Lecture 5: The decay problem

“Elementary” example: diffusive decay

☛ For a random IC with finite, non-zero correlation length in unbounded space:

$$c_{RMS}(t) \sim \frac{\ell^{d/2}}{(\kappa t)^{d/4}} c_{RMS}(0)$$

☛ This result is obtained using the CLT, or by solving

$$\mathcal{C}_t = 2\kappa \nabla^2 \mathcal{C} \qquad \text{where} \qquad \mathcal{C}(\boldsymbol{x}_2 - \boldsymbol{x}_1, t) \equiv \langle c(\boldsymbol{x}_1, t) c(\boldsymbol{x}_2, t) \rangle$$

☛ In a finite domain, the ultimate decay is exponential:

$$c(\boldsymbol{x}, t) \sim \mathrm{e}^{-\nu t}, \qquad \text{where } \nu \text{ is the smallest nonzero eigenvalue}$$

Now scalar decay with stirring

$$c_t + \boldsymbol{u} \cdot \boldsymbol{\nabla} c = \kappa \nabla^2 c$$
$$c(\boldsymbol{x}, 0) = c_0(\boldsymbol{x})$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \int c^2 \, \mathrm{d}V = -\kappa \int |\boldsymbol{\nabla} c|^2 \, \mathrm{d}V}$$

☛ WLOG $\int c \, \mathrm{d}V = 0$ so we focus on $\int c^2 \, \mathrm{d}V = 0$

The decay problem

☛ Simulations, theory and experiments all indicate that:

$$\int c^2 \, \mathrm{d}V \sim \mathrm{e}^{-\nu(\kappa)t}$$

(In a bounded domain.)

☛ For turbulent flows, and strongly chaotic flows without transport barriers, we expect fast mixing:

$$\lim_{\kappa \to 0} \nu(\kappa) = \nu_0 \neq 0$$

☛ Given $\boldsymbol{u}(\boldsymbol{x}, t)$, predict ν_0

☛ Are the statistics of the decaying field intermittent, or self-similar?

Pierrehumbert's strange eigenmode

Pergamon

Chaos, Solitons & Fractals Vol. 4, No. 6, pp. 1091–1110, 1994

0960–0779/94$7.00 + .00

0960–0779(94)E0037–P

(1994)

Tracer Microstructure in the Large-eddy Dominated Regime

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and

CHAOS VOLUME 10, NUMBER 1

(2000)

Lattice models of advection-diffusion

Raymond T. Pierrehumbert
University of Chicago, Chicago, Illinois

Numerical simulation

☛ Use the "XY renewal model":

$$\underbrace{[0\ \tau/2]\ [\tau/2\ \tau]}_{\text{The first epoch}}\ \underbrace{[\tau\ 3\tau/2]\ [3\tau/2\ 2\tau]}_{\text{The second epoch}}\ \text{etc.}$$

$$\text{First } (u,v) = (\cos(y+\phi_n), 0)\,, \text{ and then } (u,v) = (0, \cos(x+\theta_n))$$

☛ Coerce the displacements onto a grid using shift operations on rows and columns of the c-matrix

Pierrehumbert (2000)

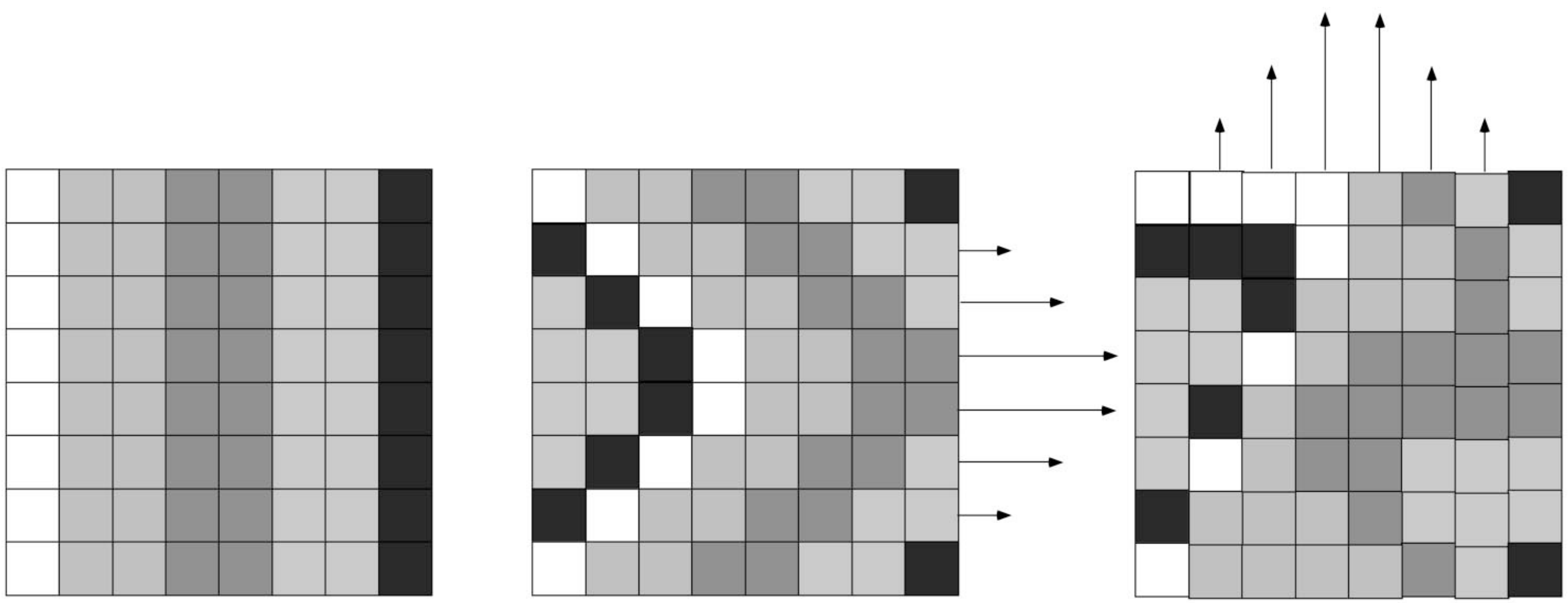
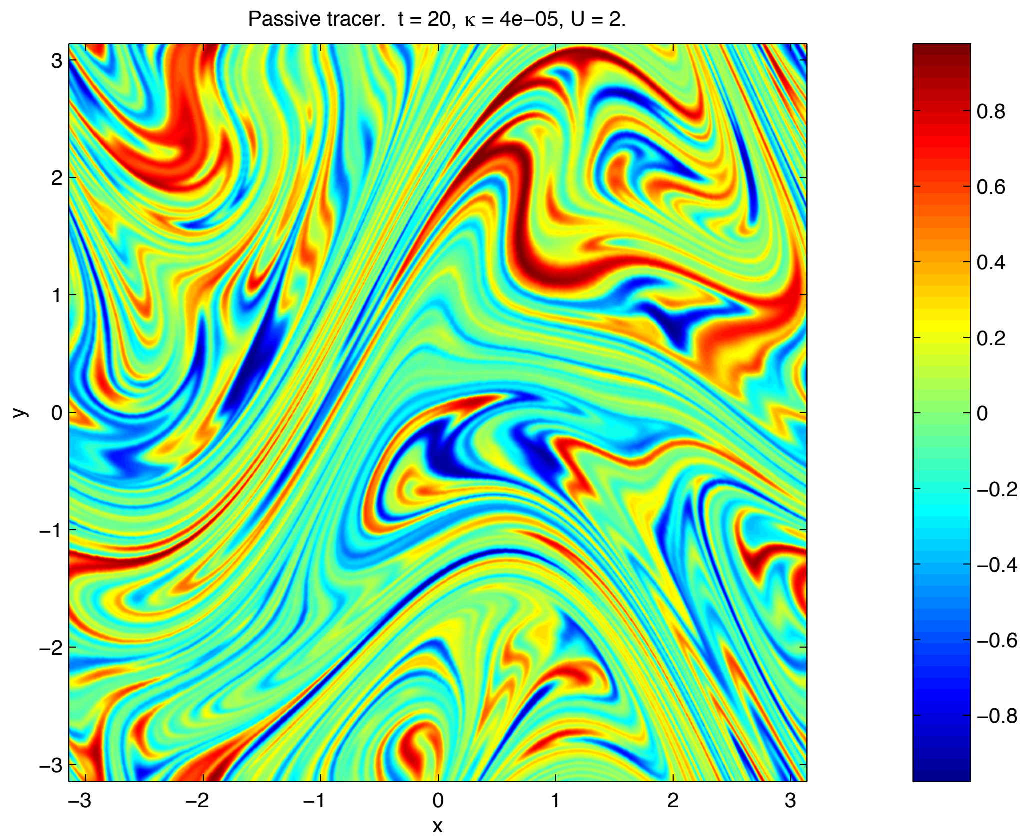
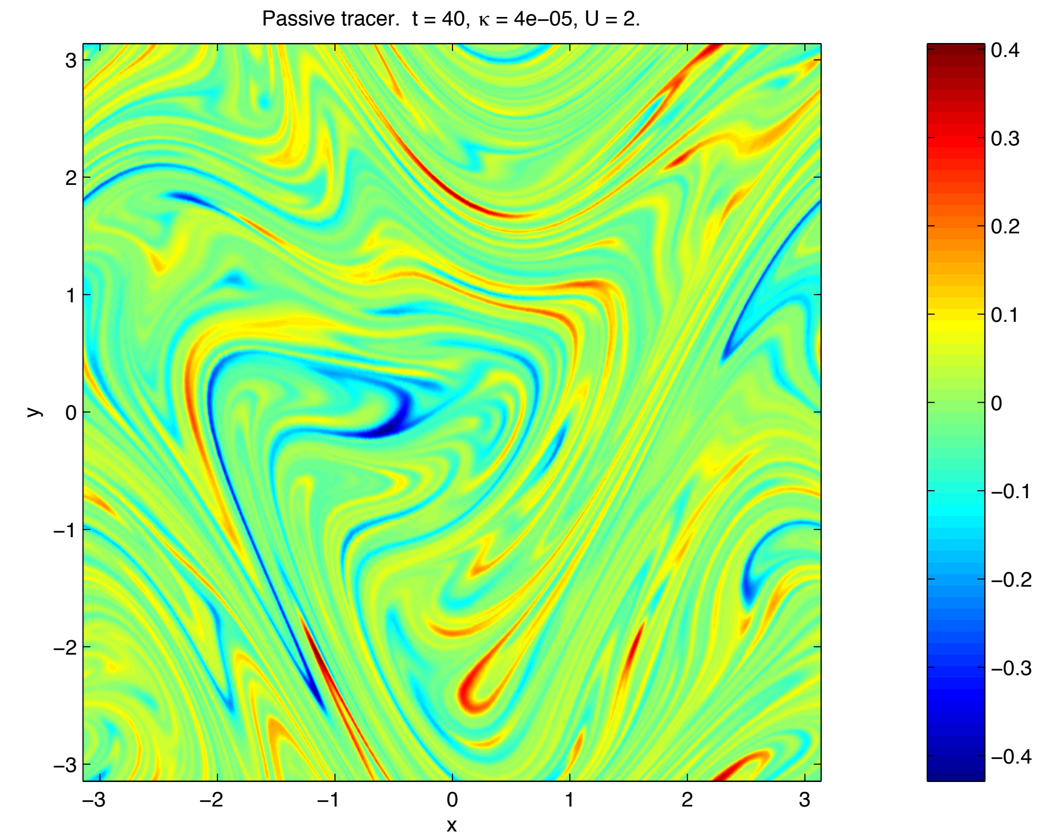


FIG. 2. Schematic of the advection step on a lattice, showing rearrangement by the composition of a shift operation in the *x*-direction followed by a shift operation in the *y*-direction.

The strange eigenmode: $c(\boldsymbol{x},t) = \mathrm{e}^{-\nu t}\hat{c}(\boldsymbol{x},t)$



t=20



t=40

- ☛ After normalization, the patterns are statistically identical.

$$\left(\int |c|^n \,\mathrm{d}A\right)^{1/n} = \mathrm{e}^{-\nu t}\underbrace{\left(\int |\hat{c}|^n \,\mathrm{d}A\right)^{1/n}}_{\text{stationary}}$$

- ☛ Decay is not intermittent i.e., pdf's are identical after re-scaling by the exponential decay.

Why is the strange eigenmode "strange"?

☛ It is not really an eigenmode: $c(\boldsymbol{x},t) = \mathrm{e}^{\nu t}\hat{c}(\boldsymbol{x},t)$

and $\hat{c}_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\hat{c} = \nu\hat{c} + \kappa\nabla^2\hat{c}$

☛ If the velocity is temporally periodic, then Floquet theory justifies the designation "eigenmode":

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}(\boldsymbol{x},t+p) \qquad \Rightarrow \qquad \hat{c}(\boldsymbol{x},t) = \hat{c}(\boldsymbol{x},t+p)$$

☛ It's still "strange" because the spatial structure is complicated...

Experiments

letters to nature (1999)

Persistent patterns in transient chaotic fluid mixing

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$$J \times B$$

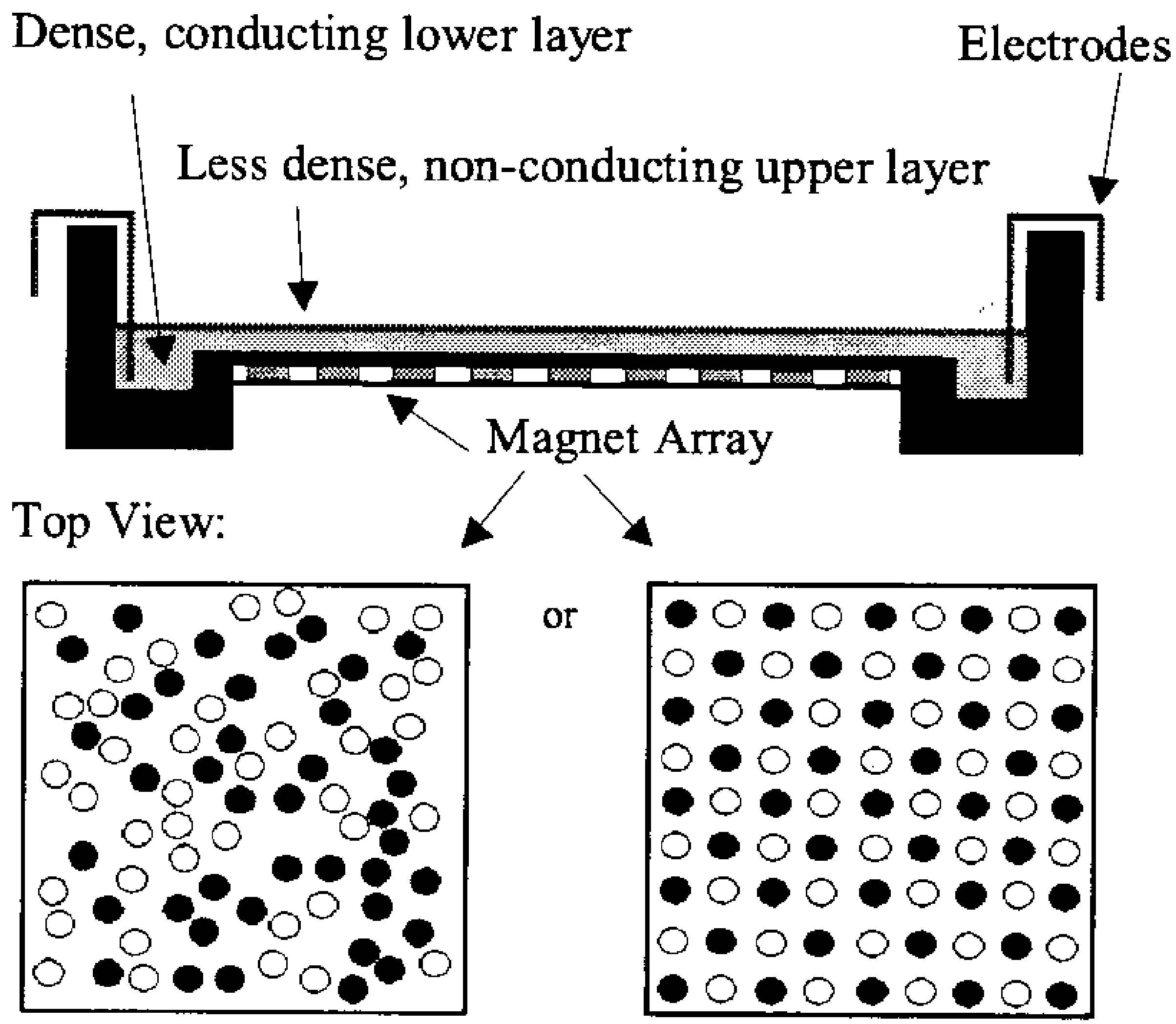

FIG. 1. Diagram of the two-dimensional, magnetically forced, time-periodic flow.

PHYSICS OF FLUIDS VOLUME 15, NUMBER 9 SEPTEMBER 2003

Mixing rates and symmetry breaking in two-dimensional chaotic flow

Greg A. Voth,[a] T. C. Saint, Greg Dobler, and J. P. Gollub[b]

The flow

Figure 1 Snapshot of the concentration field for transient magnetically forced chaotic mixing. Fluid labelled with fluorescein was initially confined to the right half of the 20 × 24 cm cell; the imaged region is 9.4 × 9.4 cm. The light intensity under ultraviolet illumination is shown after 20 periods of forcing by a time periodic electric current at 100 mHz, in the presence of a spatially periodic magnet array. The repetitive stretching and folding that is characteristic of chaotic advection is evident.

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t + p)$$

and $p \geq 5\,\mathrm{s}$

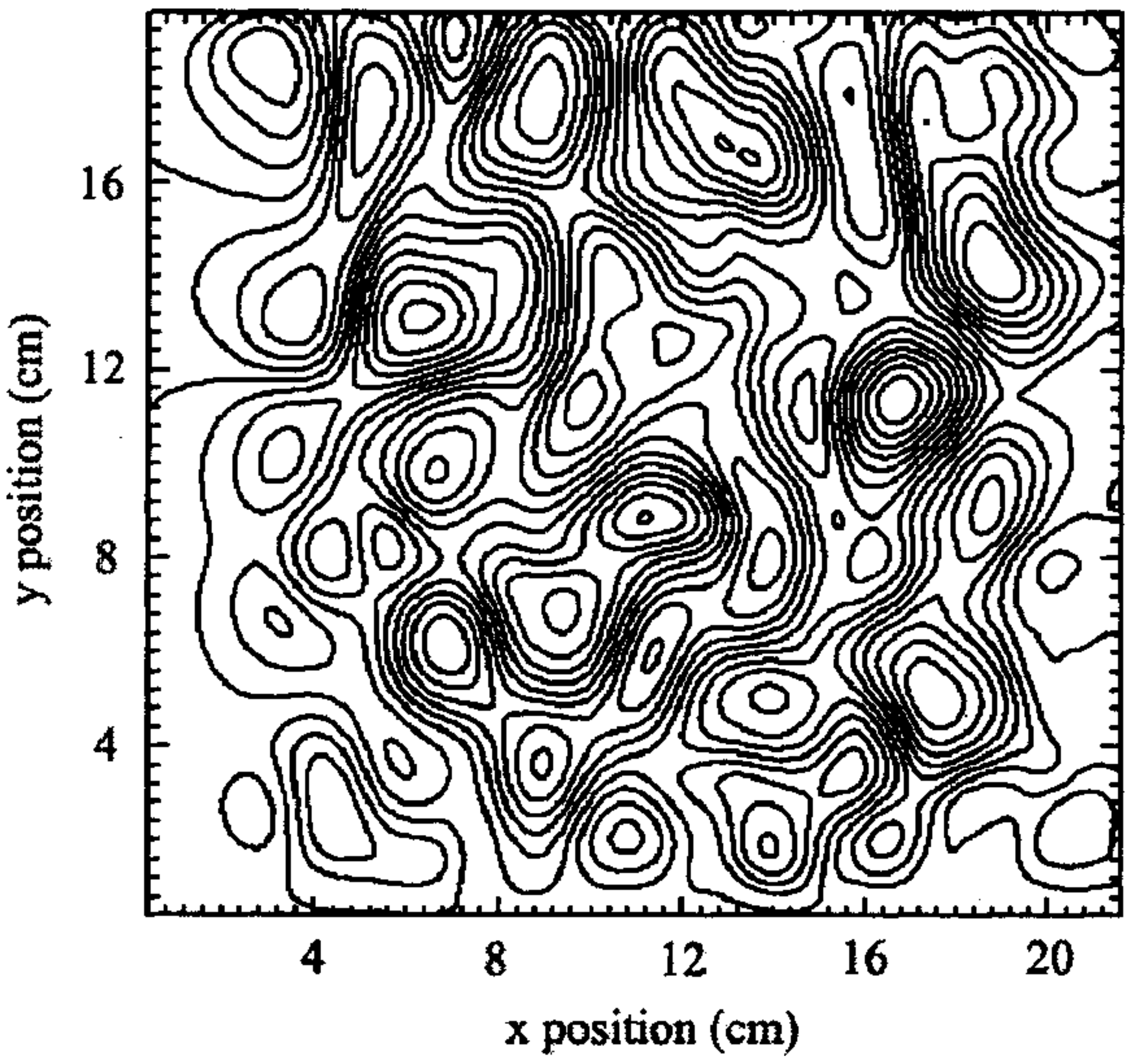

FIG. 2. Instantaneous stream function of the flow driven by the random magnet configuration at Re=100 and p=5. The flow is instantaneously a random vortex array whose form is distorted with time, but that repeats periodically.

$$\kappa_{\text{Sodium fluorescein}} = 5 \times 10^{-10}\mathrm{m}^2\ \mathrm{s}^{-1}$$

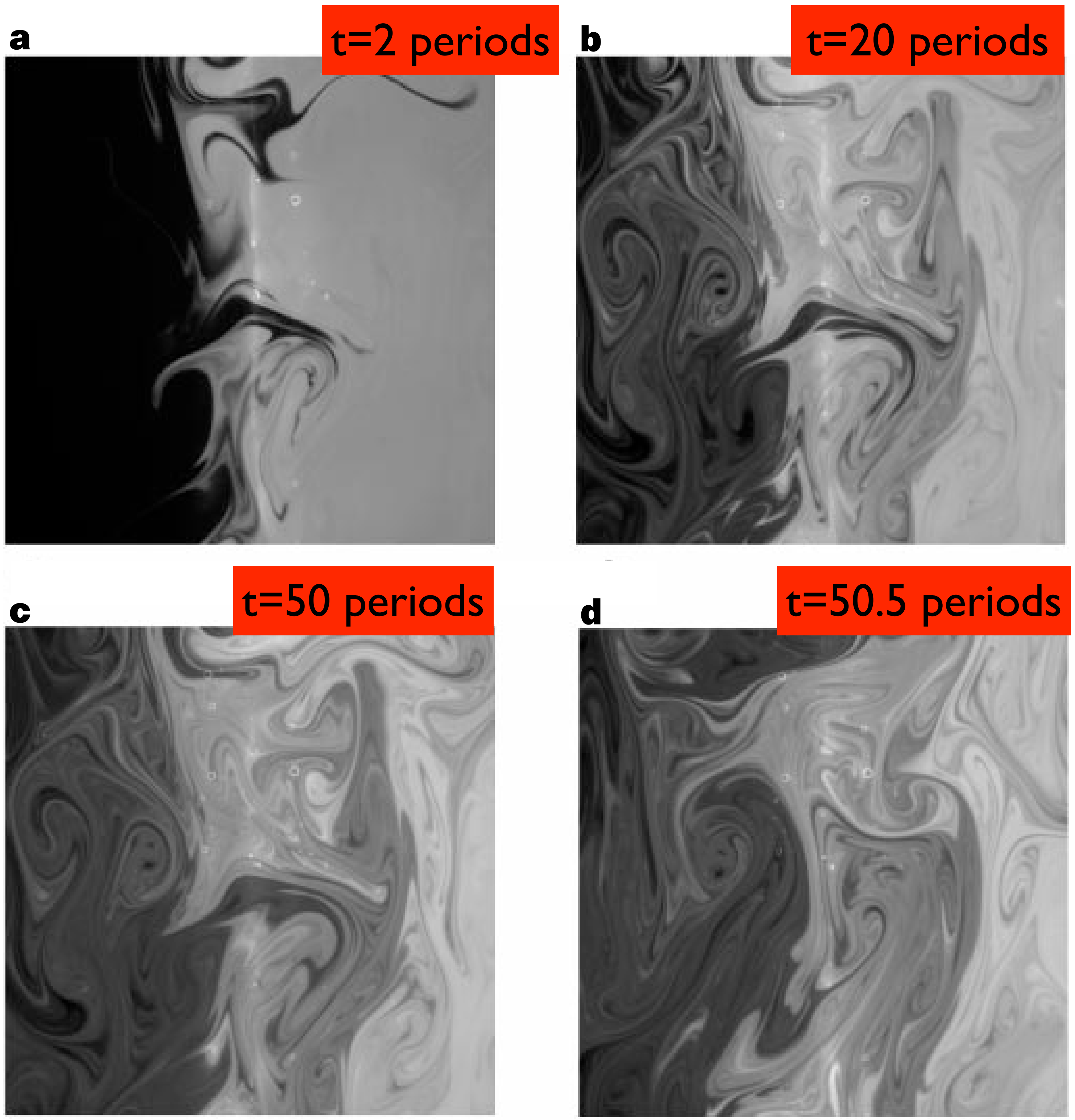

Figure 2 Attainment of a persistent mixing pattern using a disordered array of forcing magnets. The impurity field develops a complex structure after about 10 periods that repeats periodically, except for a gradual loss of contrast. **a–c**, $t = 2$, 20 and 50 periods, respectively. **d**, Sampling out of phase at 50.5 periods shows that the tracer distribution changes substantially within each cycle. The forcing frequency is 70 mHz.

☛ The velocity is periodic:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t + p)$$

so according to Floquet:

$$c(\boldsymbol{x}, t) = \mathrm{e}^{-\nu t} \hat{c}(\boldsymbol{x}, t)$$

$$\hat{c}(\boldsymbol{x}, t) = \hat{c}(\boldsymbol{x}, t + p)$$

Predicting the decay rate: the literature is a mess...

- ☛ Recent (1994-2004) analysis is by statistical physicists, who have developed "Local Lagrangian Stretching" theories.

- ☛ Traditional fluid mechanics would estimate a decay rate using eddy diffusion i.e., "Global Transport".

- ☛ It seems that truth and reconciliation has been achieved by HV:

PHYSICS OF FLUIDS **17**, 097103 (2005)

What controls the decay of passive scalars in smooth flows?

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It is easy to explain the GT prediction

☛ Use appropriate eddy diffusivity, and solve the eigenproblem....

$$\langle c \rangle_t = \kappa_e \nabla^2 \langle c \rangle$$

e.g., in a doubly periodic domain the slowest mode is

$$\langle c \rangle = \cos(k_D x) \mathrm{e}^{-\kappa_e k_D^2 t}, \quad k_D = \frac{2\pi}{L}$$

i.e. $\nu = \kappa_e k_D^2$

☛ HV show that this is the correct answer, provided there is scale separation: $k_f \gg k_D$

☛ Without scale separation, $k_f = k_D$, LLS is almost correct.

Local Lagrangian Stretching: Kraichnan 1974

☛ The model: $$c_t + \left[\gamma + s'(t)\right] x c_x - \left[\gamma + s'(t)\right] y c_y = \kappa \nabla^2 c$$

where $\langle s'(t_1) s'(t_2) \rangle = \gamma \delta(t_1 - t_2)$ in $d = 2$

☛ The solution: $$c = \frac{\alpha(t)\beta(t)}{2\pi} \exp\left[-\tfrac{1}{2}\alpha^2(t)x^2 - \tfrac{1}{2}\beta^2(t)y^2\right]$$

☛ The most basic measure of decay is: $$\left\langle \iint c^2(x,y,t)\, \mathrm{d}x\mathrm{d}y \right\rangle \propto \mathrm{e}^{-\gamma t/2}$$

☛ The LLS prediction is that for K^2 velocities: $$\boxed{\nu = \frac{\gamma}{2}}$$

γ is the Lyapunov exponent of the flow

And the statistics are intermittent...

☛Different moments decay at different rates:

$$\left\langle \iint c^{n+1}(x,y,t)\,\mathrm{d}x\mathrm{d}y \right\rangle \sim \begin{cases} \exp\left[\left(\frac{1}{2}n^2 - n\right)t'\right], & \text{if } n<1, \\ \frac{1}{2}\mathrm{e}^{-t'/2}, & \text{if } n=1, \\ (n-1)^{-1}(2\pi t')^{-1/2}\exp\left(-\frac{1}{2}t'\right), & \text{if } n>1. \end{cases}$$

above: $t' \equiv \gamma t$

☛A few "long-surviving" Gaussians dominate the averages. There are some exponentially rare realizations, for which there is no stretching, and only diffusive decay.

☛N.B. Intermittency is in contradiction to the decay via a strange eigenmode.

The LLS estimate with finite correlation time

The role of chaotic orbits in the determination of power spectra of passive scalars

Thomas M. Antonsen, Jr., Zhencan Fan, Edward Ott, and E. Garcia-Lopez
University of Maryland, College Park, Maryland 20742

Physics of Fluids vol 8, 3096 (1996)

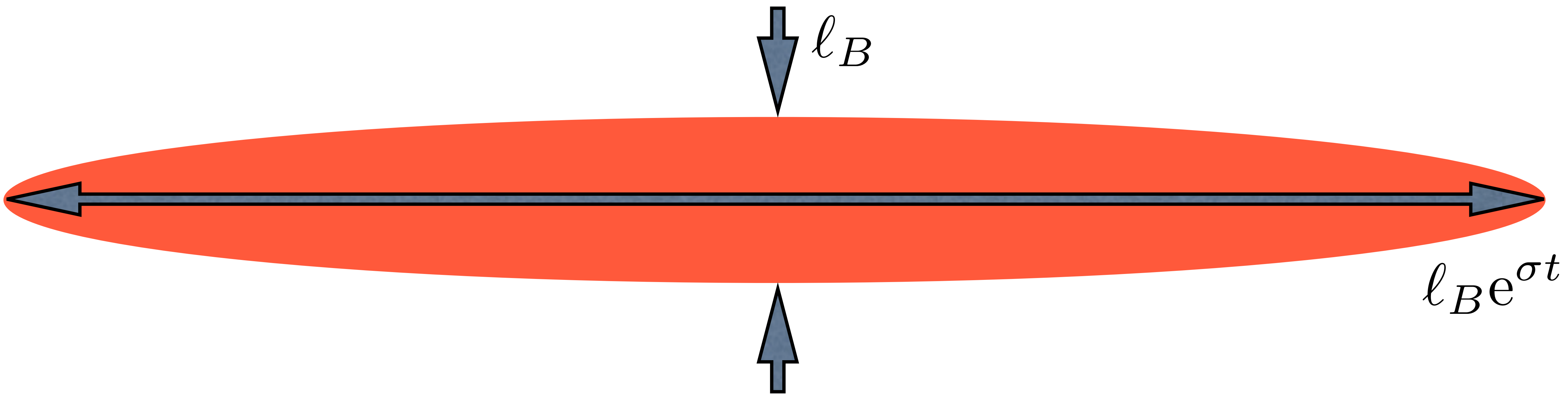
PHYSICAL REVIEW E VOLUME 60, NUMBER 4 OCTOBER 1999

Universal long-time properties of Lagrangian statistics in the Batchelor regime and their application to the passive scalar problem

E. Balkovsky and A. Fouxon
Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

☛ How is the K^2 result $\boxed{\nu = \frac{\gamma}{2}}$ affected by non-zero correlation time?

Recall the hot spot



☛ It follows that: $\displaystyle\int c^2(\boldsymbol{x},t)\,\mathrm{d}A \propto \frac{1}{\text{major axis}} = \frac{\ell_B}{\ell(t)}$

and $\therefore$ $\displaystyle\left\langle \int c^2(\boldsymbol{x},t)\,\mathrm{d}A \right\rangle \propto \langle \ell^{-1} \rangle = \mathrm{e}^{F(-1)t}$ as $t \to \infty$

☛ The major axis stretches just as an infinitesimal line element...

The LLS prediction is: $\nu = -F(-1) = -\inf_{\forall h} (h + G(h))$

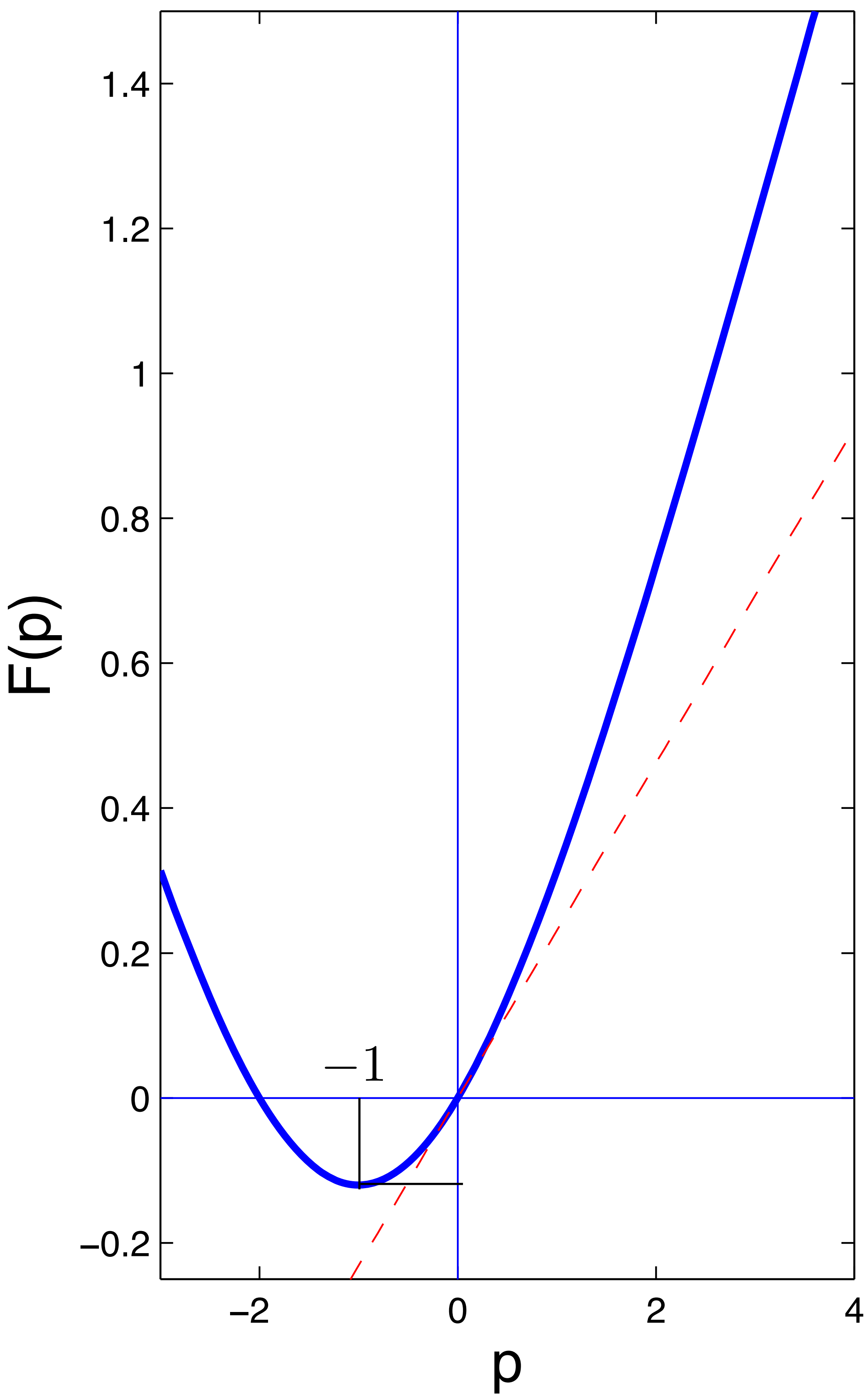
☛ Recall our definition of F(p):

$$\langle \ell^p \rangle \propto \mathrm{e}^{F(p)t} \text{ as } t \to \infty$$

☛ In the K^2 case:

$$F(p) = \tfrac{1}{2}\gamma p(p+2)$$

and $F(-1) = -\tfrac{1}{2}\gamma$ ✓



Antonsen, Fan, Ott & Garcia-lopez (1994) take a different route to the LLS prediction

$$\nu = -F(-1) = -\inf_{\forall h} \left(h + G(h)\right)$$

In wavenumber space: $c(\boldsymbol{x},t) = a(\boldsymbol{k}_0,t)\exp\left(\mathrm{i}\boldsymbol{k}(t)\cdot\boldsymbol{x}\right)$

☛ The solution of $\dfrac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}t} = -\boldsymbol{W}^{\mathsf{T}}\boldsymbol{k}$ has the form:

$$\boldsymbol{k}(t) = \left(\boldsymbol{k}_0\cdot\boldsymbol{e}_0^{+}\right)\boldsymbol{e}^{+}(t)\,\mathrm{e}^{q(t)} + \left(\boldsymbol{k}_0\cdot\boldsymbol{e}_0^{-}\right)\boldsymbol{e}^{-}(t)\,\mathrm{e}^{-q(t)}$$

so that $|\boldsymbol{k}(t)|^2 = |\boldsymbol{k}_0|^2\left(\mathrm{e}^{2q(t)}\cos^2\theta + \mathrm{e}^{-2q(t)}\sin^2\theta\right)$

☛ The solution of the transport equation $\dfrac{\mathrm{d}a}{\mathrm{d}t} = -\kappa|\boldsymbol{k}|^2 a$ is:

$$a(\boldsymbol{k}_0,t) = a(\boldsymbol{k}_0,0)\exp\left[-\kappa|\boldsymbol{k}_0|^2\int_0^t \mathrm{e}^{2q(t')}\cos^2\theta + \mathrm{e}^{-2q(t')}\sin^2\theta\mathrm{d}t'\right]$$

Now make some approximations

☛ At large times $a(\boldsymbol{k}_0, t) \approx a(\boldsymbol{k}_0, 0) \exp\left[-\kappa|\boldsymbol{k}|^2\tau\cos^2\theta\right]$ where

$$\tau(t) \equiv \int_0^t \mathrm{e}^{2(q(t')-2q(t))}\mathrm{d}t'$$

☛ Introduce the FTLE, and the large deviation function

$$q(t) = th(t) \qquad \text{and} \qquad P(h,t) \sim \mathrm{e}^{-tG(h)}$$

☛ Take an ensemble average:

$$|a(\boldsymbol{k}_0, t)|^2 \approx |a(\boldsymbol{k}_0, 0)|^2 \int_0^\infty \mathrm{d}h P(h,t) \int_0^\infty \mathrm{d}\tau M(\tau) \oint \frac{\mathrm{d}\theta}{2\pi} \exp\left[-2\kappa\cos^2\theta|\boldsymbol{k}_0|^2\mathrm{e}^{2ht}\tau\right]$$

☛ Evaluate the theta integral with Laplace's method:

$$\langle|a(\boldsymbol{k}_0, t)|^2\rangle \sim |a(\boldsymbol{k}_0, 0)|^2 \int_0^\infty \mathrm{d}h\, \mathrm{e}^{-t[h+G(h)]} \int_0^\infty \mathrm{d}\tau M(\tau) \sqrt{\frac{1}{2\pi\kappa|\boldsymbol{k}_0|^2\tau}}$$

$$\propto \exp\left[-\min_{\forall h}\left(h + G(h)\right)t\right]$$

Mixing rates and symmetry breaking in two-dimensional chaotic flow

Greg A. Voth,[a] T. C. Saint, Greg Dobler, and J. P. Gollub[b]

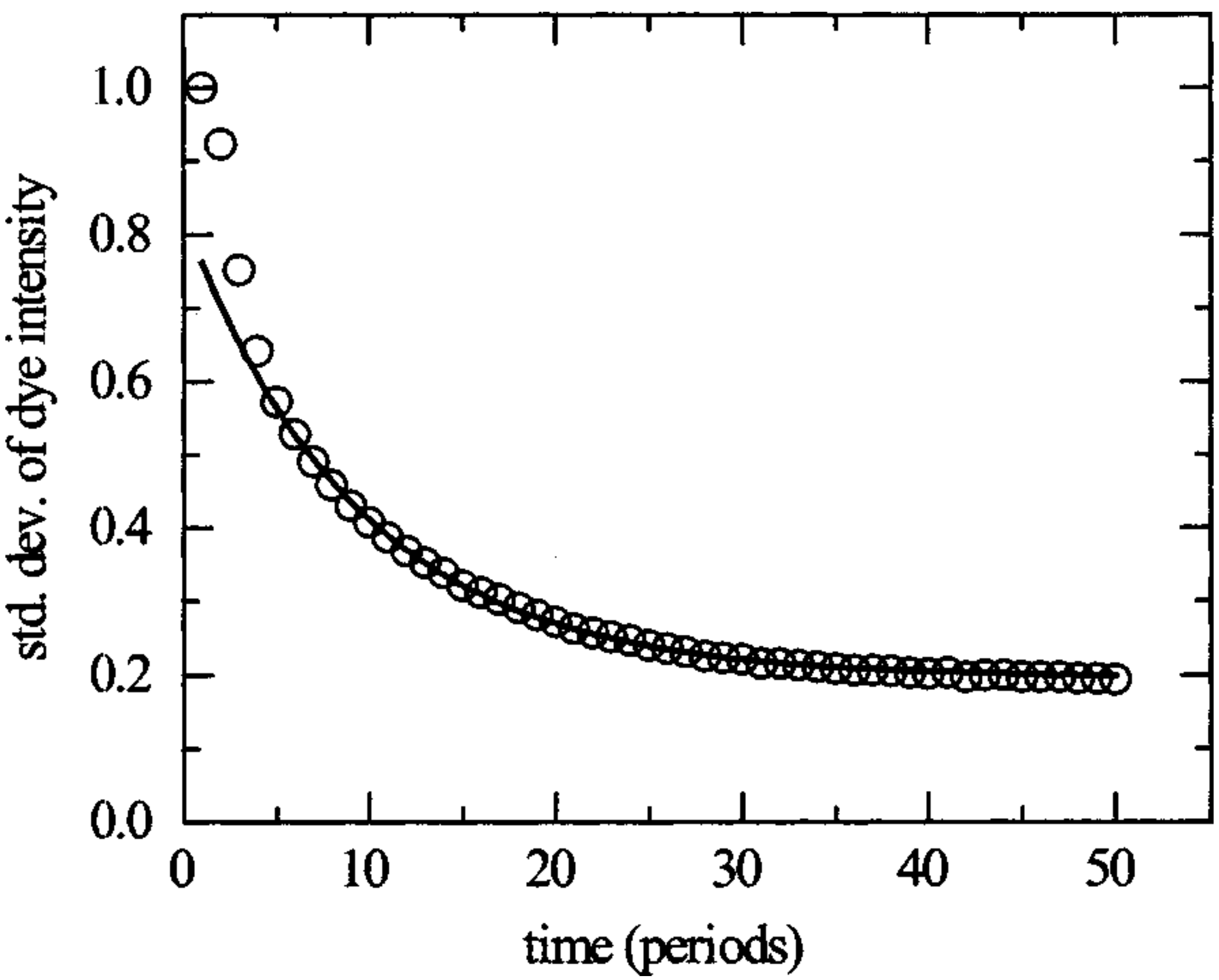


FIG. 3. Standard deviation of the dye intensity as a function of time for $p=5$, $Re=100$. The solid line shows a fit of Eq. (3) to the time range from 6 to 35 periods.

Eq. (3) $\langle C^2 \rangle^{1/2} = C_0 \exp(-Rt) + C_1$

☛ The experimental decay rate is smaller by a factor of ten than the LLS prediction:

$$\nu = F(-1) = \inf_{\forall h} \left(h + G(h)\right)$$

A global mechanism?

PHYSICS OF FLUIDS VOLUME 16, NUMBER 12 DECEMBER 2004

Scalar decay in two-dimensional chaotic advection and Batchelor-regime turbulence

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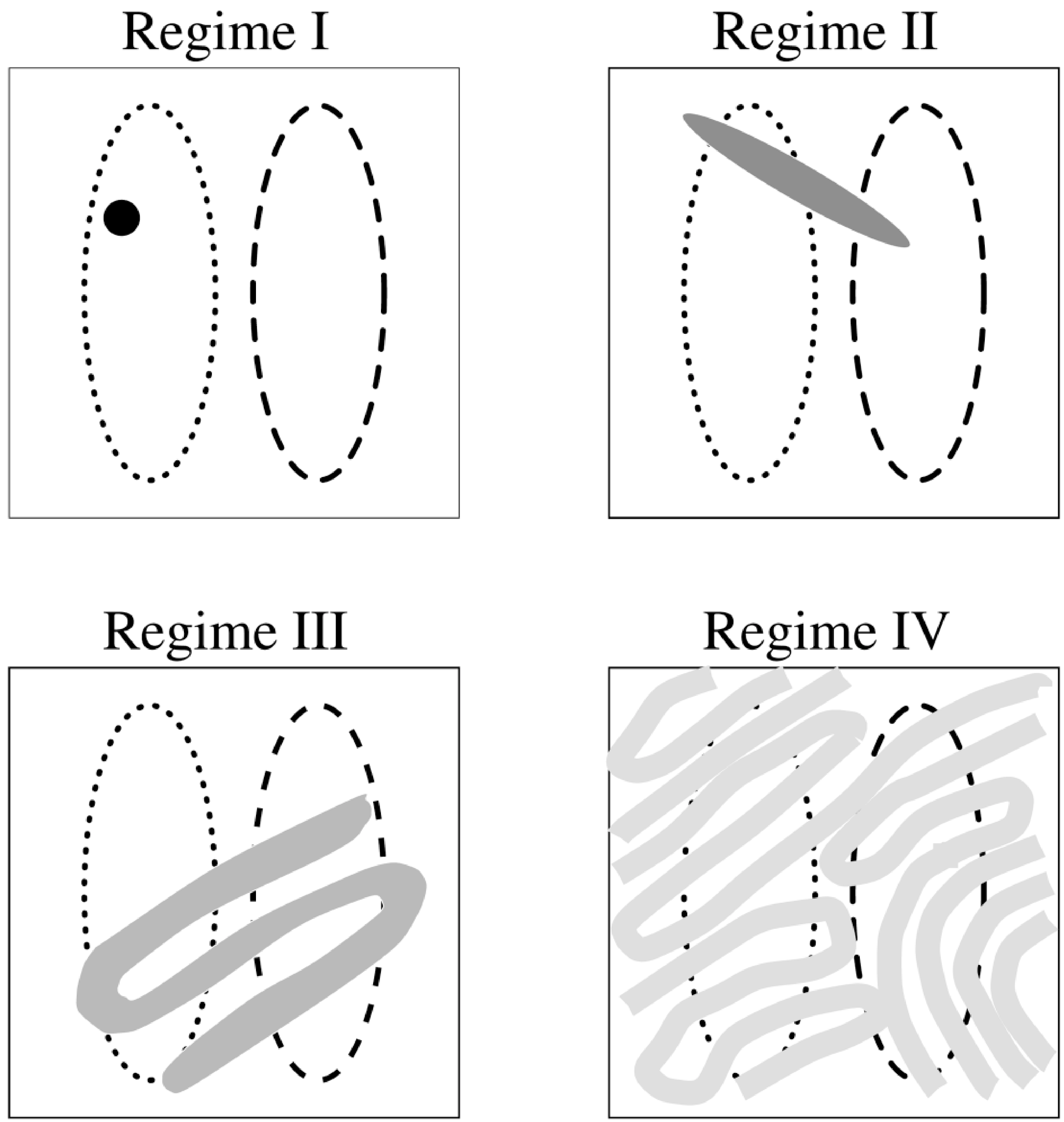


HV (2005) solve a K^2 eigenproblem

☛ Recall the velocity correlation tensor:

$$\langle u_i(\boldsymbol{x}_1, t_1) u_j(\boldsymbol{x}_2, t_2)\rangle = 2\mathcal{U}_{ij}(\boldsymbol{x}_2 - \boldsymbol{x}_1)\delta_\tau(t_2 - t_1)$$

☛ This implies:

$$\langle [u_i(\boldsymbol{x}_2, t_2) - u_i(\boldsymbol{x}_1, t_1)] [u_j(\boldsymbol{x}_2, t_2) - u_j(\boldsymbol{x}_1, t_1)]\rangle = 4\mathcal{S}_{ij}(\boldsymbol{x})\delta_\tau(t_2 - t_1)$$

where the velocity "structure function" is $\mathcal{S}_{ij}(\boldsymbol{x}) \equiv \mathcal{U}_{ij}(0) - \mathcal{U}_{ij}(\boldsymbol{x})$

above $\boldsymbol{x} \equiv \boldsymbol{x}_2 - \boldsymbol{x}_1$

☛ With a K^2 calculation, we see that: $\mathcal{C}(\boldsymbol{x}, t) \equiv \langle c(\boldsymbol{x}_1, t) c(\boldsymbol{x}_2, t)\rangle$

satisfies: $\mathcal{C}_t = 2\kappa\nabla^2\mathcal{C} + 2\mathcal{S}_{ij}\mathcal{C}_{,ij}$

Isotropic solutions of $\mathcal{C}_t = 2\kappa\nabla^2\mathcal{C} + 2\mathcal{S}_{ij}\mathcal{C}_{,ij}$

- ☛ For an isotropic flow: $\mathcal{S}_{ij}(\boldsymbol{r}) = \frac{1}{d-1}\left[r^{2-d}\left(r^{d-1}S\right)_r\delta_{ij} - r^{-1}S_r x_i x_j\right]$

- ☛ A model structure function: $S(r) = \kappa_{\mathrm{e}}'\left[1 - \frac{1}{(1+\bar{r}^2)^p}\right]$ with $\bar{r} \equiv \frac{|\boldsymbol{x}|}{a}$

- ☛ Assume isotropy: $\mathcal{C}_t = 2r^{1-d}\left[r^{d-1}K\mathcal{C}_r\right]_r$ with $K(r) \equiv \kappa + S(r)$

- ☛ Eigensolutions, $\mathcal{C}(r,t) = \mathrm{e}^{-2\nu t}C(r)$, with a model BC $C_r(r_*) = 0$

- ☛ This is a standard S-L problem: $r^{-1}\left[rKC_r\right]_r + \nu C = 0$

The LLS prediction uses $S(r) = \beta r^2 + O(r^2)$

☛ With kappa=0, the eigenproblem is: $\left(r^3 C_r\right)_r + \left(\frac{\nu}{\beta}\right) rC = 0$

☛ There is a continuous spectrum:

$$C(r) = r^{-1 \pm \mathrm{i}k}, \qquad \text{where} \qquad \frac{\nu}{\beta} = 1 + k^2$$

☛ The LLS prediction is that the decay rate is: $\min_{\forall k} \nu = \beta\,.$

The Rayleigh-Ritz objection to LLS

- Estimate the eigenvalue with: $\nu = \min_{\forall F} \frac{\int_0^{r_*} K F_r^2 r \,\mathrm{d}r}{\int_0^{r_*} F^2 r \,\mathrm{d}r}\,, \quad \text{where } F'(r_*) = 0,$

- To obtain a test function, solve: $\kappa_e \left(r F_r\right)_r + \nu_{RR} r F = 0$

$$F(r) = J_0(z_1 r / r_*)\,, \qquad \nu_{RR} = z_1^2 \kappa_e / r_*^2\,,$$ where $z_1 = 3.83171$

- The decay rate is constrained by: $\nu \le \frac{\int_0^{r_*} K F_r^2 r \,\mathrm{d}r}{\int_0^{r_*} F^2 r \,\mathrm{d}r} \le \frac{\kappa_e \int_0^{r_*} F_r^2 r \,\mathrm{d}r}{\int_0^{r_*} F^2 r \,\mathrm{d}r} = \frac{z_1^2 \kappa_e}{r_*^2}$

- In a large domain, The LLS result, $\nu = \beta$, violates this upper bound...

- Is LLS ever correct?

Small domain $\frac{r_*}{a} \ll 1$ and $S(r) \approx \beta r^2$

☛ To analyze the eigenproblem we use new variables:

$$X \equiv \ln\left(\sqrt{\frac{\beta}{\kappa}} r\right) \qquad K(r) = \kappa\left(1 + \mathrm{e}^{2X}\right)$$

☛ We have: $\left(1 + \mathrm{e}^{-2X}\right) C_{XX} + 2C_X + (1 + k^2)C = 0$

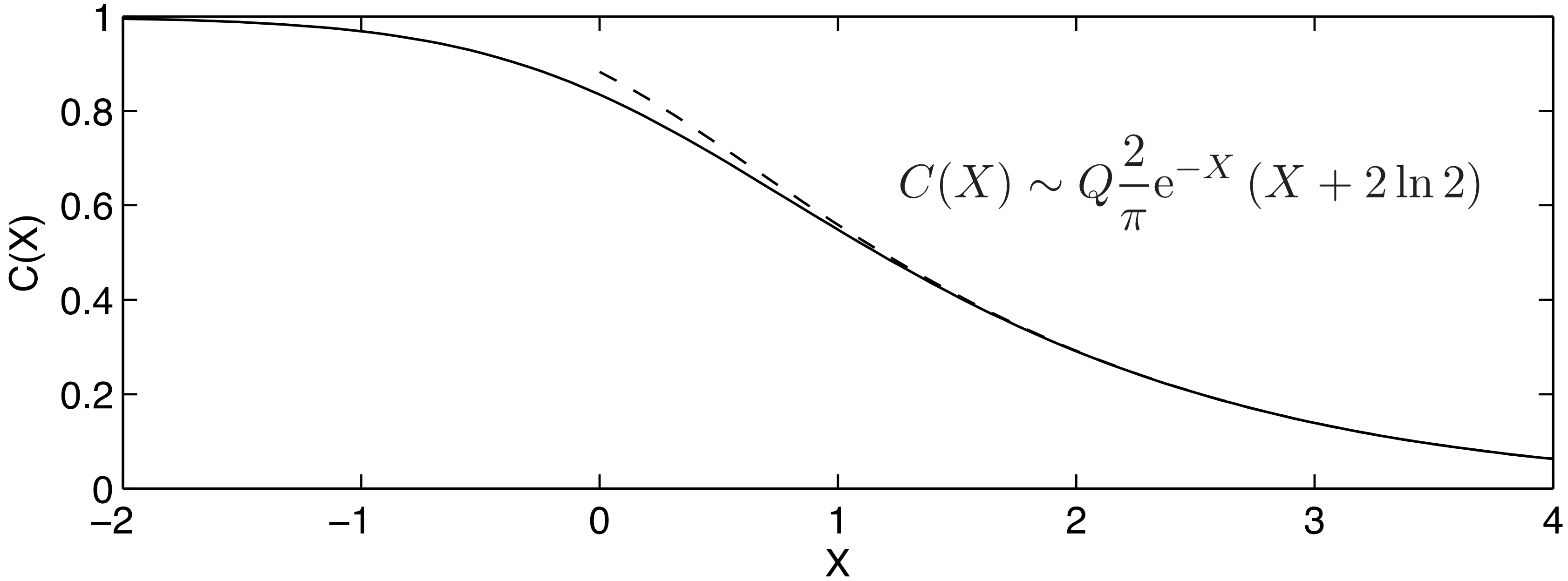
$$C(-\infty) = 1 \qquad C_X(X_*) = 0$$

☛ In the outer region, 1<<X < X_* it is easy:

$$C = \mathrm{e}^{-X}\left[k\cos\left(k\left(X - X_*\right)\right) + \sin\left(k\left(X - X_*\right)\right)\right]$$

Inner region: $X \leq 1$

- We find an analytic solution of: $\left(1+\mathrm{e}^{-2X}\right) C_{XX} + 2C_X + C = 0$



- Asymptotically matching the in the region $1 \ll X \ll k^{-1}$

  the eigenvalue is $k \approx \dfrac{\pi}{X_* - \ln 4} = \dfrac{\pi}{\ln\left(\sqrt{\frac{\beta}{\kappa}}\frac{r_*}{4}\right)}$

- So LLS is correct in a small domain, and $\nu = \beta + O\left(\dfrac{1}{\ln^2 \kappa}\right)$

Numerical solution of the eigenproblem

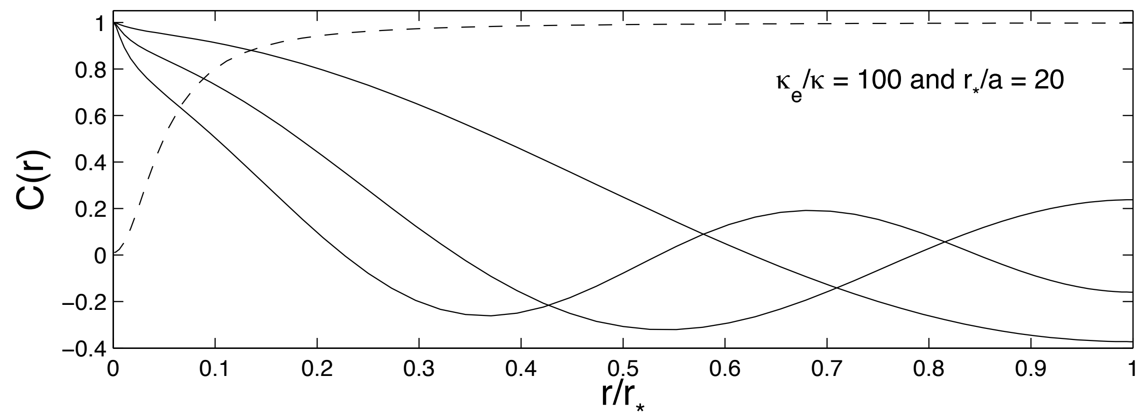


Figure 1: The solid curves are the first three eigenfunctions obtained by solving (47), with the boundary condition (48), numerically. We use the model in (44) with $p = 1$; the dashed curve is $K(r)/\kappa_{\mathrm{e}}$.

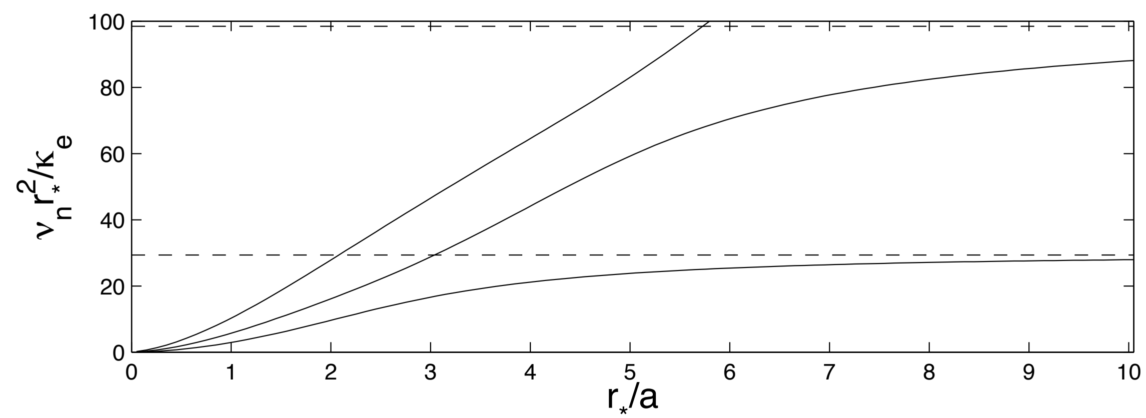


Figure 2: The first three eigenvalues of (47) and (48) as functions of r_*/a. As r_*/a increases the eignvalue 2ν approaches the asymptotic value $2z_n^2\kappa_{\mathrm{e}}/r_*^2$, where z_n is the n'th zero of J_1; the first two asymptotic values (corresponding to $z_1 = 3.83171$ and $z_2 = 7.01559$) are indicated by the dashed lines. This shows that in a large domain the decay of a passive scalar is globally controlled by κ_{e}.

THE END