Kinematic dynamo problem in a linear velocity field

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A magnetic field is shown to be asymptotically $(t \to \infty)$ decaying in a flow of finite conductivity with $v = Cr$, where $C = C_\xi(t)$ is a random matrix. The decay is exponential, and its rate does not depend on the conductivity. However, the magnetic energy increases exponentially owing to growth of the domain occupied by the field. The spatial distribution of the magnetic field is a set of thin ropes and (or) layers.

1. Introduction

A flow with uniform velocity cannot change the magnetic field in magnetohydrodynamics. So a linear velocity $v_i = c_{ik} r_k$ is the simplest velocity field in which a transformation of kinetic energy to magnetic energy seems possible.

The linear velocity field is usually understood as a local approximation to a smooth velocity field of a general type. With this meaning it was originally used in hydrodynamics to describe small-scale turbulence (Townsend 1951; Batchelor 1959; Monin & Yaglom 1975). This approximation seems to be still more attractive with respect to the kinematic dynamo problem in so far as the magnetic field is not connected in any way with the velocity, in contrast with the vorticity $\omega = \nabla \wedge v$, which, in addition, proves to be uniform. The dynamo problem in reflectionally invariant homogeneous and isotropic turbulence, put forward by Batchelor, was studied in this approximation by Saffman (1963), Moffatt & Saffman (1964), Knobloch (1977, 1978) and others. Steady magnetic configurations in linear flow were examined by Moffatt (1963) and Clarke (1964). The linear velocity field is well known in cosmology (the Hubble law). The magnetic field in the anisotropic Hubble flow was studied for instance by Zel'dovich (1965).

It was shown by Moffatt & Saffman (1964) that the kinematic dynamo equations have no exponentially growing solution in the linear velocity field for finite (even very small) conductivity. However, in spite of the magnetic-field decay, simultaneous exponential growth of the total magnetic energy is possible owing to still more rapid growth of a domain occupied by the magnetic field. The analogous effect was noticed earlier by Pearson (1959) in hydrodynamics.

The results were obtained for the case of the time-constant (determinate or random) diagonal matrix $c_{ik} (\equiv C)$.

In the present paper we will consider a more general case of the time-dependent random non-diagonal matrix C. The results are referred to the magnetic-field realization and not to its average properties. In essence they are reduced to the case

of the constant diagonal matrix. So we shall devote a section (§3) to clear this simplest case.

The technique we use to treat the general case (§4) is to reduce the kinematic dynamo problem for the differential induction equation to the 'algebraical' problem of multiplication of large number of matrices, or in other words to the calculation of multiplicative integrals.

2. Formulation of the problem

The problem is to study the evolution of an initially given magnetic field in the given flow of a conductive incompressible fluid

$$\frac{\partial H}{\partial t} + (v \cdot \nabla) H = (H \cdot \nabla) v + \nu_{\rm m} \nabla^2 H, \left.\vphantom{\frac{\partial H}{\partial t}}\right\}$$
$$\nabla \cdot H = 0, \quad H(0, r) = H_0(r), \left.\vphantom{}\right\}$$
(1)

where $\nu_{\rm m}$ is a constant magnetic diffusivity, $v(t, r)$ is the velocity field. The initial magnetic field $H_0(r)$ is usually considered to vanish at infinity, to be finite and to possess finite total energy.† Let us discuss these conditions in a more detailed way. A finite total energy does not imply yet that the magnetic field vanishes when $r \to \infty$. For example, a system of an infinite number of magnetic dipoles with vanishing magnetic moments can make $\int H^2 \, d^3 r$ limited, but a magnetic field calculated over some sequence of points (on the centres of the dipoles) going to infinity will be unlimited. Moreover, the finite limit of the total magnetic energy does not imply the absence of local singularities of the field. However, when it is assumed that the initial field is created by the current distribution in a finite volume the magnetic field will evidently vanish as $O(r^{-3})$.

The solution of the problem (1) basically depends on the form of the magnetic-field decrease at infinity. For example, when the field vanishes more slowly compared with $O(r^{-3})$, say as $O(x^{-1} \ln x)$, along one axis (which guarantees the finiteness of the total magnetic energy), exponential growth of the magnetic field can be shown to be possible.

It should be noted that the finite total energy and even the condition for vanishing of the finite field H_0 as $O(r^{-3})$ when $r \to \infty$ do not guarantee limited Fourier modes of the field (logarithmic singularities are possible).

We shall consider the velocity field

$$v = Cr, \quad {\rm tr} \, C = 0,$$
(2)

where C is a random (or determinate) matrix. The second condition is due to $\nabla \cdot v = 0$.

Let us focus attention on one rather delicate point. The matrix C for a random linear velocity field depends on time and the random variable ξ which determines a realization of v. For the given ξ we have a definite matrix function $C_\xi(t)$ describing the time evolution of the velocity field. It is natural to demand that the initial field distribution must not depend on ξ, i.e. on the future history of the velocity field.

3. Qualitative prelude

The linear velocity field implies that an infinitesimal vector δr_i connecting two close fluid elements obeys the equation

$$\frac{{\rm d}\delta r_i}{{\rm d}t} = c_{ik} \, \delta r_k \quad (i, k = 1, 2, 3).$$

† If the field does not vanish at infinity, for example, $H_x = H_0 = {\rm const}$, $H_y = H_z = 0$, then there is the exponentially growing solution of the problem (1) in the velocity field (2): $H_x = H_0 \exp(c_1 t)$.

When the matrix c_{ik} is constant and diagonal ($c_1, c_2, c_3 \neq 0$), this vector will evidently grow exponentially. By virtue of incompressibility, $c_1 + c_2 + c_3 = 0$, so that a fluid particle of spherical form stretches along one, say the first, axis ($c_1 > 0$) and compresses along the other axes ($c_2, c_3 < 0$) to form a 'rope', or squeezes along two axes ($c_1, c_2 > 0$) and compresses along the rest to form a 'pancake'.

It is known that a magnetic line embedded in an ideal conductive fluid evolves as the vector δr_i if the field was initially aligned with this vector (a frozen-in condition). One would think that this gives a solution to our problem, at least in the case of high conductivity, which is of practical interest. Say, for example, in the case $c_1 > 0$, $c_2, c_3 < 0$ that the H_1 component grows exponentially, then the other two field components should decay exponentially. However, the situation is completely different: even a very small (but finite) magnetic diffusivity results in the opposite answer.

For the purpose of making estimates it is convenient to use the Fourier space k_0 of the initial magnetic field. One should distinguish the solutions of two types.

(a) $c_1 > 0$, $c_3 < c_2 < 0$ (a rope)

In this case the scales along the k_{02}, k_{03} axes exponentially decrease in time as $k_{02} \exp(|c_2| t)$ and $k_{03} \exp(|c_3| t)$. Hence almost all the field harmonics, except for the harmonic at $k_0 = 0$ and those corresponding to k_0 in an exponentially narrow cone parallel to the k_{01} axis (figure 1 a), abruptly decay in proportion to $\exp(-\nu_m \int_0^t k^2(s)\, ds)$, i.e. as the exponent to exponent. A marked contribution to the magnetic field comes only from the harmonics of the cone $\nu_m \int_0^t k^2\, ds = O(1)$. The cone has an elliptical cross-section with semiaxes proportional to $\exp(-|c_2| t)$ and $\exp(-|c_3| t)$. Since the field is solenoidal ($kH_k = 0$) the directions of the harmonics with wave vectors of the cone are nearly orthogonal to the cone axis, or more exactly, they form an 'orthogonal cone'. The harmonics of the orthogonal cone have projections on the k_{01} axis of order $k_{01}/k_{02} \propto \exp(-|c_2| t)$ so that they grow in time as $\exp(c_1 t) \exp(-|c_2| t)$. The first factor is due to stretching along the k_{01} axis.

The magnetic field in r-space is estimated as the product of an amplitude of the growing harmonic by the volume of the cone proportional to $\exp[-(|c_2| + |c_3|) t]$. Hence

$$H(t, r) \propto \int H_k\, d^3 k_0 \propto \exp(-|c_2| t).$$

Thus the magnetic field having initial harmonics inside the cone stretches into a rope parallel to the first axis and decays asymptotically. The other fields will decay more rapidly.

However, the domain occupied by the magnetic field grows exponentially owing to stretching along the first axis. Therefore the total magnetic energy grows as

$$\int H^2\, d^3 r \propto \exp[(c_1 - 2|c_2| t)] = \exp[(|c_3| - |c_2|) t]$$

because $|c_3| > |c_2|$. Let us emphasize that the magnetic diffusivity is very important here. It stabilizes the scales along axes $2, 3$ and does not prevent the extension along axis 1.

(b) $c_1 > c_2 > 0$, $c_3 < 0$ (a pancake)

In this case the cone is constructed by wave vectors close to the (k_{01}, k_{02})-plane (see figure 1 b). Its opening decreases as $\exp(-|c_3| t)$ insofar as the cone is determined by the condition $\nu_m \int_0^t k^2\, ds \propto \nu_m |c_3|^{-1} k_{03} \exp(|c_3| t) = O(1)$. It is evident that a harmonic
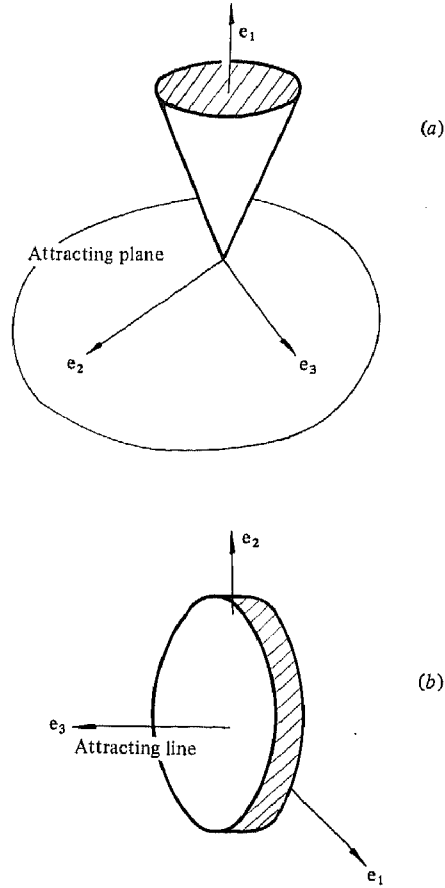
FIGURE 1. Cone of the wave vectors corresponding to the magnetic field harmonics which do not decay owing to magnetic diffusion. (a) $c_1 > 0 > c_2 > c_3$; (b) $c_1 > c_2 > 0 > c_3$.

of maximal growth is directed along the k_{01} axis and its wave vector is close to the k_{03} axis. Its amplitude grows as $\exp(c_1 t)$, so that the magnetic field in r-space vanishes asymptotically as

$$\boldsymbol{H}(t, \boldsymbol{r}) \propto \exp(c_1 t) \exp(-|c_3| t) = \exp(-c_2 t).$$

To estimate the total magnetic energy one should multiply $\boldsymbol{H}^2(t, \boldsymbol{r})$ by the volume occupied by the magnetic field. The volume grows as $\exp[(c_1 + c_2) t]$ because stretching proceeds now in the (k_{01}, k_{02})-plane. Again we have the growing total energy

$$\int \boldsymbol{H}^2 \, \mathrm{d}^3 r \propto \exp[(|c_3| - 2c_2) t] = \exp[(c_1 - c_2) t],$$

because $c_1 > c_2$.

The case $c_2 = 0$ corresponding to the plane motion $v_y = 0$ should be noted. In this case a stabilization of the field with an exponential growth of its total magnetic energy is possible. The result does not contradict the exclusion theorem for the plane motion (v_x, v_z) (Zel'dovich & Ruzmaikin 1980). In contrast with (2) this theorem implies that the velocity field vanishes at infinity. In the present case the H_y component decays

exponentially, independently of the other components owing to the condition $v_y = 0$ (see (1)). However, this decaying field is a non-decaying source for the two-dimensional field H_x, H_z owing to the exponential growth of a domain occupied by H_y.

Up to now we have considered the diagonal matrix $\boldsymbol{C}$. Let us now discuss the non-diagonal-term effects.

The usual direction of thinking is as follows. A general matrix $\boldsymbol{C}$ can be decomposed into antisymmetric and symmetric parts. The former is excluded by transition to the frame of reference rotating with constant angular velocity $\frac{1}{2}e_{ikl}c_{kl}$ (e_{ikl} is a unit pseudotensor). The surviving symmetric part can be put into diagonal form with time-independent eigenvalues. But the principal axes of this matrix will intricately (but periodically) change in time. The arising problem looks to be insurmountably as difficult in the absence of simplifying assumptions.

In reality, the non-diagonal terms of $\boldsymbol{C}$ are not so dangerous. Consider first the simple plane Couette flow, where only $c_{12} \neq 0$. The abovementioned diagonalization of this matrix by transition to the rotating frame of references results in a matrix with two equal eigenvalues of opposite signs. However, owing to the change of the principal axes, the exponential stretchings are periodically followed by exponential squeezings, so that the resulting growth of the magnetic field in the absence of the magnetic diffusivity will be linear. Any finite ν_m produces the exponential decay of the field, depending on the magnetic diffusivity.

The technique developed below ranks equally well with any matrix $\boldsymbol{C}$.

4. Method

The problem is solved as follows. First, the partial solutions

$$H(t, r) = h(t, k_0) \exp{(\mathrm{i}k(t) \cdot r)} \tag{3}$$

of the plane-wave type with the amplitude and wave vector changing in time are studied. Substituting (3) in (1) and comparing terms with the same powers of r, we have

$$\frac{\mathrm{d}k}{\mathrm{d}t} = -\boldsymbol{C}^* k, \tag{4}$$

where $\boldsymbol{C}^*$ is the matrix transpose to $\boldsymbol{C}$ (in general non-diagonal and time-dependent), and

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -\nu_m k^2 h + \boldsymbol{C}h. \tag{5}$$

The field (3) is subjected to the condition

$$k \cdot h = 0$$

which, according to (4) and (5), is conserved in time.

Let $\boldsymbol{T}_t \equiv \boldsymbol{T}(t_0, t)$ be a fundamental matrix of (4) in the interval (t_0, t). It is evident that $\boldsymbol{T}(t_0, t_0) = \boldsymbol{E}$ ($= \delta_{ik}$). The matrix $\boldsymbol{T}_t$ can be represented in the form of the Volterra multiplicative integral (see e.g. Gantmacher 1967)

$$\boldsymbol{T}_t = \prod_{s=0}^{t} (\boldsymbol{E} - \boldsymbol{C}^*(s)\,\mathrm{d}s). \tag{6}$$

Insofar as $\mathrm{tr}\,\boldsymbol{C}^* = \mathrm{tr}\,\boldsymbol{C} = 0$, $\det \boldsymbol{T}_t = 1$.

The solution of (4) and (5) is easily found in terms of the matrix T_t:

$$k(t, k_0) = T_t k_0,$$
$$h(t, k_0) = (T_t^*)^{-1} h_0 \exp\left[-\nu_m \int_0^t k^2(s, k_0) \, ds\right], \Big\} \tag{7}$$

where $h_0 \equiv h(0, k_0)$ is a Fourier spectrum of the initial magnetic field.

Then the problem (1) is solved by the plane-wave expansion

$$H(t, r) = \int h(t, k_0) \exp(ik \cdot r) \, d^3 k_0,$$

or, after substituting (7),

$$H(t, r) = \int \exp\left[i(T_t k_0 \cdot r) - \nu_m \int_0^t (T_s k_0)^2 \, ds\right] (T_t^*)^{-1} h_0 \, d^3 k_0.$$

By using the transformation $k \to T_t k$, the Parseval equality and then the reverse transformation, we obtained the following expression for the magnetic energy:

$$\int H^2 \, d^3 r = \int \exp\left[-2\nu_m \int_0^t (T_s k_0)^2 \, ds\right] [(T_t^*)^{-1} h_0]^2 \, d^3 k_0. \tag{8}$$

The multiplicative integral (6) is easily calculated in the case of the constant matrix C. The general 3×3 matrix C, according to the well-known Jordan idea, can be represented in the three forms

$$\begin{pmatrix} -c_1-c_2 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_1 \end{pmatrix}, \qquad \begin{pmatrix} -2c_1 & 0 & 0 \\ 0 & c_1 & 0 \\ 0 & 1 & c_1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Here we have taken into account that tr $C = 0$. The constants $c_1, c_2, c_3 = -c_1 - c_2$ and the corresponding basis vectors are generally non-orthogonal and complex. The calculation of (6) is reduced to the simple multiplication of matrices $E - C^* t/n$ followed by the transition $n \to \infty$. As a result we obtain for the above three forms of C

$$T_t = \begin{pmatrix} e^{(c_1+c_2)t} & 0 & 0 \\ 0 & e^{-c_2 t} & 0 \\ 0 & 0 & e^{-c_1 t} \end{pmatrix}, \quad \begin{pmatrix} e^{2c_1 t} & 0 & 0 \\ 0 & e^{-c_1 t} & -t e^{-c_1 t} \\ 0 & 0 & e^{-c_1 t} \end{pmatrix}, \quad \begin{pmatrix} 1 & -t & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix}.$$

The calculations of k and H (see (7) and (8)) with these T_t support the conclusions of §3.

5. Random matrix

Our task is to calculate the multiplicative integral (6) in the case of a random matrix. Let us assume that $C_\xi(t)$ is a matrix process satisfying the natural conditions of ergodicity. For the sake of simplicity we consider one simple model. The range of applicability of the results is apparently more broad. In particular, the matrix $C_\xi(t)$ may be of Markov type, with an exponentially decreasing binary correlation function $\langle C(t) C^*(s) \rangle$.

We consider the matrix function $C_\xi(t)$ constant in the time intervals $(0, \tau)$, $(\tau, 2\tau)$, ..., $[(n-1)\tau, n\tau]$, in which it takes the independent matrix values $C_1, C_2, ..., C_n$. Recall that tr $C_i = 0$, $i = 1, 2, ..., n$.

Such a type of process is called 'the innovation process'. It is a good approximation to a stationary process with a finite correlation time (owing to invariance $t \to t + \tau$ and the independence of the matrices C_i).

In each time interval the matrix C is constant, so the integral (6) can be calculated (see §4). Let

$$T_i = \prod_{(i-1)\tau}^{i\tau} (E - C_i^* \, ds).$$

The matrices T_i are evidently independent and $\det T_i = 1$. Evidently also

$$T_t = \prod_{i=1}^{n} T_i \tag{9}$$

when $t = n\tau$.

Consider first the action of T_t on a vector k_0 of unit length. Let $|k_n| \equiv R_n$ denote the length of the vector k_n. It is evident that the unit vectors $\kappa_n \equiv T_n k_{n-1}/|T_n k_{n-1}|$ form a Markov chain on the unit sphere. Since

$$\kappa_n = \frac{T_n k_{n-1}}{|T_n k_{n-1}|} = \frac{T_n \kappa_{n-1}}{|T_n \kappa_{n-1}|} = \dots = \frac{T_n T_{n-1} \dots T_1 k_0}{|T_n \kappa_{n-1}||T_{n-1} \kappa_{n-2}| \dots |T_1 \kappa_0|}.$$

It follows that

$$R_n \equiv |T_n T_{n-1} \dots T_1 k_0| = |T_n \kappa_{n-1}||T_{n-1} \kappa_{n-2}| \dots |T_1 \kappa_0|.$$

According to the ergodic theorem (or the strong law of large numbers) there exists with unit probability a limit (the Liapunov index)

$$\frac{1}{n\tau} \ln_{n\to\infty} R_n = \frac{1}{n\tau} \sum_{l=1}^{n} \ln |T_e \kappa_{l-1}| \equiv \gamma_1. \tag{10}$$

Let us show that γ_1 is positive.[†] Consider the function

$$f(p) = \lim_{n\to\infty} \frac{1}{n\tau} \langle R_n^p \rangle,$$

where $\langle \ \rangle$ denotes a statistical average. It is clear that

$$\gamma_1 = \frac{df}{dp} \quad \text{at} \quad p = 0.$$

The function $f(p)$ is analytical and concave. In fact, according to the Cauchy–Buniakowski inequality

$$\langle R_n^{\frac{1}{2}(p+q)} \rangle \leqslant (\langle R_n^p \rangle \langle R_n^q \rangle)^{\frac{1}{2}}.$$

After taking the logarithm and dividing by $n\tau$, we have immediately

$$f(\tfrac{1}{2}(p+q)) \leqslant \tfrac{1}{2}(f(p) + f(q)).$$

It is evident that $f(0) = 0$. We shall show also that $f(-3) = 0$ (the number 3 is of course connected with the three-dimensionality of the space under consideration). In fact, as it is clear from simple geometrical consideration (figure 2), the Jacobian of the mapping $\kappa \to T_t \kappa/|T_t \kappa|$ is $|T_t \kappa|^{-3} \equiv R_n^{-3}$ (for any unimodular matrix T_t). So the integral over the sphere of unit radius κ is $4\pi = \int d\sigma = \int R_n^{-3} d\sigma_0$, where $d\sigma_0$ and $d\sigma$ are the area elements before and after the mapping. After averaging

$$\int \langle R_n^{-3} \rangle \, d\sigma_0 = \int \exp\left[nf(-3) + o(1)\right] d\sigma_0 \approx 4\pi \exp\left[nf(-3)\right].$$

Hence $f(-3) = 0$.

[†] For degenerate matrices C, for example $C = AUA^{-1}$, where U is a random matrix of rotation, A is a constant matrix, it is possible that $\gamma_1 = 0$. However, this case is of no interest, since the inclusion of the magnetic diffusivity immediately results in exponential decay of the field (cf. the end of §3).
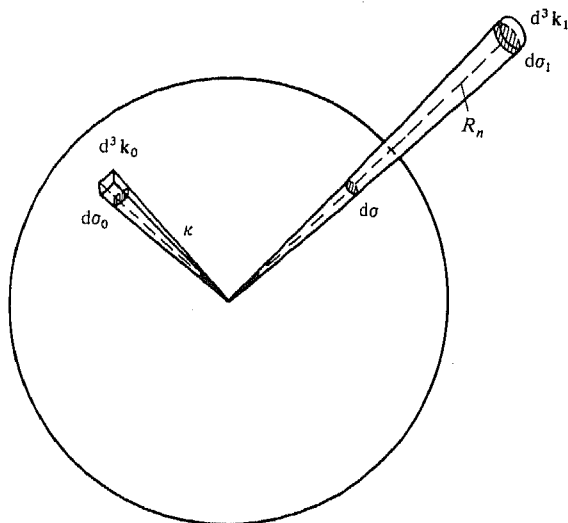
FIGURE 2. $d\sigma_0$ and $d\sigma$ are the area elements on the sphere before and after the mapping, and $d\sigma_1$ is an area element resulting from $d\sigma_0$ due to the 3-dimensional mapping $\kappa \to \boldsymbol{T}_t \kappa$. By definition $J \equiv d\sigma/d\sigma_0 = (d\sigma/d\sigma_1)(d\sigma_1/d\sigma_0)$. It is clear that $d\sigma/d\sigma_1 = R_n^{-2}$ because they have the same solid angle. Then $d\sigma_1/d\sigma_0 \equiv d\kappa\, d\sigma_1/(d\kappa\, d\sigma_0) = R_n^{-1}\, d^3k_1/d^3k_0 = R_n^{-1}$ because the mapping $\kappa \to \boldsymbol{T}_t \kappa$ has unit Jacobian (det $\boldsymbol{T}_t = 1$). As a result $J = R_n^{-3}$.

Let us show now that $f(p) \not\equiv 0$. Consider a non-random matrix $\boldsymbol{T}_0$ and a neighbourhood $\delta\boldsymbol{T}$ of it for which $\lambda_T > \lambda_0 > 1$, λ_T is the absolute value of the main eigenvalue of the matrix $\boldsymbol{T}$. This neighbourhood can be so constructed that (owing to ergodicity) the matrices $\boldsymbol{T}_i$ belong to it with a non-zero probability $\rho > 0$. Then

$$\langle R_n^p \rangle \geqslant \rho^n \lambda_0^{np}.$$

Hence
$$f(p) \geqslant \frac{1}{\tau} \ln \rho\lambda_0^p = \frac{\ln \rho}{\tau} + \frac{\ln \lambda_0}{\tau} p,$$

i.e. $f(p) \to \infty$ when $p \to \infty$.

Thus the function has the form shown on figure 3; thus proving the statement $\gamma_1 > 0$.

To illustrate this important and surprising result consider the area-preserving stretchings $(x \to \lambda x, y \to \lambda^{-1}y, \lambda > 1)$ of an initial spherical domain onto an ellipsoid by a random matrix of unit determinant (figure 4). The direction of the stretching is randomly changed at every step. However, a measure of the vector directions with the increasing length is clearly greater than one-half. In fact, a value of the transform vector is $\lambda^2 k_x^2 + \lambda^{-2} k_y^2 = k^2(\lambda^2 \cos^2 \phi + \lambda^{-2} \sin^2 \phi)$. It is greater than $k^2 = k_x^2 + k_y^2$ when $\cos \phi \geqslant (1+\lambda^2)^{-\frac{1}{2}}$.

In order to penetrate more deeply into the structure of $\boldsymbol{T}_t$ it is necessary to consider its action not only on the vectors but also on the planes. To proceed we apply the technique used by Furstenberg (1963) and Tutubalin (1972).

By orthogonalization of rows of $\boldsymbol{T}_t = \boldsymbol{T}_{n\tau}$, say beginning from above, we can decompose this matrix onto the product of a lower triangular matrix $\boldsymbol{\Delta}$ with real positive elements on its diagonal and an orthogonal matrix $\boldsymbol{U}$ (rotation of matrix)

$$\boldsymbol{T}_{n\tau} = \boldsymbol{U}(n)\, \boldsymbol{\Delta}(n). \tag{11}$$
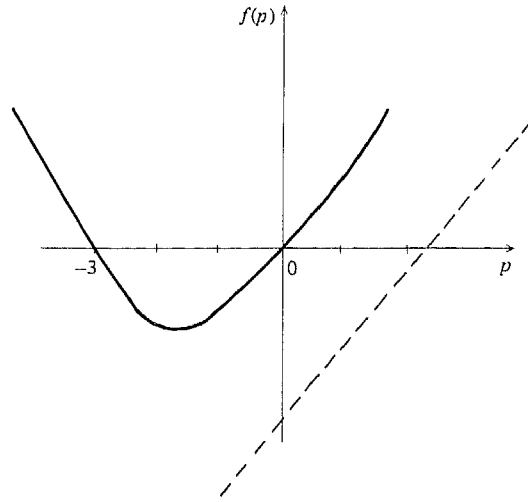
FIGURE 3. The form of the subsidiary function $f(p)$. The derivative of this function at $p = 0$ is the Liapunov index γ_1. The straight line $(\ln \rho + \ln \lambda_0 p)/\tau$ is a lower estimate of $f(p)$.

The diagonal elements $\varDelta_{\alpha\alpha}, \alpha = 1, 2, 3$, play a determining role. To investigate these elements we use the displacement $n \to n+1$:

$$T_{(n+1)\tau} = T_{n+1} T_{n\tau} = (T_{n+1} U(n)) \Delta(n).$$

The matrix in parentheses can also be decomposed as in (11), so that

$$T_{(n+1)\tau} = \tilde{u}(n) \tilde{\Delta}(n) \Delta(n),$$

i.e. $\Delta(n+1) = \tilde{\Delta}(n) \Delta(n)$. According to the definition of matrix multiplication the diagonal elements $\varDelta_{\alpha\alpha}(n+1)$ are the products of corresponding diagonal elements $\tilde{\varDelta}_{\alpha\alpha}(n)$ and $\varDelta_{\alpha\alpha}(n)$, so that

$$\ln \varDelta_{\alpha\alpha}(n+1) = \ln \tilde{\varDelta}_{\alpha\alpha}(n) + \ln \varDelta_{\alpha\alpha}(n).$$

Therefore one has

$$\ln \varDelta_{\alpha\alpha}(n) = \sum_{j=1}^{n} \ln \tilde{\varDelta}_{\alpha\alpha}(j).$$

Thus

$$\Delta(n) = \begin{pmatrix} e^{\mu_3 n\tau} & 0 & 0 \\ \dots & e^{\mu_2 n\tau} & 0 \\ \dots & \dots & e^{\mu_1 n\tau} \end{pmatrix},$$

where

$$\mu_1 + \mu_2 + \mu_3 = 0,$$

because $\det \Delta(n) = 1$.

We considered above the matrix C of general type when the function $C_\xi(t)$ covers with non-zero probability density the whole algebra of matrices with $\operatorname{tr} C = 0$. So $\mu_1 \neq \mu_2 \neq \mu_3$ with unit probability.

In the degenerate cases, for example when $C_\xi(t)$ is a matrix of random rotation around a given axis, μ_α can vanish.
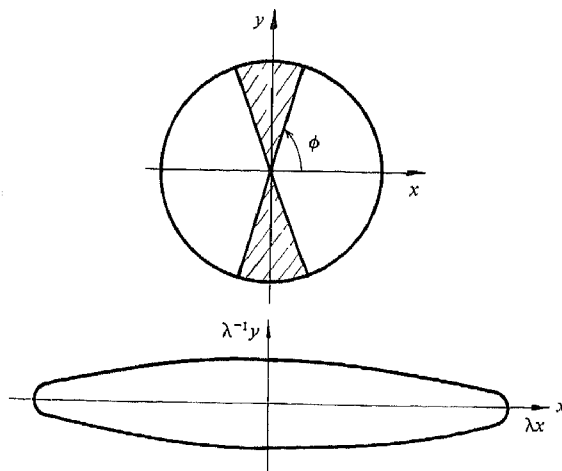
FIGURE 4. When a sphere is stretched into an ellipsoid by an unimodular matrix T (det $T = 1$) the measure of the squeezing directions (shaded regions) is less than one-half: $\phi = \arccos(1+\lambda^2)^{-\frac{1}{2}} > \frac{1}{4}\pi$.

According to the ergodic theorem the limits $\mu_\alpha \to \gamma_\alpha$ exist. Let us show that γ_1 coincides with the Liapunov index (10). To this end we stretch the vector

$$a = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

by the matrix $T_{n\tau}$ decomposed by (11). The norm of the resulting vector is

$$|T_n \dots T_1 a| = \exp(\mu_1 n\tau)|U(n) a| = \exp(\mu_1 n\tau) \underset{n \to \infty}{\to} \exp(\gamma_1 t).$$

By stretching a vector with the matrices $T_1^{-1}, \dots, T_n^{-1}$ one can obtain the limit $\mu_3 \to \gamma_3 < 0$. The same is evident from the condition

$$\gamma_1 + \gamma_2 + \gamma_3 = 0.$$

Note that γ_α are non-random constants. However, taking account of corrections results of course in a random (but asymptotically ($t \to \infty$) vanishing) contribution, i.e.

$$\Delta_{\alpha\alpha} = \exp(\gamma_\alpha t + \eta_\alpha t^{\frac{1}{2}}),$$

where η_α are random Gaussian variables.

Now we construct a basis e_α, corresponding to $\gamma_1, \gamma_2, \gamma_3$. At first note that (10) is valid in the sense of unit probability, i.e. not for all k (even for the matrix C of general type). For every n there exists a very narrow (vanishing as $n \to \infty$) cone of k_n contracting as $\exp(-|\gamma_3| n\tau)$. To construct the direction vector of this cone we subject a vector k to the transformation $T_1^{-1} \dots T_n^{-1}$, where T_n^{-1} is the reciprocal matrix to T_n. The resulting vector will evidently stretch as $\exp(|\gamma_3| t)$. Define the vector

$$e_3^{(n)} = \frac{T_1^{-1} T_2^{-1} \dots T_n^{-1} k}{|T_1^{-1} T_2^{-1} \dots T_n^{-1} k|}.$$

This is the singular vector which will not stretch but decrease (as $\exp(|\gamma_3| t)$) under the action of $T_n T_{n-1} \dots T_1$. One can ensure that this vector will not change strongly under the action of the next matrix T_{n+1}. In fact,

$$|T_{n+1}(T_n \dots T_1) e_3^{(n+1)}| \propto \exp(-|\gamma_3| t),$$

so that the vector $T_n \dots T_1 e_3^{(n+1)}$ is also contracting, i.e. it lies inside the cone around the vector $e_3^{(n)}$. Hence there exists a limit $e_3^{(n)} \to e_3$.

To consider the next basis vector e_2 one should consider the action of the reciprocal matrices on a plane containing the vector e_3. Thus one can find the plane of maximal contraction. The vector e_2 can be taken as the vector belonging to this plane and orthogonal to e_3. The rest basis vector e_1 is constructed as the vector orthogonal to e_3 and e_2.

It is clear that the basis e_α depends on a complete trajectory of the process $C_\xi(t)$, i.e. on the random variable ξ, and it cannot be defined at the initial moment. However, in spite of random character of the basis, it does not depend on the time.

So by considering the problem (1), (2) with the random matrix function $C_\xi(t)$ at the basis (e_1, e_2, e_3) we reduce it to the above case of the constant matrix C. Hence the main conclusions of §3 are valid in the stochastic case.

6. Conclusions and remarks

The magnetic field decays exponentially in time (say as $\exp(-|\gamma_2|t)$, $\gamma_2 \neq 0$) uniformly in r when $H_0 = O(r^{-3})$ as $r \to \infty$.

The total magnetic energy $\int H^2 \, d^3 r$ grows exponentially as $\exp[(|\gamma_3| - |\gamma_2|)t]$.

In the two-dimensional stochastic flow $(v_x, v_z; \gamma_2 = 0)$ a study similar to the analysis presented in §3 shows that the magnetic field decays owing to the random corrections $(\eta_\alpha t^{\frac{1}{2}})$ to the rate of growth (cf. the non-random case in §3).

The spatial distribution of the magnetic field is a set of thin ropes and (or) layers (see §3). In such an intermittent distribution the mean field is evidently much smaller than the concentrated field inside a rope. For instance, in the rope case $\langle H \rangle \sim R_m^{-1} HN$, where N is the number of ropes in the volume.

The spectrum of the magnetic field is anisotropic. The magnetic ropes are associated in the k-space with pancake configurations; the layers are associated with k-ropes.

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