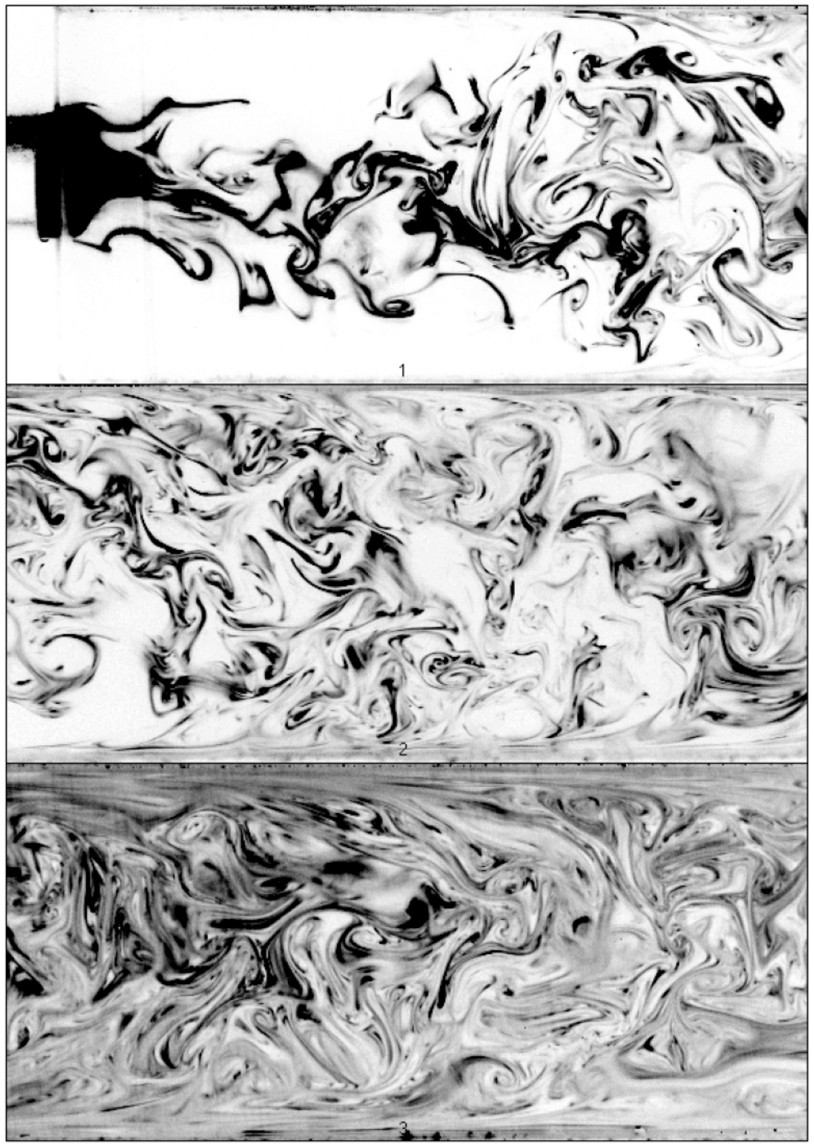
Lecture 4: line element stretching

Multiplicative random variables, large deviations, statistics of line elements, Lyapunov exponents, Kraichnan-Kazantsev model

An ensemble of line elements

Villermaux & Duplat



Villermaux & Duplat

FIG. 1. Mixing of a dye discharging from a jet of diameter $d = 8$ mm in a square ($L \times L$ with $L = 3$ cm) duct. From 1 to 3, successive instantaneous planar cuts of the scalar field at increasing downstream locations in the duct showing the progressive uniformization of the dye concentration.

☛ Stirring results in exponential growth of the length of infinitesimal material line elements.

☛ We want to understand the statistical properties of an ensemble of elements in turbulent, chaotic or stochastic flows.

☛ The simplest hypothesis is too simple.

$$\mathrm{pdf}(\ell, t) = \mathrm{e}^{-\gamma t} P\left(\mathrm{e}^{-\gamma t}\ell\right)$$

(Batchelor 1952)

☛ Start with an excursion into multiplicative random variables and large deviation theory.

Multiplicative random variables and large deviation theory

Random multiplicative processes: An elementary tutorial

S. Redner
Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215

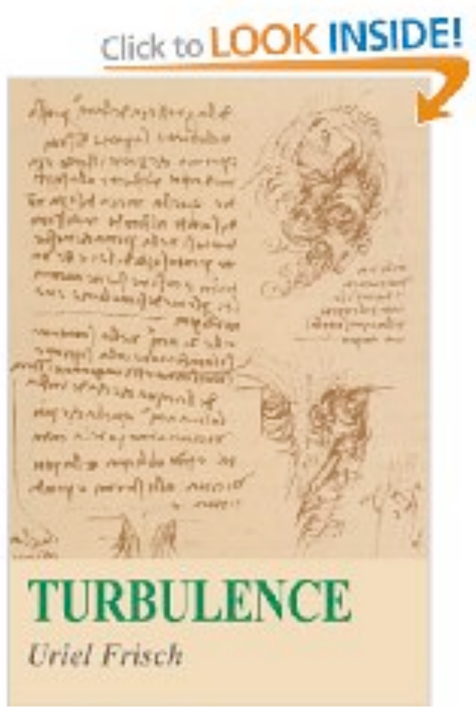
267 Am. J. Phys. **58** (3), March 1990

An Introduction to Large Deviations for Teletraffic Engineers

John T. Lewis[1] Raymond Russell[1]

November 1, 1997

Frisch



Multiplicative random variables $P = m_1 m_2 \cdots m_N$

☛ Example 1: $m_k = 0$ or $m_k = 2$ with probability one half.

$$\langle P \rangle \equiv \frac{\text{sum all the } P\text{'s from different realizations}}{\text{number of realizations}} = 1$$ and $$P_{\mathrm{mp}} = 0$$

☛ Example 2: $m_k = \alpha^{\pm 1}$ with probability one half.

$$\langle P \rangle = \left(\frac{\alpha + \alpha^{-1}}{2} \right)^N$$ and $$P_{\mathrm{mp}} = 1$$

☛ Extreme events are exponentially rare, but exponentially larger, than typical events.

Take the logarithm: $\ln P = \ln m_1 + \ln m_2 + \cdots + \ln m_N$

☛ The law of large numbers implies: $\boxed{P_{\text{mp}} = \mathrm{e}^{\langle \ln P \rangle} \, .}$

☛ Recall Example 2: $m_k = \alpha^{\pm 1} \quad \Rightarrow \quad P_{\text{mp}} = 1$

☛ The central limit theorem is valid, but not powerful enough:

$$\text{pdf}_{\text{CLT}}(S) = \frac{1}{\sqrt{2\pi N \ln^2 \alpha}} \exp\left(- \frac{S^2}{2N \ln^2 \alpha} \right)$$

, $\text{pdf}(S) \approx \text{pdf}_{\text{CLT}}(S)$ where $S \equiv \ln P$

☛ Moments of P are dominated by the non-CLT tails $\langle P^\beta \rangle = \langle \mathrm{e}^{\beta S} \rangle$

Homework/Discussion

Problem 6.1. Your broker offers you a stock whose value doubles every year with probability 3/8, or halves with probability 5/8. To justify his commission, he argues that the expected multiplier is

$$\langle m \rangle = \frac{3}{8} \times 2 + \frac{5}{8} \times \frac{1}{2} = \frac{17}{16}\,.$$

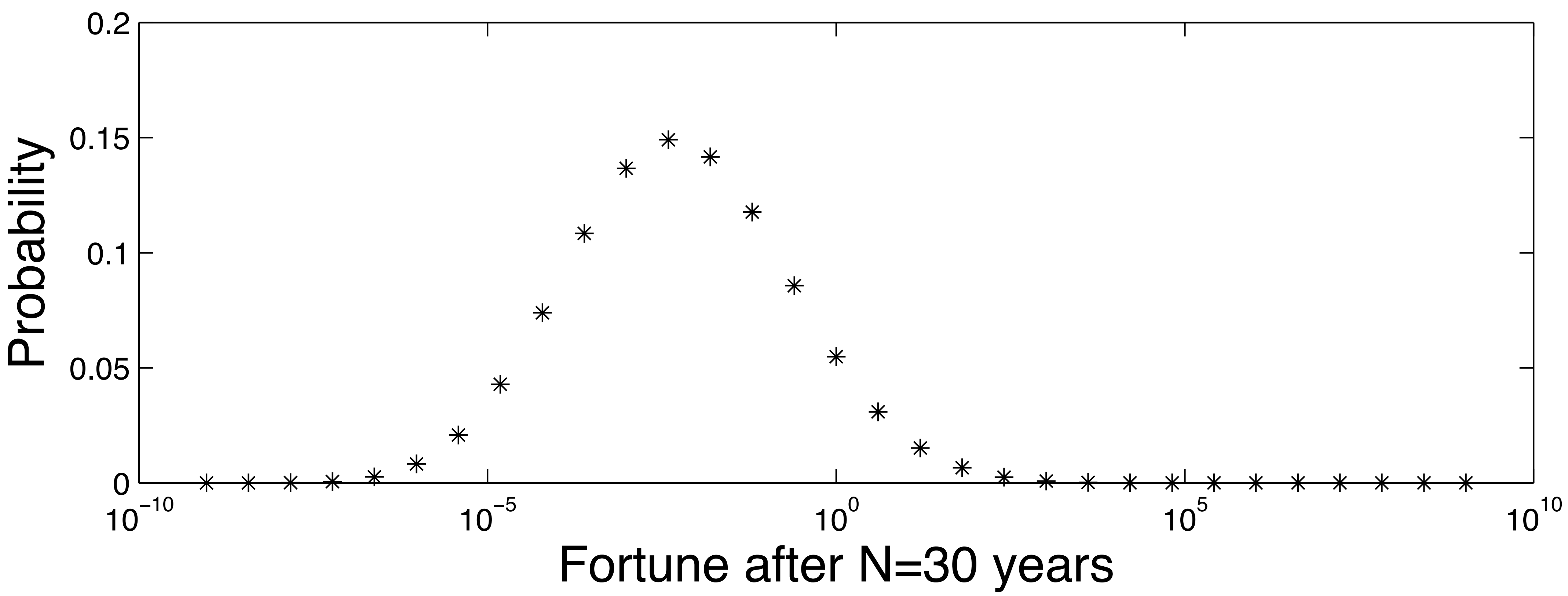
If you hold this stock for 30 years what is the probability that your return exceeds $(17/16)^{30} = 6.1641$? How about less than one? Should you sell the family farm and buy this stock?

Homework/Discussion

Problem 6.1. Your broker offers you a stock whose value doubles every year with probability 3/8, or halves with probability 5/8. To justify his commission, he argues that the expected multiplier is

$$\langle m \rangle = \frac{3}{8} \times 2 + \frac{5}{8} \times \frac{1}{2} = \frac{17}{16} \,.$$

If you hold this stock for 30 years what is the probability that your return exceeds $(17/16)^{30} = 6.1641$? How about less than one? Should you sell the family farm and buy this stock?

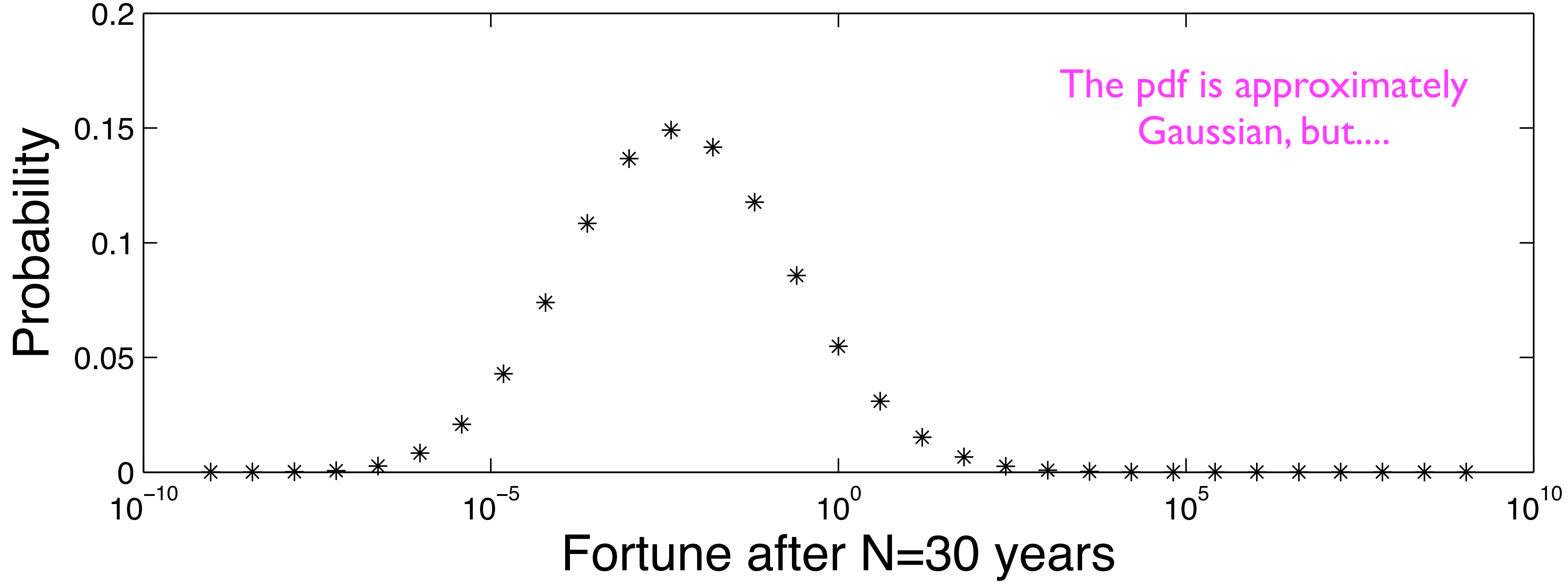


Homework/Discussion

Problem 6.1. Your broker offers you a stock whose value doubles every year with probability 3/8, or halves with probability 5/8. To justify his commission, he argues that the expected multiplier is

$$\langle m \rangle = \frac{3}{8} \times 2 + \frac{5}{8} \times \frac{1}{2} = \frac{17}{16}\,.$$

If you hold this stock for 30 years what is the probability that your return exceeds $(17/16)^{30} = 6.1641$? How about less than one? Should you sell the family farm and buy this stock?



Large deviation theory

- ☛ Consider a random sum $S_N = a_1 + a_2 + \cdots + a_N$

- ☛ A "normal deviation" is

$$S_N = N\bar{a} \pm \sqrt{N} \times (\text{a random something})$$

- ☛ A "large deviation" is a rare event, such as $S_N = Nx$

- ☛ The main result of large deviation theory is

$$\boxed{\mathrm{pdf}\left(S_N\right) = \exp\left[-NG\left(\frac{S_N}{N}\right) + o(N)\right]\ .}$$

- ☛ The CLT is a special case.

The binomial distribution

$a_k = 0 \text{ or } 1\,,\ \text{with probability one half}$

☛ The exact pdf is $\text{prob}(S_N = xN) = \left(\frac{1}{2}\right)^N \frac{N!}{(xN)!(N(1-x))!}$

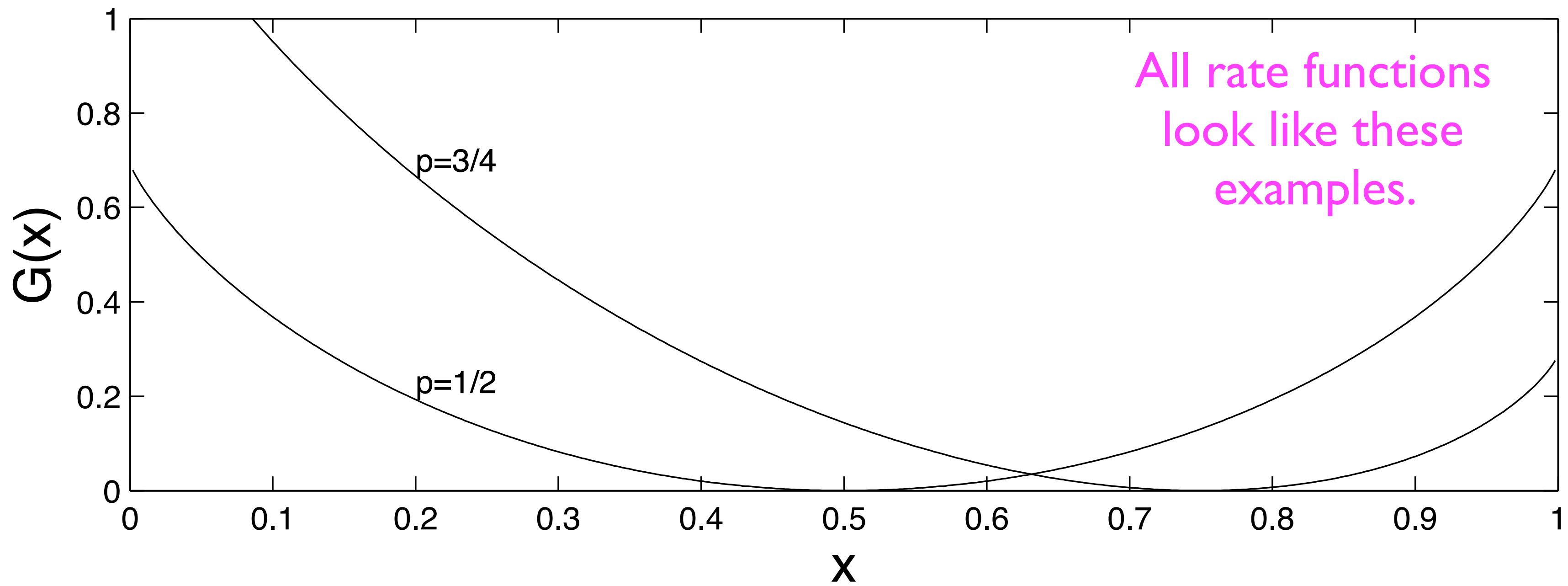


Figure 2: The rate function (48) for the binomial model; p is the probability that $a_k = 1$, and $1 - p$ the probability that $a_k = 0$.

☛ Courtesy of Stirling $\text{prob}(S_N = xN) \approx \frac{\exp\left[-N\left(\ln 2 + x\ln x + (1-x)\ln(1-x)\right)\right]}{\sqrt{2\pi N x(1-x)}}$

Existence of the rate function

☛ A definition: $$L_N(x) \equiv \ln\left[\text{prob}\left(\sum_{i=1}^{n} a_i > Nx\right)\right]$$

☛ Then prove the limit exists:

$$\lim_{N\to\infty} \frac{L_N(x)}{N}$$

and

$$-G(x) \equiv \sup \frac{L_N(x)}{N}$$

☛ The key result is super-additivity

$$L_{N_1+N_2} \geq L_{N_1} + L_{N_2}$$

The sequence is not monotonic

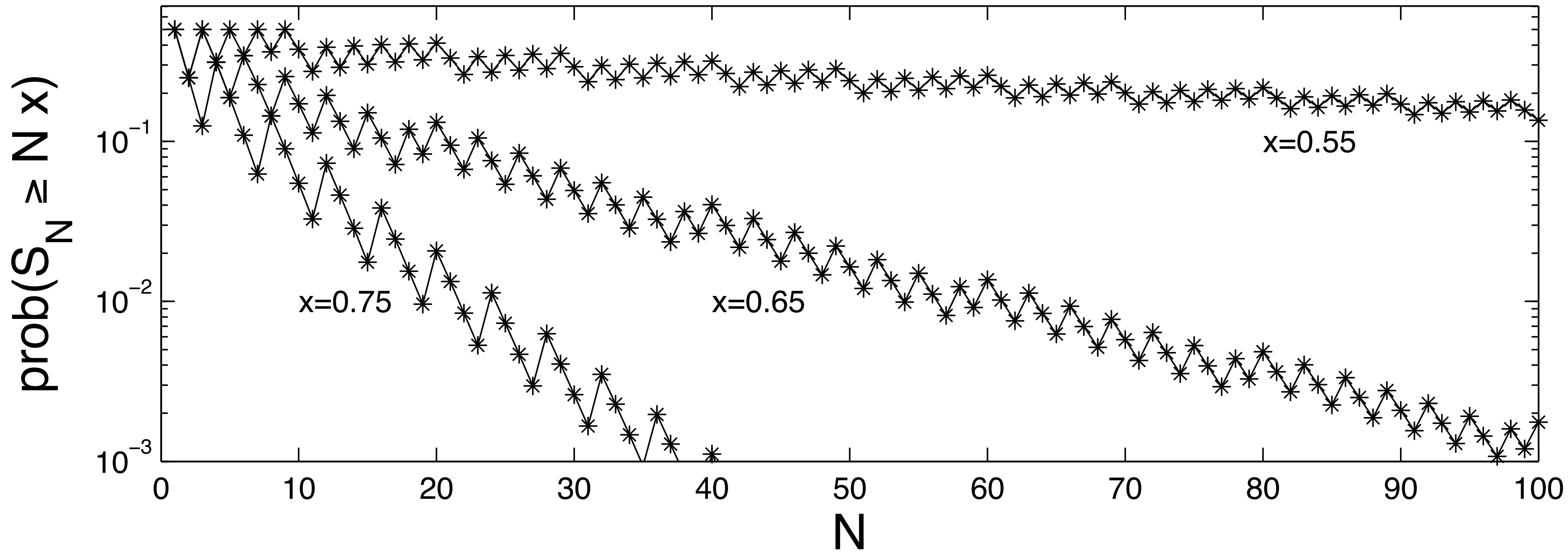


Figure 3: The sequence $L_N(x)$ at three values of x, using the binomial model with $\text{prob}(a_k = 0) = \text{prob}(a_k = 1) = 0.5$. The probabilities are computed by iteration of the `conv` command in matlab. There is a wiggly exponential decrease of $\text{prob}(S_N > xN)$ as $N \to \infty$.

$$L_N(x) \equiv \ln\left[\text{prob}\left(\sum_{i=1}^{n} a_i > Nx\right)\right]$$

and

$$-G(x) \equiv \sup \frac{L_N(x)}{N}$$

The CGF is the Legendre dual of the rate function

- The rate function G is the Legendre transform of the cumulant generating function, and vice versa.

The CGF: $$\mathrm{e}^{F(\beta)} = \int_{-\infty}^{\infty} \mathrm{e}^{\beta a} \mathrm{pdf}(a)\, \mathrm{d}a$$

- Legendre duals:

$$G(x) = \sup_{\beta} \left(\beta x - F(\beta)\right) \qquad F(\beta) = \sup_{x} \left(\beta x - G(x)\right)$$

Recall: $$S_N = a_1 + a_2 + \cdots + a_N$$

Line element stretching: general results

J. Fluid Mech. (1984), *vol.* 144, *pp.* 1–11
Printed in Great Britain

Kinematic dynamo problem in a linear velocity field

By YA. B. ZEL'DOVICH, A. A. RUZMAIKIN,
Keldysh Institute of Applied Mathematics, Academy of Sciences of the USSR,
Miusskaya ploshchad, Moscow, 125047

S. A. MOLCHANOV AND D. D. SOKOLOFF
Moscow State University, Moscow, 117234

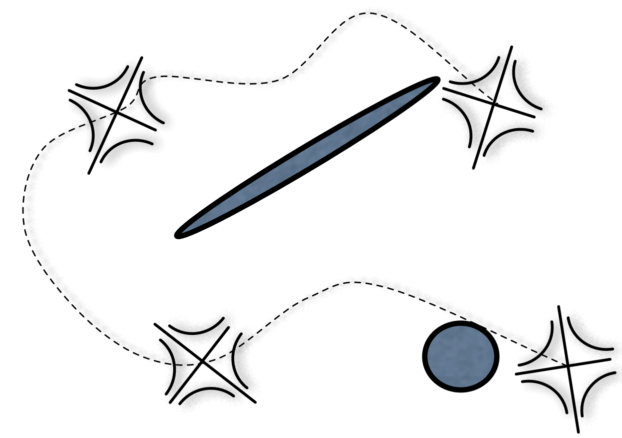
J. Fluid Mech. (1990), *vol.* 215, *pp.* 45–59
Printed in Great Britain

Turbulent stretching of line and surface elements

By I. T. DRUMMOND AND W. MÜNCH
Department of Applied Mathematics and Theoretical Physics, University of Cambridge,
Silver Street, Cambridge CB3 9EW, UK

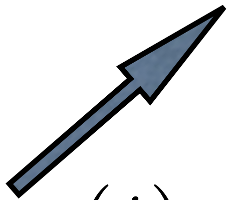
The “local stretching model”

- Focus on small scales, and elaborate Townsend’s hot-spot model.



- Solve the line-element equation:

$$\partial_t \boldsymbol{\xi} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{\xi} = \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \boldsymbol{u}$$



$\boldsymbol{\xi}(\boldsymbol{x}(t), t)$

$\boldsymbol{x}(t)$

No molecular diffusion (yet)

- In this Lagrangian frame we have a stochastic differential equation:

$$\dot{\boldsymbol{\xi}} = \boldsymbol{W}(t)\boldsymbol{\xi}$$

- We desire the statistical properties of line-element lengths.

Notation: $\ell(t) = |\boldsymbol{\xi}(t)|$

$$h(t) \equiv \frac{1}{t} \ln \left(\frac{\ell(t)}{\ell_0} \right)$$

Definition of the Lyapunov exponent

☛ For the moment, we use the definition:

$$\gamma_{\mathrm{Lpv}} \equiv \lim_{t\to\infty} \frac{1}{t} \left\langle \ln\left(\frac{\ell(t)}{\ell_0}\right)\right\rangle$$

☛ Using the golden rule for multiplicative processes:

$$\boxed{\ell_{\mathrm{mp}} = \ell_0 \mathrm{e}^{\gamma_{\mathrm{Lpv}} t}}$$

☛ According to Batchelor, all elements would stretch at this rate. This is not exactly true - we need a more complete characterization of stretching statistics.

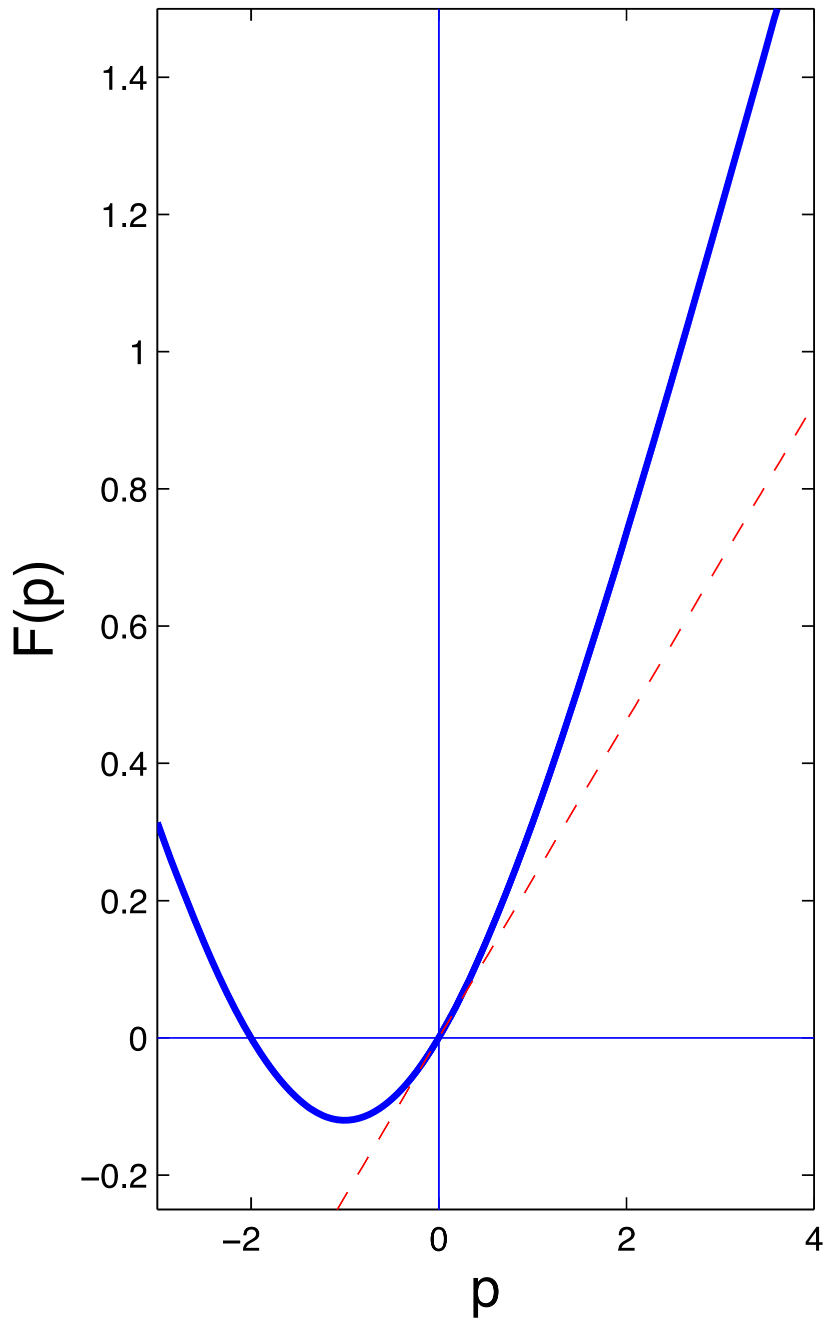
The big picture for line-element stretching

- $$F(p) \equiv \lim_{t\to\infty} \frac{1}{t} \log \left\langle \left(\frac{\ell}{\ell_0}\right)^p \right\rangle$$

  or $$\langle \mathrm{e}^{ph} \rangle = \mathrm{e}^{tF(p)}\,, \qquad \text{as } t \to \infty$$

  F(p) is the CGF.

- $$\gamma_{\mathrm{Lpv}} = \lim_{p\to 0} \frac{F(p)}{p}$$



- F(p) is convex, and

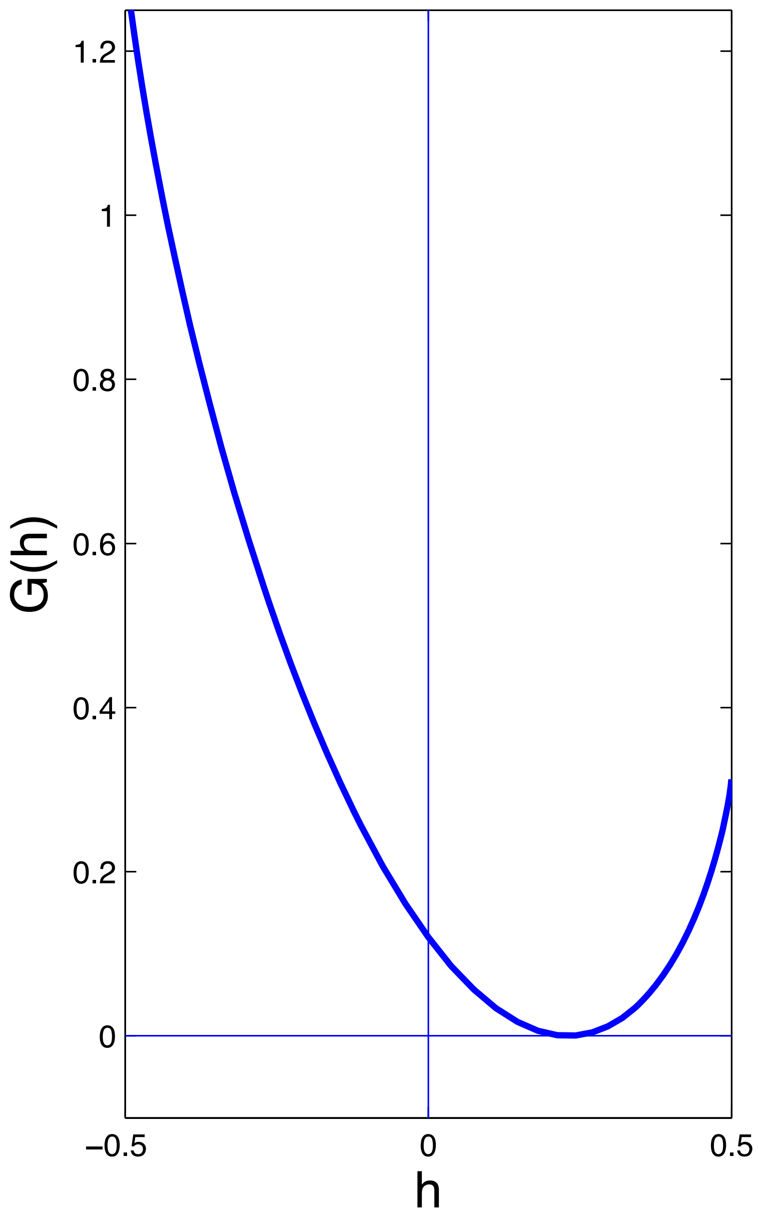
$$F(0) = 0\,, \quad F'(0) > 0\,, \quad F(-d) = 0\,,$$ and $$\underbrace{F(p) = F(-2-p)}_{d=2}$$

The other half of the big picture

- Large deviation theory gives

$$\mathrm{pdf}(h) \approx \sqrt{\frac{tG''(h)}{2\pi}}\,\mathrm{e}^{-tG(h)}$$

and $$G\left(\gamma_{\mathrm{Lpv}}\right) = 0$$



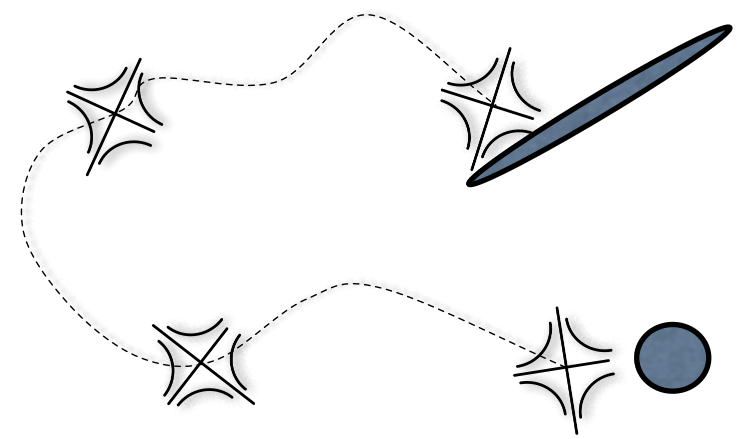
- F(p) and G(h) contain equivalent information:

$$G(h) = \sup_{\forall p}\left(ph - F(p)\right) \qquad F(p) = \sup_{\forall h}\left(ph - G(h)\right)$$

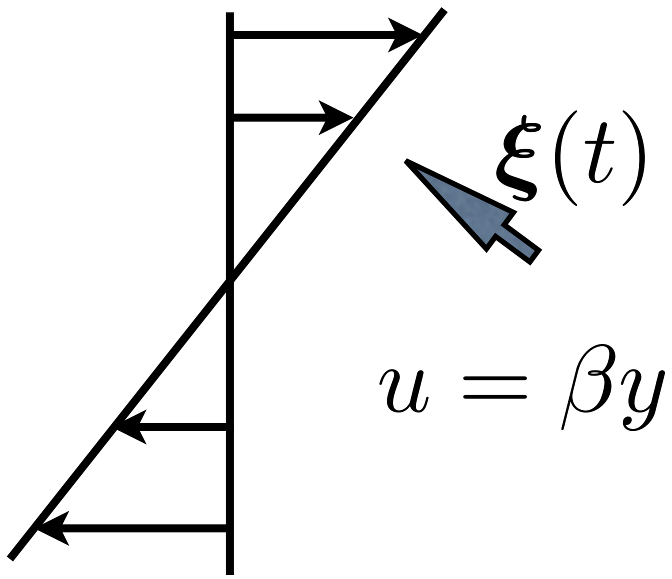
An example illustrating the main features of the big picture

Renewing Couette flow

☛ The illustration at right is misleading: hyperbolic points are not necessary for exponential stretching.



☛ Even Couette flow can produce exponential stretching, provided there is "random realignment".



☛ Use a renewal model,

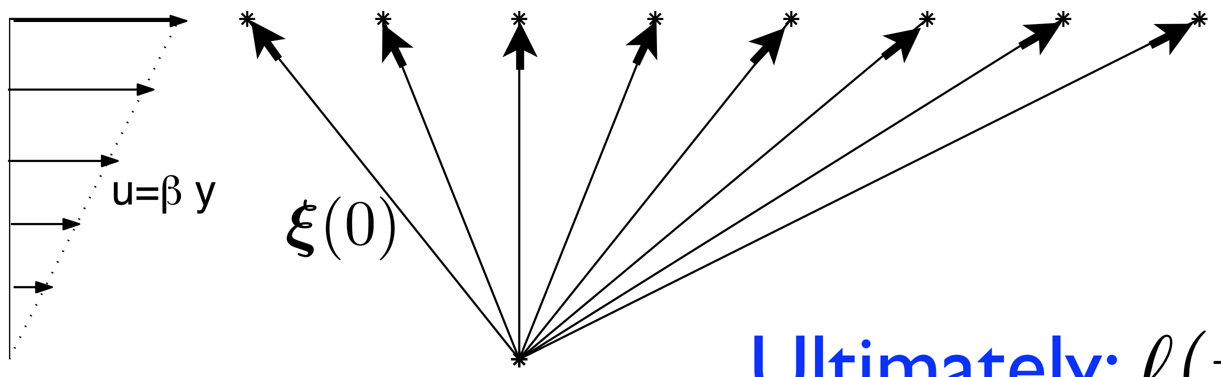
$$[0 \leq t < \tau] \quad [\tau \leq t < 2\tau] \quad [2\tau \leq t < 3\tau]$$

the first epoch   the second epoch   the third epoch

with randomly change direction at the end of each epoch.

Couette flow $\dot{\boldsymbol{\xi}} = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} \boldsymbol{\xi}$

☛ Recall elementary Couette flow - a material line moves like this:



Ultimately: $\ell(t) \equiv |\boldsymbol{\xi}(t)| \propto \ell_0 \times \beta t$

☛ The solution of the line-element equation is:

$$\boldsymbol{\xi}(t) = \ell_0 \begin{pmatrix} \cos\theta + \beta t \sin\theta \\ \sin\theta \end{pmatrix}$$

and $\ell^2(t) = \left[1 + \beta t \sin 2\theta + \beta^2 t^2 \sin^2\theta\right] \ell_0^2$

Renewing Couette flow

☛ At the end of each epoch,

$$[0 \le t < \tau] \quad [\tau \le t < 2\tau] \quad [2\tau \le t < 3\tau]$$

randomly rotate the direction of the Couette flow.

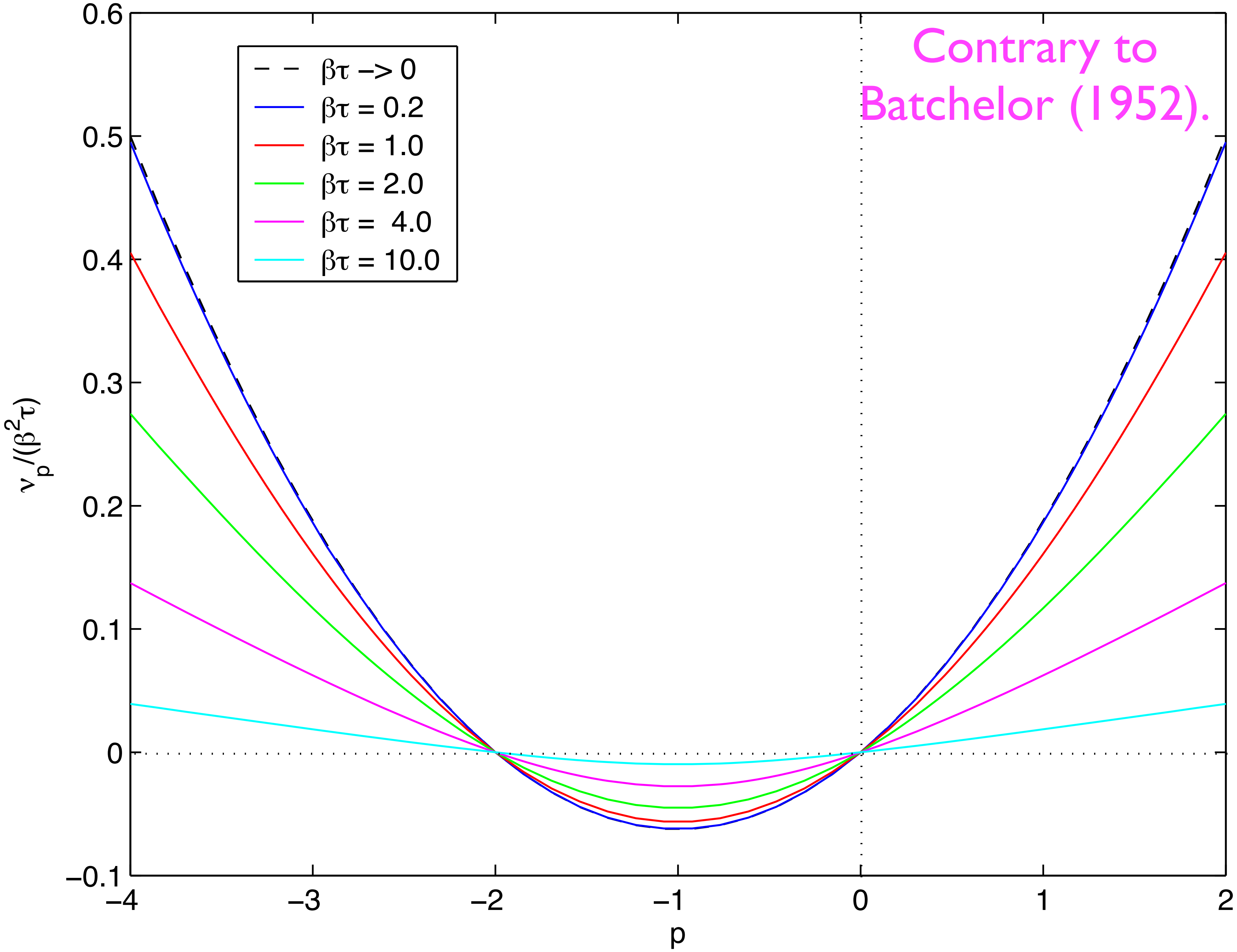
☛ At the end of the n'th epoch, $\ell(n\tau) = \prod_{k=1}^{n} \underbrace{\sqrt{1 + \beta\tau \sin 2\theta_k + \beta^2\tau^2 \sin^2\theta_k}}_{\equiv m(\theta_k)} \, \ell_0$

☛ Since this is a random product: $\ell_{\mathrm{mp}} = \mathrm{e}^{\langle \ln \ell \rangle} = \mathrm{e}^{n\langle m(\theta) \rangle} = \mathrm{e}^{t\gamma_{\mathrm{Lpv}}}$

$$\gamma_{\mathrm{Lpv}} = \frac{1}{2\tau} \oint \ln\left(1 + \beta\tau \sin 2\theta + \beta^2\tau^2 \sin^2\theta\right) \frac{\mathrm{d}\theta}{2\pi}$$

or $\quad \gamma_{\mathrm{Lpv}} = \dfrac{1}{2\tau} \ln\left(1 + \dfrac{\beta^2\tau^2}{4}\right)$

The CGF of Renewing Couette flow



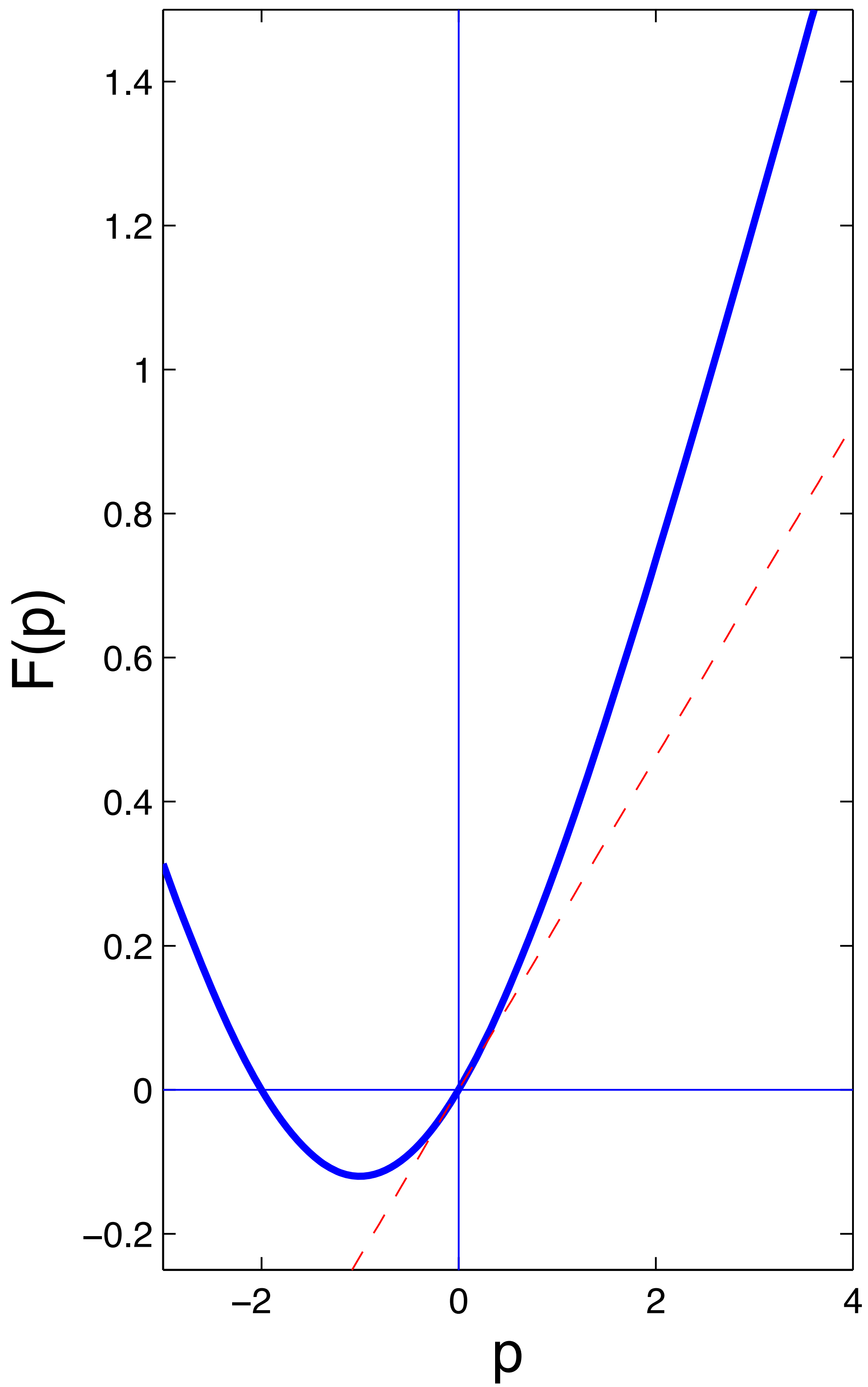
$$F(p) = \frac{1}{\tau} \ln \left[\oint \left(1 + \beta\tau \sin 2\theta + \beta^2 \tau^2 \sin^2 \theta \right)^{p/2} \frac{\mathrm{d}\theta}{2\pi} \right]$$

(Legendre functions)

Properties of F(p): some "proofs"

$$F(p) \equiv \lim_{t\to\infty} \frac{1}{t} \log \left\langle \left(\frac{\ell}{\ell_0} \right)^p \right\rangle$$

F(p) is convex



$$F(p) \equiv \lim_{t\to\infty} \frac{1}{t} \log \left\langle \left(\frac{\ell}{\ell_0} \right)^p \right\rangle$$

- ☛ It is evident that F(0)=0.
- ☛ Use Cauchy-Schwarz:

$$\left\langle \ell^{\frac{1}{2}(p+q)} \right\rangle < \left(\langle \ell^p \rangle \langle \ell^q \rangle \right)^{\frac{1}{2}}$$

$$\Rightarrow \quad F\left(\frac{p+q}{2}\right) < \tfrac{1}{2}\left(F(p) + F(q)\right)$$

ln isotropic flow $\gamma_{\mathrm{Lpv}} = F'(0) > 0$

☛ Since the line-stretching equation is linear:

$$\boldsymbol{\xi}(t) = \boldsymbol{J}(t)\boldsymbol{\xi}(0) \quad \Rightarrow \quad \ln\left(\frac{\ell}{\ell_0}\right) = \tfrac{1}{2}\ln\left(\boldsymbol{e}^\intercal \boldsymbol{J}^\intercal \boldsymbol{J}\boldsymbol{e}\right), \qquad |\boldsymbol{e}|^2 = 1$$

and $\det \boldsymbol{J} = 1$ where $\boldsymbol{\xi}(0) = \ell_0 \boldsymbol{e}$

☛ Three things we know about: $\boldsymbol{J}^\intercal \boldsymbol{J}$

☛ Use Jensen's inequality log(avg) > avg(log):

$$\begin{aligned} \left\langle \ln\left(\frac{\ell}{\ell_0}\right)\right\rangle_{\boldsymbol{e}} &= \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\ln\left(\lambda_1 e_1^2 + \lambda_2 e_2^2 + \lambda_3 \mathrm{e}_3^2\right)\mathrm{d}S \\ &\geq \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\left(e_1^2 \ln\lambda_1 + e_2^2 \ln\lambda_2 + e_3^3 \ln\lambda_3\right)\mathrm{d}S \\ &= \frac{1}{6}\ln(\lambda_1\lambda_2\lambda_3) = 0 \end{aligned}$$

In isotropic flow $\gamma_{\rm Lpv} = F'(0) > 0$

☛ Since the line-stretching equation is linear:

$$\boldsymbol{\xi}(t) = \boldsymbol{J}(t)\boldsymbol{\xi}(0) \quad \Rightarrow \quad \ln\left(\frac{\ell}{\ell_0}\right) = \tfrac{1}{2}\ln\left(\boldsymbol{e}^{\mathsf{T}}\boldsymbol{J}^{\mathsf{T}}\boldsymbol{J}\boldsymbol{e}\right), \qquad |\boldsymbol{e}|^2 = 1$$

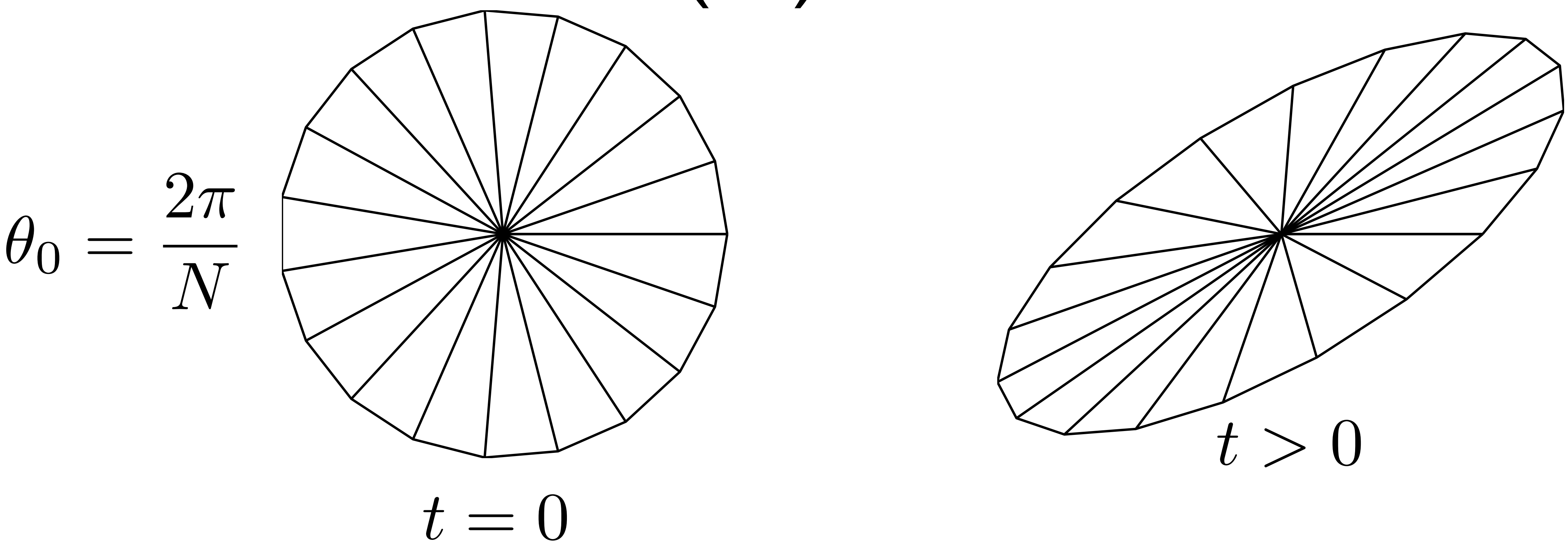
and $\det \boldsymbol{J} = 1$ where $\boldsymbol{\xi}(0) = \ell_0 \boldsymbol{e}$

☛ Three things we know about: $\boldsymbol{J}^{\mathsf{T}}\boldsymbol{J}$ symmetric, det =1, and eigenvalues are positive.

☛ Use Jensen's inequality log(avg) > avg(log):

$$\begin{aligned} \langle \ln\left(\frac{\ell}{\ell_0}\right)\rangle_{\boldsymbol{e}} &= \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\ln\left(\lambda_1 e_1^2 + \lambda_2 e_2^2 + \lambda_3 \mathrm{e}_3^2\right) \mathrm{d}S \\ &\geq \frac{1}{4\pi}\int_{|\boldsymbol{e}|=1} \tfrac{1}{2}\left(e_1^2 \ln\lambda_1 + e_2^2 \ln\lambda_2 + e_3^3 \ln\lambda_3\right) \mathrm{d}S \\ &= \frac{1}{6}\ln(\lambda_1\lambda_2\lambda_3) = 0 \end{aligned}$$

$F(-d) = 0$



☛ Incompressibility (conservation of area in *d*=2) is the key:

$$A = \ell_0^2 \theta_0 = \ell_n^2(t)\theta_n(t) \quad n = 1\,, 2\,, \cdots N$$

$$\Rightarrow \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\ell_0}{\ell_n(t)} \right)^2 = 1$$