).$$

- The transformed equation $c_t = \kappa \mathrm{e}^{-2\sigma t} c_{aa} + \kappa \mathrm{e}^{2\sigma t} c_{bb}.$

- A plane wave solution

$$c = \exp\left[-\kappa k^2 \left(\frac{1-\mathrm{e}^{-2\sigma t}}{2\sigma}\right) - \kappa l^2 \left(\frac{\mathrm{e}^{2\sigma t}-1}{2\sigma}\right)\right] \cos\left(\mathrm{e}^{-\sigma t}kx + \mathrm{e}^{\sigma t}ly\right)$$

- Only waves with wave vectors in an exponentially shrinking ellipse survive.



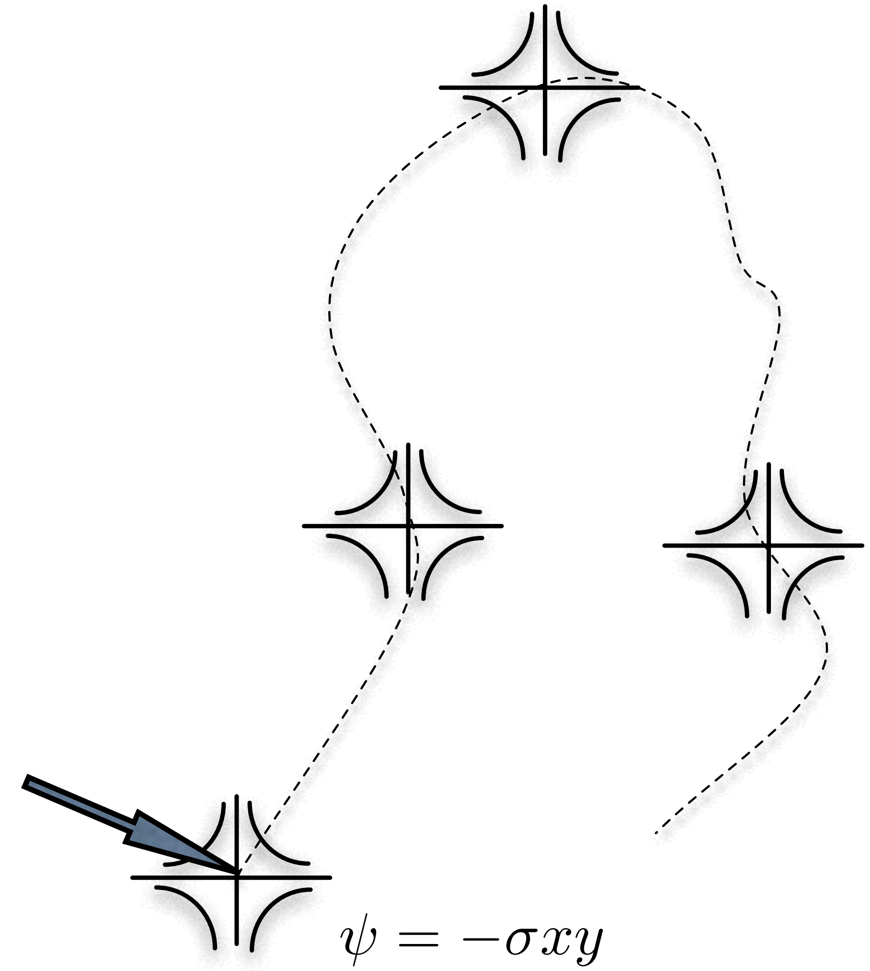
$$k^2 + l^2\mathrm{e}^{2\sigma t} = \ell_B^{-2}$$

Solution 3: exponential decay of a hot spot

☛ We consider "An instantaneous liberation at a point in the fluid of a finite quantity of heat", and follow the hot spot in a Lagrangian frame.

(Townsend 1951)

$$c(x, y, 0) = \delta(x)\delta(y)$$

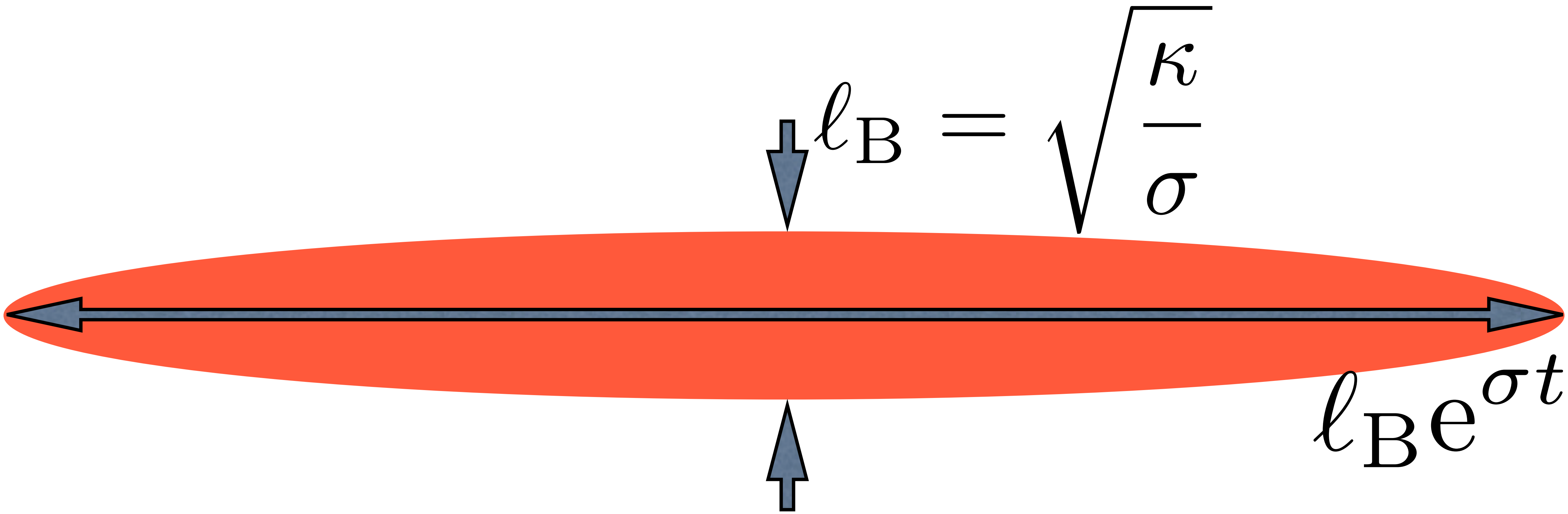


☛ The solution is

$$c(x, y, t) = \frac{1}{2\pi ab} \exp\left[-\frac{x^2}{2a^2} - \frac{y^2}{2b^2}\right]$$

$$a^2 \equiv \frac{\kappa}{\sigma}\left(e^{2\sigma t} - 1\right), \qquad b^2 \equiv \frac{\kappa}{\sigma}\left(1 - e^{-2\sigma t}\right)$$

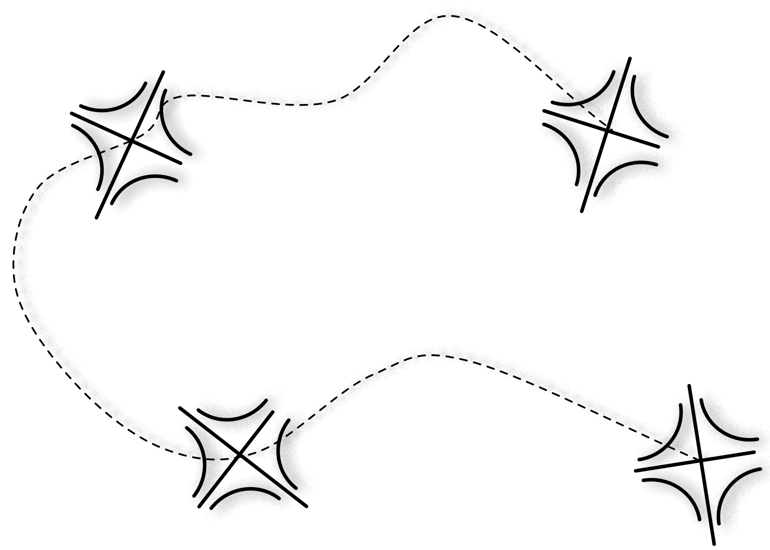
Anatomy of the hot spot



$$\max_{\forall \boldsymbol{x}} c(\boldsymbol{x}, t) \propto \mathrm{e}^{-\sigma t}$$

(Independent of kappa)

Solution 4: randomly re-orienting strain (Homework - this is a difficult one)



☛ Improve the hot-spot model by randomly re-orienting the strain.

☛ Use a renewal model,

$$[0 \leq t < \tau] \quad [\tau \leq t < 2\tau] \quad [2\tau \leq t < 3\tau] \quad \text{etc.}$$

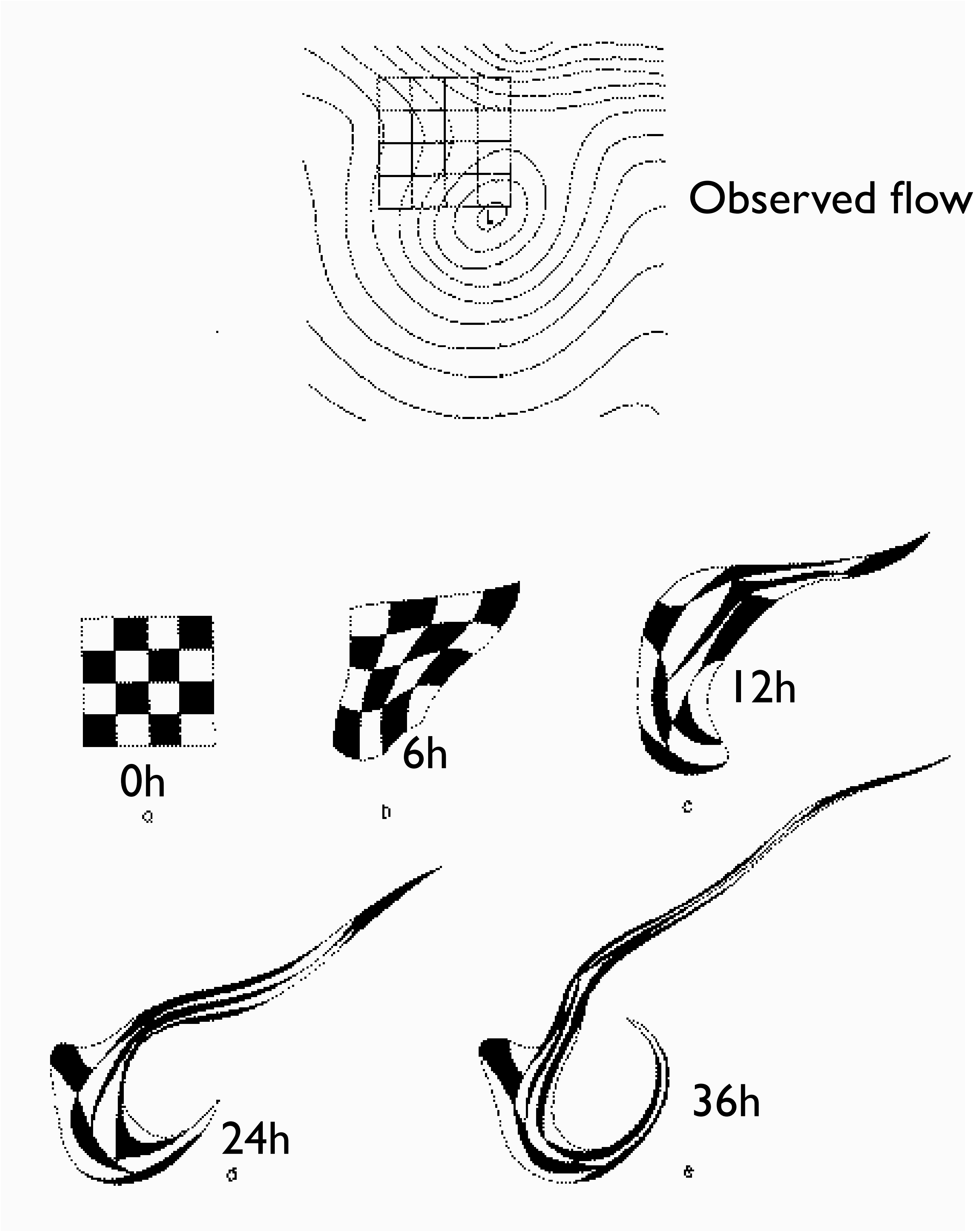
the first epoch   the second epoch   the third epoch

and at the end of each epoch, randomly rotate the strain.

☛ Your mission: $\langle \max_{\forall \boldsymbol{x}} c(\boldsymbol{x}, t) \rangle \propto \mathrm{e}^{-\gamma t}$ $\qquad \gamma = \sigma f(\sigma\tau)$????

THE END

Exponential stretching in other interesting flows



Welander (1955)

Welander's suminagashi visualization

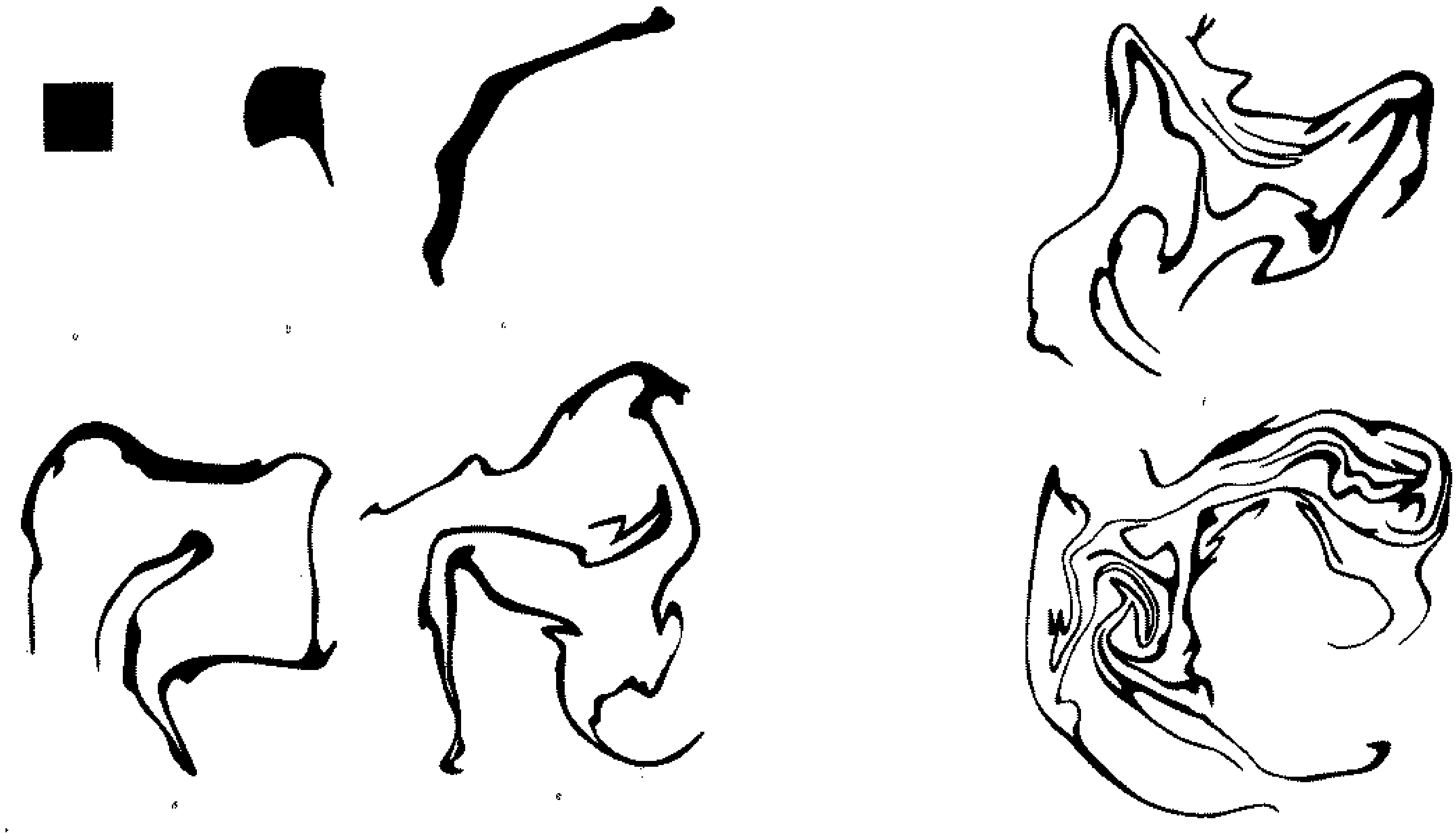


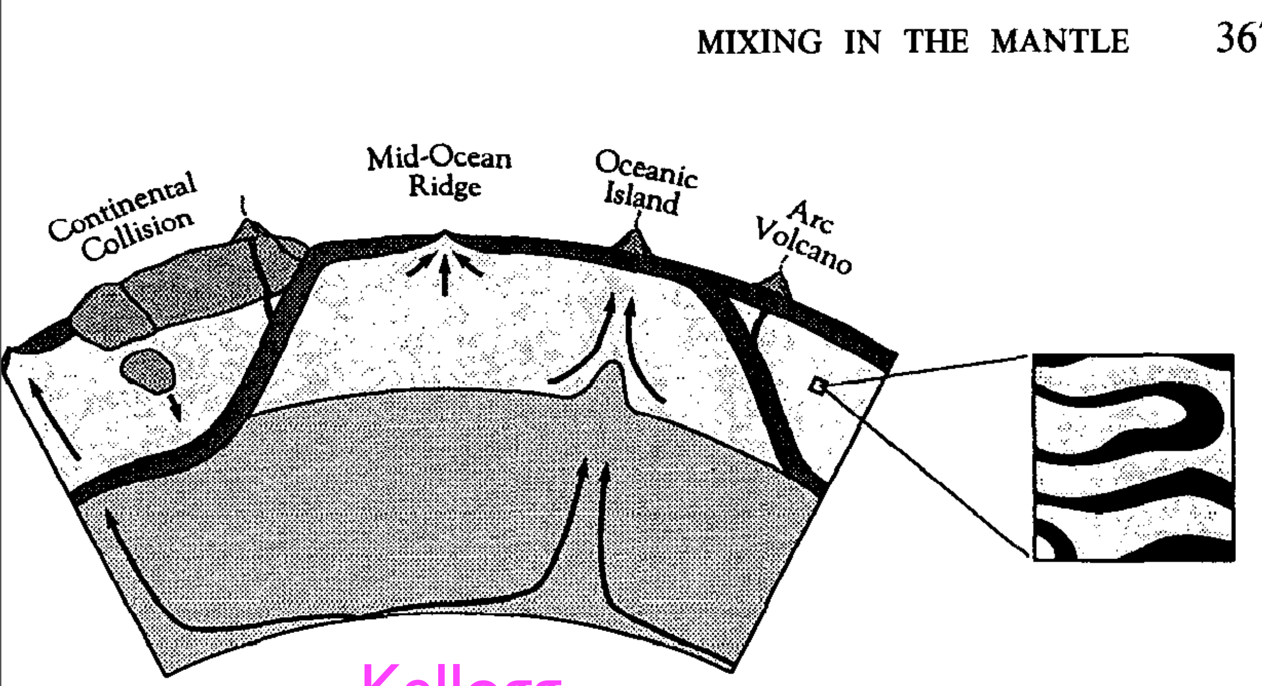
Fig. 3. Observed deformation of a fluid element.

The picture shows the observed deformation of a small, coloured square element of a fluid surface. A rectangular vessel of dimensions 50 × 30 × 30 cm filled with water to half the depth was used for the experiment. On the water surface was put a film of butanol, which was divided into square elements by means of a metal grid. One or several of these elements were coloured with methyl-red and the water was set into horizontal motion. The grid was then quickly taken away and the fluid was left to move undisturbed. To keep the motion two-dimensional, the whole fluid mass was set into a slow basic rotation before the initial disturbance was created.

Tellus VII (1955), 2

The marble-cake mantle

(Allegre & Turcotte)



Kellogg

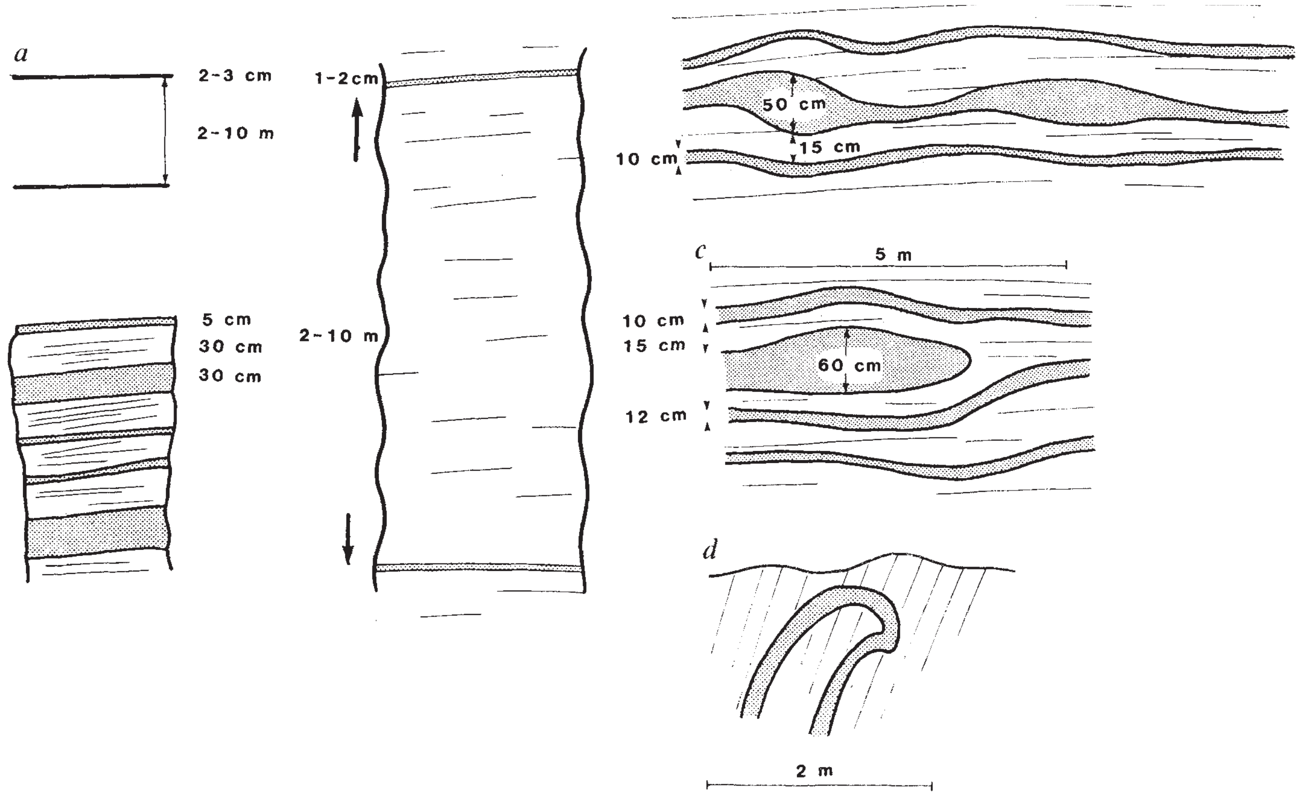


Fig. 2 Occurrences of pyroxenite layers in the Beni Bousera high-temperature peridotite. Grey, pyroxenite; white, lherzolite with foliation. *a*, Occurrences in an outcrop with no folding; *b–d*, occurrences with folding and boudinage.

☛ Fine-scale filaments, generated by mantle convection, are exposed at the Earth's surface as in high-temperature peridotites.