Parr, R. G. & Crawford, B. L. 1948 *J. Chem. Phys.* **16**, 1049.
Rosen, N. 1931 *Phys. Rev.* **38**, 255, 2099.
Rosen, N. & Ikehara, S. 1933 *Phys. Rev.* **43**, 5, 671.
Rosen, P. 1950 *J. Chem. Phys.* **18**, 1182.
Sklar, A. L. 1939 *J. Chem. Phys.* **7**, 990.
Sklar, A. L. & Lyddane, R. H. 1939 *J. Chem. Phys.* **7**, 374.
Slater, J. C. 1928 *Phys. Rev.* **32**, 349.
Sugiura, Y. 1927 *Z. Phys.* **45**, 484.
Van Vleck, J. H. & King, G. W. 1939 *Phys. Rev.* **55**, 1215.
Van Vleck, J. H. & Sherman, A. 1935 *Rev. Mod. Phys.* **7**, 168.
Wilson, E. B. & Lindsay, R. B. 1935 *Phys. Rev.* **47**, 681.

The effect of homogeneous turbulence on material lines and surfaces

By G. K. Batchelor, *Trinity College, University of Cambridge*

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A material line consisting always of the same particles of fluid is lengthened by the convective action of turbulent motion, at a rate which is shown to be exponential at a sufficient time after the initial choice of the line. The area of material surface elements also increases exponentially, although with a different exponential coefficient. It is then a consequence of conservation of mass of the fluid that the normal distance between two neighbouring parallel material surfaces decreases at an exponential rate which is simply related to the rate at which material lines lengthen.

These kinematical results are relevant to the convective action of the turbulence on the distribution of two kinds of local property of the fluid, viz. quantities, represented by θ, of which the total amount in a material volume of the fluid remains constant (e.g. mass density of a foreign substance), and quantities, represented by $\mathbf{F}$, of which the total flux across a material surface remains constant (e.g. vorticity). $\mathbf{F}$ is proportional, at all times, to the vector representing a material line element and $\mathbf{G}$, $= \nabla\theta$, is proportional to the vector representing a material surface element, and hence they both increase exponentially in time. The last two sections of the paper are concerned with the combined effect on $\mathbf{F}$ and $\mathbf{G}$ of convection and molecular diffusion, and a condition for the effect of the convection to be dominant is obtained. In the case of $\mathbf{F}$, convection is found to be dominant if the appropriate molecular diffusivity is small compared with the kinematic viscosity of the fluid, and $\mathbf{F}^2$ will then increase indefinitely, unless some additional effect comes into play (as it does in the case of magnetic field strength). In the case of $\mathbf{G}$, convection will be dominant if the initial distribution of $\mathbf{G}$ has a large enough length scale; the effect of the convection is to decrease this length scale so that ultimately the two effects reach an equilibrium. The length scale of the ultimate distribution of $\mathbf{G}$, as obtained herein, is different from that put forward recently by Obukhoff.

1. Introduction

One of the basic properties of the turbulent motion of fluids is its diffusive character. The velocity of any particle, or small element of volume of the fluid, varies randomly with time as it follows the motion, and as a consequence the particle tends (statistically) to move away from its initial position. Likewise the difference between the velocities of two fluid particles varies randomly with time and the particles tend to move apart, however close they may be initially. This diffusion process can be regarded mathematically as a consequence of the fact that the mean square of an integral (viz. the displacement) of a random function (viz. the velocity) increases, except in rather special cases, as the range of integration increases. These effects of turbulent motion have been examined in papers (Batchelor 1949, 1952) which in some measure are forerunners of the present paper.

A manifestation of the tendency for neighbouring particles to separate which has some important consequences is that material lines (i.e. lines consisting always of the same fluid particles) tend to lengthen as they follow the motion. It may be that the presence of boundaries prevents the shortest distance between two particles from tending to increase indefinitely, but neighbouring particles always tend to separate. Material lines tend to lengthen without limit, irrespective of the presence of boundaries, since, however long the line may become, each element of it continues to lengthen. It will be found later that analogous remarks may be made about the effect of the motion on material surfaces. The importance of these kinematical effects lies in the fact that there are certain local physical properties of the fluid whose variation with time is simply related to that of material lines and surfaces. To take a well-known example, in the absence of molecular viscosity lines of vorticity follow the motion and the ratio of the magnitude of the vorticity to the length of a material line element coincident with the local vortex line remains constant. The effect of the turbulent motion on the distribution of these physical quantities will be taken up in later sections; for the moment we shall consider the purely kinematical properties of material lines and surfaces.

Suppose that an arbitrary line is drawn in the fluid at any instant t_0 and that the particles along the length of this line define the material line under consideration. Then if s represents (instantaneously) length along this line, and $\mathbf{u}(P, t)$ is the vector velocity at time t of a particle P on the line, the rate at which the length $L(t)$ of the line between any two particles P_1 and P_2 on the line is increasing is

$$\frac{\partial L(t)}{\partial t} = \int_{P_1}^{P_2} \left[\frac{\partial \mathbf{u}(P, t)}{\partial s}\right] . d\mathbf{s}. \tag{1·1}$$

We are interested, as always, only in the mean value (denoted by an over-bar) of this quantity over a large number of trials, or realizations of the motion, for each of which the material line has the prescribed initial position and shape. We shall assume for simplicity that the turbulence is spatially *homogeneous*, so that in fact only the initial shape (including orientation) of the material line need be prescribed. At the instant t_0 at which the line is selected, the mean value of this rate of extension is zero, because $\mathbf{u}(P, t)$ is then a velocity at a definite point in space and, together with all its derivatives in definite directions, has a zero mean; whatever the initial shape of the line, it neither lengthens nor shortens, on the average. At subsequent times $\overline{\partial L(t)/\partial t}$ is not zero, in part because the integrand of (1·1) is a function of time and in part because the path of integration is changing.

In view of the homogeneity of the turbulence the statistical behaviour of all elements of the above line is the same, except for any effects which may be due to their different initial orientation. The histories of different elements of the line will be statistically related if they are close enough, but these relations do not enter into a discussion of the average total length and they will not be considered here. Consequently it will be sufficient to confine our discussion to the statistical behaviour of the material line element whose direction and length at the initial instant t_0 are specified by the vector $\delta\mathbf{L}(t_0)$. This element will be supposed short enough to remain straight and to experience a rate of strain which is uniform over its length at all the values of t under consideration (larger values of t requiring the choice of a smaller value of $\delta L(t_0)$). The rate of extension of the line element is then

$$\frac{1}{\delta L(t)}\frac{\partial\,\delta L(t)}{\partial t} = \frac{\left[\dfrac{\partial \mathbf{u}(P,t)}{\partial s}\right].\delta\mathbf{L}(t)}{\delta L(t)} = l_i l_j \frac{\partial u_i(P,t)}{\partial x_j}, \tag{1·2}$$

where P is any particle on the line element and has the instantaneous position $\mathbf{x}$, and $l(t)$ is the unit vector specifying the direction of the line element (i.e. $\delta\mathbf{L}(t) = \mathbf{l}\,\delta L(t)$).

With this reduction to a consideration of short line elements over which the stretching motion is uniform, the problem is equivalent to the relative diffusion of two particles which initially are separated by the vector $\delta\mathbf{L}(t_0)$. The relative diffusion of two fluid particles, and those aspects of the relative diffusion of a cloud of marked fluid which can be related to the behaviour of two particles, have been considered in a previous paper (Batchelor 1952). In that paper no consideration was given to the particular case of two particles which are so close together—or of clouds whose linear dimensions are so small—that the value of the spatial derivative of the velocity is the same at the (simultaneous) positions of the particles,† for the reason that such a condition is unlikely to be realized with practical methods of marking and observing particular fluid particles. Had this case been considered, it would have been seen that the uniformity of the velocity derivative leads to appreciable simplification of the integrals expressing the relative diffusion of a cloud. For instance, the rate of increase of the relative dispersion tensor $D_{ij}(t)$ of a cloud of marked fluid in a given trial is

$$\begin{aligned}\frac{\mathrm{d}D_{ij}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\iint y_i(t)\,y_i(t)\,\mathrm{d}V'\,\mathrm{d}V'' \\ &= \iint [y_i(t)\,v_j(t) + y_j(t)\,v_i(t)]\,\mathrm{d}V'\,\mathrm{d}V'',\end{aligned} \tag{1·3}$$

where $\mathbf{y}(t)$ and $\mathbf{v}(t)$ are the relative position and relative velocity vectors of the two (moving) volume elements of marked fluid, $\mathrm{d}V'$ and $\mathrm{d}V''$, and both integrations are

† The condition for which is that $1/\delta L(t)$ must be large compared with the wave-numbers of Fourier components—in a spatial Fourier analysis of the velocity distribution—making a significant contribution to the mean-square rate of strain. For turbulence at large Reynolds numbers the condition becomes

$$\delta L(t) \ll \nu^{\frac{3}{4}}\epsilon^{-\frac{1}{4}},$$

where ν is the kinematic viscosity and ϵ is the energy dissipation per unit mass of fluid.

over the cloud of marked fluid. Then if the whole cloud is experiencing homogeneous strain at all times, we have

$$\frac{\mathrm{d}D_{ij}(t)}{\mathrm{d}t} = \iint \left[y_i(t)\, y_k(t) \frac{\partial u_j(t)}{\partial x_k} + y_j(t)\, y_k(t) \frac{\partial u_i(t)}{\partial x_k} \right] \mathrm{d}V' \mathrm{d}V''$$

$$= D_{ik}(t) \frac{\partial u_j(t)}{\partial x_k} + D_{jk}(t) \frac{\partial u_i(t)}{\partial x_k}, \qquad (1\cdot4)$$

where $\partial u_i(t)/\partial x_j$ is an Eulerian velocity derivative which is evaluated at the instantaneous position of the cloud.

Difficulties of observation of marked particles which are very close together do not worry us in the present connexion, since we are concerned here with the changes in the total length of a material line, which is made up of the changes in a large number of infinitesimal line elements, unlike the changes in the shortest distance between two fluid particles. Consequently, as already stated, $\delta L(t_0)$ will be supposed so small that $\partial u_i(t)/\partial x_j$ is uniform over the length $\delta L(t)$ at all the values of t under consideration, and the relation (1·2) will be used in the remainder of the paper.

2. The rate of extension of material line elements

As has already been remarked, the mean value of the expression $l_i l_j \dfrac{\partial u_i(P,t)}{\partial x_j}$ in (1·2) is initially zero, because at $t = t_0$ $\mathbf{l}$ does not fluctuate and $\dfrac{\partial u_i(P,t)}{\partial x_j}$ is a velocity derivative at a fixed point in a given direction. This influence of the definiteness, or non-random character, of the initial direction of the line element will gradually be lost as t increases, and it is a reasonable assumption that the probability distribution of $l_i l_j \dfrac{\partial u_i(P,t)}{\partial x_j}$ ultimately becomes independent of the vector $\mathbf{l}(t_0)$. A more general hypothesis of this kind has already been put forward in the earlier paper (Batchelor 1952); it was assumed there that the probability distribution of the vector $\mathbf{Y}(t)$ joining two particles ceases to be explicitly dependent on the initial separation $\mathbf{Y}(t_0)$ (except for the effect of $|\mathbf{Y}(t_0)|$ on the virtual origin of t) before the two particles are so far apart that they wander independently, provided that $|\mathbf{Y}(t_0)|$ is sufficiently small. The hypothesis in the present context is equivalent to the special case† $|\mathbf{Y}(t_0)| \to 0$ of this general hypothesis and clearly is a weaker assumption. In the more general hypothesis, the intermediate state was referred to as 'quasi-asymptotic' to distinguish it from the later state in which the particles are far apart and wander independently; the effect of letting $|\mathbf{Y}(t_0)| \to 0$ is to make the 'quasi-asymptotic' state of infinite duration, so that we can now drop the prefix 'quasi'.

What then will be the nature of the variation of $l_i l_j \dfrac{\partial u_i(P,t)}{\partial x_j}$ as t increases? We can ignore the effects of the decay of the turbulent energy, since, provided the

† The line element considered herein is not of zero length initially, but the limit $\delta L(t_0) \to 0$ has already been taken, in effect, by means of the assumption of a uniform rate of strain over the length of the line element.

Reynolds number of the turbulence is large enough, the eddies which dominate the stretching of a line element are small and have time scales small compared with the time scale of the decay. The magnitude of $l_i l_j \dfrac{\partial u_i(P,t)}{\partial x_j}$ is bounded, and oscillations in its mean properties as t increases are out of the question. The only real possibility is that $l_i l_j \dfrac{\partial u_i(P,t)}{\partial x_j}$, like $\mathbf{l}(t)$, becomes a stationary random function (s.r.f.) of t when $t-t_0$ is so large that the influence of the initial definite direction of $\delta\mathbf{L}(t)$ has been lost.

Using the intuitive belief that lines moving with the fluid *continue* to lengthen, on the average, we shall suppose that the above s.r.f. has a positive mean value ζ, i.e. that

$$\overline{\frac{1}{\delta L(t)}\frac{\partial\,\delta L(t)}{\partial t}} = \overline{l_i l_j \frac{\partial u_i(P,t)}{\partial x}} \to \zeta \tag{2·1}$$

as $t-t_0 \to \infty$, where ζ is a positive constant determined by the parameters describing the turbulence. It seems to be difficult to establish rigorously the conditions under which ζ is in fact non-zero,† but the validity of this property for all except rather special kinds of turbulent motion is strongly suggested by the following argument. Since the only effect of the turbulence is to produce a homogeneous strain which varies with time, and since the influence of the initial length has been removed by letting, in effect, $\delta L(t_0)\to 0$, there is no length other than $\overline{\delta L(t)}$ on which the statistical characteristics of $\delta\mathbf{L}(t)$ can depend, and a similarity in the shape of the probability density function of $\delta\mathbf{L}(t)$ at different values of t seems inevitable. In other words, it seems that $\dfrac{\delta\mathbf{L}(t)}{\overline{\delta L(t)}}$, like $\mathbf{l}(t)$ and $l_i l_j \dfrac{\partial u_i(P,t)}{\partial x_j}$, is (asymptotically) a s.r.f. of t. The same remarks will apply to $\mathbf{l}.\delta\mathbf{L}(t) = \delta L(t)$, so that the probability density function of $\delta L(t)$ is of the form

$$\frac{1}{\overline{\delta L(t)}}\,Q\left(\frac{\delta L(t)}{\overline{\delta L(t)}}\right). \tag{2·2}$$

Hence

$$\begin{aligned}\overline{\frac{1}{\delta L(t)}\frac{\partial\,\delta L(t)}{\partial t}} &= \frac{\partial}{\partial t}\int \ln \delta L(t)\,Q\left(\frac{\delta L(t)}{\overline{\delta L(t)}}\right)\mathrm{d}\frac{\delta L(t)}{\overline{\delta L(t)}} \\ &= \frac{\partial}{\partial t}\left[\overline{\ln\frac{\delta L(t)}{\overline{\delta L(t)}}} + \ln\overline{\delta L(t)}\right] \\ &= \frac{1}{\overline{\delta L(t)}}\frac{\partial\,\overline{\delta L(t)}}{\partial t},\end{aligned} \tag{2·3}$$

which shows that if $\overline{\delta L(t)}$ *continually* increases as t increases, ζ is a positive non-zero constant. Another way of stating the case is to note the additional deduction from (2·2) that

$$\overline{\frac{1}{\delta L(t)}\frac{\partial\,\delta L(t)}{\partial t}} = \frac{1}{2\,\overline{\delta L^2(t)}}\frac{\partial\,\overline{\delta L^2(t)}}{\partial t} = \frac{1}{\overline{\delta L^2(t)}}\int_{t_0}^{t}\overline{v_i(t)\,v_i(t')}\,\mathrm{d}t'; \tag{2·4}$$

† The proof would have to take detailed account of the properties of the turbulence, since, as will be noted in § 6, it is possible for the existence of particular stress systems acting on the fluid to prevent certain material lines from being lengthened on the average, even though the motion is still turbulent. Unless otherwise stated, such stresses will be supposed not to exist.

the integral in (2·4) is zero at $t = t_0$, is necessarily positive at small values of $t-t_0$, and can return to zero at large values of $t-t_0$ only in the very unlikely event that $v_i(t)$ and $v_i(t')$ are negatively correlated over a large range of values of $t-t'$.

We can of course regard the assumption that $\zeta > 0$ as equivalent to a restriction of the applicability of the analysis to cases of turbulent flow for which the integral in (2·4) is positive for all values of t; however, in view of our present lack of information about Lagrangian mean values like $\overline{v_i(t)\, v_i(t')}$, this criterion is effectively only a restatement of the assumption.

From (2·1) and (2·3), we find†

$$\frac{1}{\overline{\delta L(t)}} \frac{\partial\, \overline{\delta L(t)}}{\partial t} \to \zeta \tag{2·5}$$

as $t-t_0 \to \infty$, so that

$$\overline{\delta L(t)} \to \delta L(t_0)\, \mathrm{e}^{(t-t_1)\zeta}, \tag{2·6}$$

where t_1 is an effective origin of the diffusion process which takes account of the fact that the two sides of (2·5) are not equal at values of t near $t = t_0$. It is instructive to look also at the variation of $\delta L(t)$ in a single trial. It follows from (1·2) that

$$\delta L(t) = \delta L(t_0) \exp\left[\int_{t_0}^{t} l_i l_j \frac{\partial u_i(P, t')}{\partial x_j}\, \mathrm{d}t'\right]. \tag{2·7}$$

As $t-t_0 \to \infty$, the contribution to the integral in (2·7) from the mean value of the integrand tends to $(t-t_0)\,\zeta$, whereas the contribution from the fluctuation of the integrand (which is a s.r.f. of t) about its mean is an oscillating quantity with zero (temporal) mean and a root-mean-square which increases as $(t-t_0)^{\frac{1}{2}}$ (like, for instance, the displacement $\mathbf{X}(t)$ of a single particle in non-decaying turbulence). Hence the order of magnitude of $\delta L(t)$ at large values of $t-t_0$ is determined by the mean value of the integrand in (2·7) and is

$$\delta L(t) \sim \delta L(t_0)\, \mathrm{e}^{(t-t_2)\zeta}, \tag{2·8}$$

where the constant t_2 is given by the convergent integral

$$t_2 - t_0 = \int_{t_0}^{\infty} \left[1 - \overline{l_i l_j \frac{\partial u_i(P, t')}{\partial x_j}} \Big/ \zeta\right] \mathrm{d}t',$$

in agreement with (2·6). The asymptotic behaviour (2·8) will not always be valid, because it is possible that the contribution to the integral in (2·7) from the fluctuation of the integrand about its mean is of order $(t-t_0)$ for *some* line elements in any one realization of the field, and for any given line element in *some* realizations of the

† Another meaning of the assumption leading to (2·5) emerges from the exact relation

$$\frac{\partial\, \overline{\delta L(t)}}{\partial t} = \overline{\delta L(t)\, l_i l_j \frac{\partial u_i(P, t)}{\partial x_j}};$$

the mean value of $l_i l_j \dfrac{\partial u_i(P, t)}{\partial x_j}$ at large values of $t-t_0$ is defined to be ζ, so that (2·5) is valid provided the covariance of the fluctuations, about the means, of $\delta L(t)$ and $l_i l_j . \dfrac{\partial u_i(P, t)}{\partial x_j}$ is ultimately negligible by comparison with the product of the means.

field. Indeed, if we consider the line elements which are initially the radii of the small sphere, which at subsequent times is drawn out into an ellipse, we see that it is necessary that some of the line elements are ultimately shorter than they were initially. However, the probability that a particular line element will experience a net contraction in a particular realization decreases as $t-t_0$ increases, and the order of magnitude relation (2·8) is to be interpreted in a similar statistical sense.

It has already been remarked (Batchelor 1952) that the relative diffusion of particles which are sufficiently close together is a process dominated by the small eddies which are the subject of the universal similarity theory put forward by Kolmogoroff and others. The same remark applies, with greater force, to the process of extension of material line elements. Hence provided the Reynolds number of the turbulence is sufficiently large we can infer that ζ is determined by the parameters ν (= kinematic viscosity of the fluid) and ϵ (= viscous dissipation of energy per unit mass of fluid) alone, i.e. that

$$\xi = c(\epsilon/\nu)^{\frac{1}{2}}, \tag{2·9}$$

where c is a universal constant of order of magnitude unity. The meaning of (2·9) is simply that ζ is proportional to, and of the same order of magnitude as, the root-mean-square of any of the Eulerian velocity (spatial) derivatives. Another, and a well-substantiated, prediction of the universal similarity theory is that the motion associated with the smaller eddies is isotropic; the extension of material line elements should therefore be an isotropic process.

3. The rate at which the area of a material surface element increases

Many of the general remarks which have been made about the tendency for material lines to lengthen as they follow the motion apply equally to material surfaces, and need not be repeated. We can proceed immediately to a calculation of the rate at which the area of an element of a material surface tends to increase. The surface element will be assumed to have an initial area $\delta A(t_0)$ whose linear dimensions are so small that spatial derivatives of the fluid velocity are uniform over the surface element at the values of t under consideration. The surface element thus experiences a homogeneous strain which varies with time. This being so, the ratio of the area at time t to the initial area is independent of the initial shape of the element, and we can base the choice of the initial shape on convenience. The surface element will be chosen to be initially a parallelogram defined by the two (intersecting) material line elements $\delta \mathbf{b}(t_0)$ and $\delta \mathbf{c}(t_0)$, to which the considerations of the previous section apply; at subsequent times the surface element will still be a parallelogram, with the (vector) area

$$\delta \mathbf{A}(t) = \delta \mathbf{b}(t) \times \delta \mathbf{c}(t). \tag{3·1}$$

In the previous section it was established that the sides of the parallelogram, $\delta b(t)$ and $\delta c(t)$, both tend to increase exponentially with t when $t-t_0$ is large enough. The determination of the rate at which the area $\delta A(t)$ tends to increase therefore requires a determination of the effect of the turbulence on the angle between the line elements vectors $\delta\mathbf{b}(t)$ and $\delta\mathbf{c}(t)$. At first sight the effects of the turbulence on the lengths of material line elements and on the angles between them do not seem to be related. However, as soon as we consider more than two intersecting material line elements, from which a solid figure can be formed, it is clear that the changes in the lengths of the elements and in the angles between them must be related by the need to conserve the mass of fluid enclosed by the figure. We shall therefore regard the surface element (3·1) as being one of the faces of a parallelepiped formed by the three (intersecting) material line elements $\delta\mathbf{a}(t)$, $\delta\mathbf{b}(t)$, $\delta\mathbf{c}(t)$, the other two faces being

$$\delta\mathbf{B}(t) = \delta\mathbf{c}(t)\times\delta\mathbf{a}(t) \quad \text{and} \quad \delta\mathbf{C}(t) = \delta\mathbf{a}(t)\times\delta\mathbf{b}(t),$$

and make use of the fact that the mass $\rho\delta V$, where

$$\delta V = \delta\mathbf{a}\,.\,(\delta\mathbf{b}\times\delta\mathbf{c}) = [\delta\mathbf{A}\,.\,(\delta\mathbf{B}\times\delta\mathbf{C})]^{\frac{1}{2}}, \tag{3·2}$$

remains constant during the diffusion process.

If $\alpha(t)$ is the angle between $\delta\mathbf{b}(t)$ and $\delta\mathbf{c}(t)$, and $\theta(t)$ is the angle between $\delta\mathbf{a}(t)$ and the plane of $\delta\mathbf{b}(t)$ and $\delta\mathbf{c}(t)$ (with similar meanings for $\beta(t)$, $\gamma(t)$ and $\phi(t)$, $\psi(t)$), we have

$$\delta V = \delta a\,\delta b\,\delta c \sin\alpha\sin\theta.$$

With the assumption that the mean density $\bar{\rho}$ is constant and that $\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\rho-\bar{\rho}}{\bar{\rho}}\right)^2$ is negligible, we find, by logarithmic differentiation of this (constant) quantity, that

$$\overline{\frac{1}{\sin\alpha}\frac{\partial\sin\alpha}{\partial t}}+\overline{\frac{1}{\sin\theta}\frac{\partial\sin\theta}{\partial t}} = -\left(\overline{\frac{1}{\delta a}\frac{\partial\,\delta a}{\partial t}}+\overline{\frac{1}{\delta b}\frac{\partial\,\delta b}{\partial t}}+\overline{\frac{1}{\delta c}\frac{\partial\,\delta c}{\partial t}}\right)\to -3\zeta, \tag{3·3}$$

as $t-t_0\to\infty$, according to (2·1). One of the two terms on the left-hand side of (3·3) must be negative, so that either $\sin\alpha$ or $\sin\theta$ tends to decrease indefinitely with increase of t. The arguments that led, in § 2, to the conclusion that the probability distribution of $\delta L(t)$ is asymptotically self-preserving in shape apply also to the angles α and θ, etc., so that relations like (2·3) are valid; hence either $\overline{\sin\alpha}$ or $\overline{\sin\theta}$ must decrease exponentially in t. If $\overline{\sin\theta}$ decreases exponentially, the associated quantities $\overline{\sin\phi}$ and $\overline{\sin\psi}$ decrease (asymptotically) at an equal rate, and we can show from the geometry of the figure that $\overline{\sin\alpha}$, $\overline{\sin\beta}$ and $\overline{\sin\gamma}$ must decrease at the same rate. For we have the identity

$$\sin^2\alpha\sin^2\theta = 1-\cos^2\theta-\cos^2\phi-\cos^2\psi+2\cos\theta\cos\phi\cos\psi,$$

which becomes, when θ, ϕ and ψ are all small,†

$$\theta^2\sin^2\alpha = \tfrac{1}{2}(\theta^2\phi^2+\phi^2\psi^2+\psi^2\theta^2)-\tfrac{5}{4}(\theta^4+\phi^4+\psi^4), \tag{3·4}$$

and, provided $\overline{\theta^4}$ is of the same order of smallness as $(\overline{\theta^2})^2$, the two sides of (3·4) can be of the same order of smallness as $t\to\infty$ only if $\overline{\sin^2\alpha}$, and therefore $\overline{\alpha^2}$, is of the

† Or close to π, in which case θ (say) is replaced in the relation (3·4) by $\pi-\theta$.

same order of smallness as $\overline{\theta^2}$. The same is found to be true if we infer from (3·3) that $\overline{\frac{1}{\sin\alpha}\frac{\partial \sin\alpha}{\partial t}} < 0$. Hence

$$\overline{\frac{1}{\sin\alpha}\frac{\partial \sin\alpha}{\partial t}} = \overline{\frac{1}{\sin\theta}\frac{\partial \sin\theta}{\partial t}} = -\tfrac{3}{2}\zeta \tag{3·5}$$

and
$$\sin\alpha \sim \mathrm{e}^{-\frac{3}{2}t\zeta}, \quad \sin\theta \sim \mathrm{e}^{-\frac{3}{2}t\zeta}, \tag{3·6}$$

when $t-t_0$ is large, with identical results for β, γ, ϕ and ψ.

The effect of the turbulence is thus to make the parallelepiped tend to a long narrow shape. All the sides tend to become parallel, whereas the normals to the faces tend to become coplanar. The probability distributions of the angles α, β and γ are ultimately concentrated (symmetrically about $\frac{1}{2}\pi$) near 0 and π, so that two intersecting line elements tend to become, and stay, parallel or anti-parallel, and as t increases the transitions from the one state to the other become more infrequent. An immediate further result, from (3·1) and (3·5), is that

$$\overline{\frac{1}{\delta A(t)}\frac{\partial\, \delta A(t)}{\partial t}} = \overline{\frac{\partial}{\delta B(t)}\frac{\partial \delta B(t)}{\partial t}} = \overline{\frac{1}{\delta C(t)}\frac{\partial\, \delta C(t)}{\partial t}} = \tfrac{1}{2}\zeta, \tag{3·7}$$

which, together with the now-standard hypothesis of self-preservation of the shape of the probability distribution of $\delta A(t)$ as t increases, shows that areas of material surface elements tend to increase according to the asymptotic law

$$\delta A(t) \sim \mathrm{e}^{\frac{1}{2}t\zeta}. \tag{3·8}$$

To complete the display of the duality of the results for line and surface elements, we may also note that if $\alpha'(t)$ is the angle between $\delta\mathbf{B}(t)$ and $\delta\mathbf{C}(t)$, and $\theta'(t)$ is the angle between $\delta\mathbf{A}(t)$ and the plane of $\delta\mathbf{B}(t)$ and $\delta\mathbf{C}(t)$,

$$\overline{\frac{1}{\sin\alpha'}\frac{\partial \sin\alpha'}{\partial t}} = \overline{\frac{1}{\sin\theta'}\frac{\partial \sin\theta'}{\partial t}} = -\tfrac{3}{4}\zeta. \tag{3·9}$$

It will be explained in later sections that the connexion between these results and the convection of certain physical (scalar) quantities lies in the fact that level-surfaces of these quantities are material surfaces, i.e. they move with the fluid. It will be found useful to know the effect of the turbulence on the perpendicular distance between two neighbouring level-surfaces, since this distance determines the gradient of the physical quantity. The above analysis does in fact supply this information, since we can regard any two parallel faces of the parallelepiped (say those of area $\delta\mathbf{b}(t)\times\delta\mathbf{c}(t)$) as lying in the two neighbouring level-surfaces; the perpendicular distance between the surfaces is then $\delta a \sin\alpha$, and the variation with time is given by (2·1) and (3·5) as

$$\overline{\frac{1}{\delta a \sin\alpha}\frac{\partial\, \delta a \sin\alpha}{\partial t}} = -\tfrac{1}{2}\zeta. \tag{3·10}$$

The mean value of the shortest distance between two neighbouring material surfaces thus decreases ultimately as $\mathrm{e}^{-\frac{1}{2}t\zeta}$, as compared with $\mathrm{e}^{t\zeta}$ for the length of material line elements, and $\mathrm{e}^{\frac{1}{2}t\zeta}$ for the area of material surface elements.

4. Application to the effect of turbulence on certain physical properties of the fluid

The physical quantities of the fluid to which this section refers fall into two complementary classes. First, scalar quantities (having the character of mass densities) of which the amount associated with a material volume element is conserved during the motion, and secondly, vector quantities of which the flux across a material surface element is conserved during the motion. Obvious examples† of these two classes of quantity, which will be typified by the symbols θ and $\mathbf{F}$, are the mass of some foreign substance per unit mass of fluid, and vorticity (the constancy of flux being, in this case, the constancy of angular momentum of the fluid). It is assumed that the spatial distribution of some local property of the fluid falling into one of these classes is specified initially, and the problem is to determine the statistical effect of the turbulence on the subsequent distribution of the property.

The relations which express the conservation laws governing the variation of θ and $\mathbf{F}$ are

$$\frac{\mathrm{D}\,\theta\rho\,\delta V}{\mathrm{D}t} = 0, \tag{4·1}$$

$$\frac{\mathrm{D}\,\mathbf{F}.\delta\mathbf{A}}{\mathrm{D}t} = 0, \tag{4·2}$$

where $\dfrac{\mathrm{D}}{\mathrm{D}t} = \dfrac{\partial}{\partial t} + \mathbf{u}.\nabla$ is the operator giving the temporal rate of change as the fluid particle to which θ or $\mathbf{F}$ refers is followed in its motion ($\mathrm{D}/\mathrm{D}t$ and $\partial/\partial t$ are, of course, identical for quantities like $\delta\mathbf{A}$ and δV which are defined only in relation to the local particle), $\delta V(t)$ and $\delta\mathbf{A}(t)$ are material volume and surface elements, and $\rho(t)$ is the density of the fluid in δV. Now the mass of the material volume element is also conserved, so that we have

$$\frac{\mathrm{D}\rho\delta V}{\mathrm{D}t} = \frac{\mathrm{D}\rho\delta\mathbf{L}.\delta\mathbf{A}}{\mathrm{D}t} = 0, \tag{4·3}$$

where $\delta\mathbf{L}(t)$ is the vector describing any material line element which intersects the surface element $\delta\mathbf{A}(t)$. Comparison of (4·1) and (4·2) with (4·3) shows that the changes in θ and $\mathbf{F}$ are described by

$$\theta(t) = \text{const.}, \quad \mathbf{F}(t) \propto \rho(t)\,\delta\mathbf{L}(t), \tag{4·4}$$

provided the line element $\delta\mathbf{L}(t)$ initially has the same direction as $\mathbf{F}(t)$. The value of $\dfrac{1}{\rho}\mathbf{F}(t)$ for the particle changes in the same way, both in magnitude and direction, as a material line element initially coincident (locally) with the line of $\mathbf{F}$. Thus level-surfaces of θ are material surfaces, and lines of $\mathbf{F}$ are material lines, i.e. they each move with the fluid.

The temporal variation in the mean value of transportable quantities such as θ has already been considered fully (Batchelor 1949). The aspect of the diffusion which is of interest in the present context is the change in shape of small volumes of marked fluid which experience homogeneous strain. This will include the tendency

† In the absence of molecular diffusion, the effects of which are to be considered in §§5 and 6.

for the thin sheet of marked fluid between two level-surfaces of θ to distort and become thinner. Hence the parameter of the distribution of θ to which the considerations of the previous sections are relevant is the vector gradient, $\nabla\theta$, $= \mathbf{G}(\mathbf{x}, t)$ say. The magnitude of $\mathbf{G}$, as a function of time for a moving particle, is proportional to the reciprocal of the distance between the two local neighbouring level-surfaces. Hence if $\delta\mathbf{A}(t)$ is an element of a level-surface of θ, we have

$$\mathbf{G}(t) \propto \frac{\delta\mathbf{A}(t)}{\delta V}, \tag{4·5}$$

where δV is the material volume element in the shape of a cylinder with ends formed by elements $\delta\mathbf{A}$ of the two neighbouring level-surfaces of θ, and consequently

$$\mathbf{G}(t) \propto \rho(t)\, \delta\mathbf{A}(t). \tag{4·6}$$

Equations (4·4) and (4·6) together yield the interesting relation

$$\mathbf{F}(t)\,.\,\mathbf{G}(t) \propto \rho^2 \delta\mathbf{L}\,.\,\delta\mathbf{A} \propto \rho(t). \tag{4·7}$$

The quantity θ has served its purpose as a preliminary to the introduction of the vector $\mathbf{G}$ and now drops into the background. It appears from (4·4) and (4·6) that the convective properties of the quantities represented by $\mathbf{F}$ and $\mathbf{G}$ are fundamental, and complementary to each other in virtue of the requirement of conservation of mass. The correspondence between $\mathbf{F}$ and $\mathbf{G}$ on the one hand, and material line and surface elements on the other, may also be stated in tensor language, perhaps with greater clarity. The material surface element is, properly speaking, an antisymmetrical tensor of the second order, of which the typical element is

$$\delta A_{ij}(t) = \tfrac{1}{2}[\delta b_i(t)\, \delta c_j(t) - \delta b_j(t)\, \delta c_i(t)], \tag{4·8}$$

where $\delta b_i(t)$ and $\delta c_i(t)$ are the vectors describing the sides of the parallelogram comprising the surface element. The flux of any second-order tensor $F_{ij}(t)$ through the surface element is $F_{ij}\delta A_{ij}$, which involves only the antisymmetrical part of F_{ij}. Hence if $F_{ij}(t)$ describes a local property of the fluid which satisfies conservation of flux, we see that

$$F_{ij}(t)\, \delta A_{ij}(t) = \text{const.} \tag{4·9}$$

for a material element, which is equivalent to (4·2) if $\mathbf{F}$ is the adjoint vector representing the antisymmetrical part of F_{ij}. Also, if G_k is the k-component of the gradient of a local property θ which is conserved,

$$G_k(t)\, \delta L_k(t) = \text{const.} \tag{4·10}$$

expresses the constancy of the difference in the values of θ at the ends of the material line element represented by the vector δL_k. Now the conservation of mass is expressed by

$$\rho(t)\, \epsilon_{ijk}\, \delta A_{ij}(t)\, \delta L_k(t) = \text{const.}, \tag{4·11}$$

where ϵ_{ijk} is the alternating tensor, from which the solutions of (4·9) and (4·10) are found to be

$$F_{ij}(t) \propto \rho(t)\, \epsilon_{ijk}\, \delta L_k(t), \quad G_k(t) \propto \rho(t)\, \epsilon_{ijk}\, \delta A_{ij}(t), \tag{4·12}$$

and

$$\epsilon_{ijk}\, F_{ij}(t)\, G_k(t) \propto \rho(t), \tag{4·13}$$

corresponding to (4·4), (4·6) and (4·7).

The qualitative changes in the spatial distributions of θ, $\mathbf{F}$ and $\mathbf{G}$ are now clear from the results established in the preceding sections. Material line elements tend to lengthen, and consequently the magnitude of $\mathbf{F}$ tends to increase; neighbouring level surfaces tend to come together, and consequently the magnitude of $\mathbf{G}$ tends to increase; the angle between a material line element and the plane of an intersecting material surface element tends to decrease, and consequently $\mathbf{F}$ and $\mathbf{G}$ tend to become perpendicular; in all cases at an exponential rate with the exponents described in §§ 2 and 3.

In the two following sections, the aim will be to examine the modifications of these predictions which are necessary when the effect of molecular diffusion on the distributions of θ, $\mathbf{F}$ and $\mathbf{G}$ is taken into account. Before doing so, it is necessary to write down the equations describing the convective changes in these quantities in an Eulerian, or fixed, frame of reference. For θ we have

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \left(\frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i}\right)\theta = 0, \tag{4·14}$$

and hence, for the gradient,

$$\frac{\partial G_i}{\partial t} + \frac{\partial}{\partial x_i}\left(u_j \frac{\partial \theta}{\partial x_j}\right) = \frac{\mathrm{D}G_i}{\mathrm{D}t} + G_j \frac{\partial u_j}{\partial x_i} = 0. \tag{4·15}$$

For $\mathbf{F}$ we have from the conservation relation (4·2)

$$\begin{aligned}
\frac{\mathrm{D}F_i}{\mathrm{D}t}\,\delta A_i &= -F_i \frac{\mathrm{D}\,\delta A_i}{\mathrm{D}t} \\
&= -F_i\,\epsilon_{ijk}\left(\frac{\partial\,\delta b_j}{\partial t}\,\delta c_k + \delta b_j \frac{\partial\,\delta c_k}{\partial t}\right) \\
&= -F_i\,\epsilon_{ijk}\left(\frac{\partial u_j}{\partial x_l}\,\delta b_l \delta c_k + \frac{\partial u_k}{\partial x_l}\,\delta b_j \delta c_l\right) \\
&= F_i\,\epsilon_{ijk}\frac{\partial u_k}{\partial x_l}\,(\delta b_l \delta c_j - \delta b_j \delta c_l) \\
&= F_i\,\epsilon_{ijk}\frac{\partial u_k}{\partial x_l}\,\epsilon_{mlj}\,\delta A_m \\
&= (\delta_{km}\delta_{il} - \delta_{kl}\delta_{im})\,F_i \frac{\partial u_k}{\partial x_l}\,\delta A_m \\
&= F_i \frac{\partial u_k}{\partial x_i}\,\delta A_k - F_i \frac{\partial u_k}{\partial x_k}\,\delta A_i,
\end{aligned} \tag{4·16}$$

where $\delta\mathbf{b}$ and $\delta\mathbf{c}$ are again the line elements representing the sides of the parallelogram surface element $\delta\mathbf{A}$. Since (4·16) must be true for all material surface elements, the equation for F_i is

$$\frac{\mathrm{D}F_i}{\mathrm{D}t} - F_j \frac{\partial u_i}{\partial x_j} + F_i \frac{\partial u_j}{\partial x_j} = 0. \tag{4·17}$$

Equation (4·17) is well known (Prim & Truesdell 1950) as a criterion for the strength of a vector-tube of the field **F** to remain constant as the tube follows the motion of the field. With the aid of the continuity equation, (4·17) can also be written as

$$\frac{\mathrm{D}F_i/\rho}{\mathrm{D}t} - \frac{F_j}{\rho}\frac{\partial u_i}{\partial x_j} = 0. \tag{4·18}$$

5. Convection and conduction of the quantities F and G

It will be assumed that the changes in the distributions of θ, **G** and **F** due to molecular (or perhaps electronic) diffusion satisfy the heat conduction equation, with a diffusivity k. Hence the Eulerian equations for the propagation of θ, **G** and **F** are†

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = k\nabla^2\theta, \tag{5·1}$$

$$\frac{\mathrm{D}G_i}{\mathrm{D}t} + G_j\frac{\partial u_j}{\partial x_i} = k\nabla^2 G_i, \tag{5·2}$$

$$\frac{\mathrm{D}F_i}{\mathrm{D}t} - F_j\frac{\partial u_i}{\partial x_j} = k\nabla^2 F_i. \tag{5·3}$$

Although the expression of the effect of molecular diffusion in the Eulerian form is simple, the expression in the Lagrangian form is too complicated to be of any use. We are thus faced with the difficulty of describing the combined effects of convection and conduction, which separately are simple only when expressed in the Lagrangian and Eulerian forms respectively. The conclusions of the previous sections that the effect of convection is to magnify spatial variations of **F** and **G** warn us that a neglect of molecular diffusion would become more serious as t increases, so that a complete determination of the effect of turbulent motion on physical properties represented by **F** and **G** requires the difficulty to be faced, whatever the consequent limitations of the analysis.

Qualitatively the effect of molecular diffusion is two-fold: it smooths the spatial distribution of the quantity concerned, and it changes the character of the convective effect of the turbulent motion. When molecular diffusion exists level-surfaces of θ and lines of **F** do not follow the motion exactly, although they may do so approximately to an extent which depends on the magnitude of the effect of molecular diffusion. It will normally be the case that convective effects are dominant, or at any rate that strong conductive effects exist only as a *consequence* of the existence of large spatial gradients produced by the convection, and we can safely assume that the effect of convection is to tend to increase the magnitudes of **F** and **G** as before, although perhaps not at the rate indicated by the analysis of the previous sections. The effect of molecular diffusion is of course to decrease the variations of **F** and **G** about the spatial averages, so that the two mechanical processes have opposing tendencies. The determination of which effect is dominant,

† When ρ is not uniform the effect of molecular motion on $\mathbf{F}/\rho$ will not be describable by a heat conduction equation, in general, so that the variability of ρ which was permitted in the previous sections must now be abandoned; as a consequence $\partial u_k/\partial x_k = 0$.

and under what conditions, is the kind of problem to which the preceding kinematical considerations are a preliminary.

So far as the convective effect of the turbulence is concerned, it was found sufficient to confine attention to the statistical history of a single material element. However, the effect of molecular conduction depends on the spatial distributions of **F** and **G**, and some assumption about their initial spatial distribution must therefore be made. There are many particular assumptions which might be made, each of them appropriate to important physical problems. However, the emphasis is here on general effects, and the most suitable assumption seems to be that **F** and **G** are stationary random functions of position at all times, like the velocity of the fluid. This is not so far from the conditions of many real situations as it might seem. We have observed that the components of the turbulence that have greatest convective effect on **F** and **G** are of small length-scale, and this in turn implies that the components (in a spatial Fourier analysis) of **F** and **G** that are most affected by the convection—and in which there is most interest—have a similarly small length-scale. Whatever the nature of the variation of **F** and **G** on a large length-scale, any departures from statistical homogeneity will necessarily be relatively unimportant for the small-scale components of **F** and **G**.

With the assumption that **F** and **G** are s.r.f.'s of position, the equations for the rate of change of mean-square of these quantities become

$$\frac{1}{2}\frac{\mathrm{d}\overline{\mathbf{F}^2}}{\mathrm{d}t} = \overline{\mathbf{F}^2 f_i f_j \frac{\partial u_i}{\partial x_j}} - k\,\overline{\left(\frac{\partial F_i}{\partial x_j}\right)^2}, \tag{5·4}$$

$$\frac{1}{2}\frac{\mathrm{d}\overline{G^2}}{\mathrm{d}t} = -\overline{\mathbf{G}^2 g_i g_j \frac{\partial u_i}{\partial x_j}} - k\,\overline{\left(\frac{\partial G_i}{\partial x_j}\right)^2}, \tag{5·5}$$

where f_i and g_i are the direction cosines of the vectors **F** and **G**. In the absence of molecular diffusion—i.e. when $k = 0$—lines of **F** and level-surfaces of θ are material lines and surfaces; then if $\delta\mathbf{a}$ is a material line element initially parallel to the local value of **F**, and if $\delta\mathbf{b}$ and $\delta\mathbf{c}$ are material line elements initially lying in the local level-surface of θ, the vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}\times\delta\mathbf{c}$ will continue to be parallel to **F** and **G** at all subsequent times, and their magnitudes will be proportional to the magnitudes of **F** and **G** at all times. Hence when $k = 0$ we can use all the kinematical relations obtained in §§ 2 and 3, and we see (cf. equations (2·4)) that as $t - t_0$ increases and the influence of the initial distributions of **F** and **G** is lost

$$\overline{\mathbf{F}^2 f_i f_j \frac{\partial u_i}{\partial x_j}} \to \overline{\mathbf{F}^2}\,.\,\overline{f_i f_j \frac{\partial u_i}{\partial x_j}} = \zeta\overline{\mathbf{F}^2}, \tag{5·6}$$

$$\overline{\mathbf{G}^2 g_i g_j \frac{\partial u_i}{\partial x_j}} \to \overline{\mathbf{G}^2}\,.\,\overline{g_i g_j \frac{\partial u_i}{\partial x_j}} = -\tfrac{1}{2}\zeta\overline{\mathbf{G}^2}. \tag{5·7}$$

It is a reasonable assumption that when k is finite, but sufficiently small, the convective effects are represented approximately—or, at any rate, to the correct order of magnitude—by the expressions (5·6) and (5·7). The criterion for k to be 'sufficiently small' would appear to be that the viscous effects, as measured by, say, the contribition to $\mathrm{d}\overline{\mathbf{F}^2}/\mathrm{d}t$ in (5·4), or to $\mathrm{d}\overline{\mathbf{G}^2}/\mathrm{d}t$ in (5·5), are small compared with the convective effects. This will be the case provided

$$\frac{\zeta}{k} \gg \overline{\left(\frac{\partial F_i}{\partial x_j}\right)^2} \Big/ \overline{\mathbf{F}^2} \quad \text{and} \quad \overline{\left(\frac{\partial G_i}{\partial x_j}\right)^2} \Big/ \overline{\mathbf{G}^2}. \tag{5·8}$$

Hence if the power spectra of **F** and **G** are such as to satisfy

$$\zeta \overset{*}{\lambda}{}^2/k \gg 1, \tag{5·9}$$

where $\overset{*}{\lambda}$ is the length representing either $\left[\overline{\left(\frac{\partial F_i}{\partial x_j}\right)^2} \Big/ \overline{\mathbf{F}^2}\right]^{-\frac{1}{2}}$ or $\left[\overline{\left(\frac{\partial G_i}{\partial x_j}\right)^2} \Big/ \overline{\mathbf{G}^2}\right]^{-\frac{1}{2}}$, the expressions (5·6) and (5·7) are valid and they dominate the right-hand sides of (5·4) and (5·5), with the results

$$\overline{\mathbf{F}^2(t)} \sim \overline{\mathbf{F}^2(t_0)}\, \mathrm{e}^{(t-t_0)\zeta}, \quad \overline{\mathbf{G}^2(t)} \sim \overline{\mathbf{G}^2(t_0)}\, \mathrm{e}^{\frac{1}{2}(t-t_0)\zeta}. \tag{5·10}$$

On the other hand, it follows, for any initial distribution of **F** and **G**, that if k is large enough the conduction terms in (5·4) and (5·5) are larger than the convection terms and $\overline{\mathbf{F}^2}$ and $\overline{\mathbf{G}^2}$ necessarily decrease with time.

In view of (2·9), and with the aid of the known (exact) relation $\epsilon = \nu \overline{\left(\frac{\partial u_i}{\partial x_j}\right)^2} = \nu \overline{\boldsymbol{\omega}^2}$, where $\omega = \nabla \times \mathbf{u}$ is the turbulent vorticity, the condition (5·9) can also be written as

$$(\overline{\boldsymbol{\omega}^2})^{\frac{1}{2}} \overset{*}{\lambda}{}^2/k \gg 1. \tag{5·11}$$

The meaning of (5·11) is simply that a dimensionless number analogous to a Reynolds number has to be large if convection is to be dominant, which is the kind of condition to be expected from the general form of the equations (5·6) and (5·7); what is not so immediately evident is that $\overset{*}{\lambda}$ and $\overset{*}{\lambda}(\overline{\boldsymbol{\omega}^2})^{\frac{1}{2}}$ are the representative length and velocity from which the number is formed.

6. The conditions under which amplification of F and G will occur

We have not yet introduced any assumption about the initial spectra of **F** and **G**, although it has now emerged as being relevant (through the quantity $\overset{*}{\lambda}$) to the relative importance of the convection and the conduction. It is doubtful if the initial spectra of **F** and **G** have more than a temporary importance, since the non-linear convection process represents an interaction between the distributions of **F** and **G** on the one hand and that of the velocity **u** on the other which will quickly alter the spectra of **F** and **G**. What is of interest is the general form—and in particular the value of $\overset{*}{\lambda}$—of the spectra of **F** and **G** after the convection has acted for long enough to produce a substantial change in the initial spectra of **F** and **G**.

It seems probable that the effect of the convection is to build up the spectrum of **F** (but not of **G**, for reasons to be given below) most strongly in the range of wave-numbers in which most of the turbulent rate of strain lies. To make the picture concrete, if the lines of **F** are initially fairly straight (i.e. if the length-scale of the initial distribution of **F** is large compared with the length-scale of the distribution of **u**) it is clear that the radii of curvature of the twists and turns that will develop in the line of **F** have a length-scale of the same order of magnitude as the wave-length of that Fourier component of the velocity distribution for which the density of contributions to the mean-square rate of strain (or, equivalently, in homogeneous turbulence, to the mean-square vorticity) is a maximum. Likewise, if many turns of small radius of curvature are present initially, these small-scale components of **F** will persist but will not be amplified so rapidly as those on the same scale as the vorticity.

If it be granted that the effect of the convection (if dominant over conduction) is to make the spectrum of **F** develop a maximum (of ever-increasing height) at wave-numbers near that at which the maximum of the vorticity spectrum lies, we can make an estimate of $\overset{*}{\lambda}$. Provided the vorticity spectrum (and the spectrum of **F**) falls off with sufficient rapidity beyond the maximum—as is now generally accepted as a consequence of the work of Kolmogoroff and others—the length $\overset{*}{\lambda}$ will be of the same order of magnitude as the wave-length at which the two spectral maxima lie. The maximum of the vorticity spectrum is known to lie, for turbulence at large Reynolds numbers, at a wave-length of the order of $(\nu^3/\epsilon)^{\frac{1}{4}}$, and hence

$$\overset{*}{\lambda} \sim (\nu^3/\epsilon)^{\frac{1}{4}} \qquad (6\cdot1)$$

at times not too close to t_0. The condition (5·11) thus becomes

$$\nu \gg k. \qquad (6\cdot2)$$

There is a simple way of looking at this interesting result. Vorticity is an example of the class of vector quantities represented by **F** and is the one whose molecular diffusivity (viz. ν) is such that the convective and conductive effects are necessarily in equilibrium; the turbulence adjusts itself until this condition is satisfied. Hence any other quantity in this class, being subject to the same convective effect as is vorticity, will be amplified if the appropriate diffusivity k is considerably less than ν and suppressed by conduction if k is considerably greater than ν. It is not easy to see what will happen when k is of the same order of magnitude as ν, since several unknown factors of order unity occur in the above argument, but the consequences of either small or large values of k/ν are clear.

Since the equation (5·4) is linear in **F**, and the above remarks all apply irrespective of the magnitude of **F**, it follows that when the diffusivity k satisfies (6·2) the fluid in turbulent motion is *unstable* to the introduction of small amounts of the quantity represented by **F**, in the sense that the magnitude of **F** will be amplified exponentially with respect to time; this possibility has already been considered (Batchelor 1950) with an argument along the above lines for the special case of magnetic field strength, which satisfies the equation (5·3) when the fluid is electrically conducting (the diffusivity k then being equal to $1/4\pi\mu\sigma$, where μ is the magnetic permeability and σ is the conductivity). Furthermore, the amplification will proceed indefinitely, unless some change in the assumed conditions occurs, since the criterion (5·11) or (6·2) will continue to be satisfied when the magnitude of **F** is large. In practice there will usually be some back reaction which restricts the ultimate magnitude of **F**. In the case of magnetic field strength **H**, the stresses exerted on the medium (which are of the second order in **H**) ultimately become large enough to modify the velocity distribution and to *prevent* the lines of **H** from being extended even though they continue to be material lines (apart from any effect of conduction); this is an interesting case in which some material lines are extended in the normal way, but others, owing to the selective effect of the electromagnetic stresses, retain the same average length. It was found possible in the paper cited to make an estimate of the ultimate equilibrium value of $\overline{\mathbf{H}^2}$.

The situation is different in the case of quantities represented by **G**. The convection has no effect on the value of $\overline{\theta^2}$, which can only be diminished by the conduction. The only possible way in which $\overline{\mathbf{G}^2}$ can be increased by the convection is for the (constant) area under the spectrum of θ to be rearranged—in general, to be shifted in the direction of larger wave-numbers—in such a way that its second moment is increased. Hence, if the initial spectrum of **G** is such that (5·11) is satisfied, the convection will dominate the changes in **G**, and $\overline{\mathbf{G}^2}$ will eventually increase exponentially (with exponent $\frac{1}{2}\zeta t$) as a result of a continual flow of the area under the spectrum of θ in the direction of larger wave-numbers. The length-scale of the distribution of **G** is thus continually decreasing (by contrast with that of **F** which does not change); kinematically this arises from the fact that neighbouring material surfaces approach each other indefinitely, whereas material lines lengthen through an increase in the total number of turns of a certain length-scale. The length $\overset{*}{\lambda}$ (for **G**) thus continually decreases, and there will come a time when the condition (5·11) ceases to be satisfied. At this stage the effect of conduction has been so enhanced by the convective breaking up of the temperature distribution into a fine-grained structure that it is able to reduce the value of $\overline{\mathbf{G}^2}$ as fast as convection increases it. There is again an instability in the sense that any small initial value of $\overline{\mathbf{G}^2}$ is amplified, provided the condition (5·11) is satisfied by the initial θ-perturbation, but the action of the amplification is such as to bring about its own ultimate cessation.

The length-scale of the ultimate distribution of **G** is not large enough to satisfy (5·11) and is therefore given by†

$$\overset{*}{\lambda} \sim (k^2/\overline{\omega^2})^{\frac{1}{4}} = (k^2\nu/\epsilon)^{\frac{1}{4}}. \qquad (6\cdot3)$$

In this ultimate equilibrium state, the spectrum of **G** should have a maximum at a wave-number of the order of magnitude of the reciprocal of the length (6·3) and should fall off rapidly thereafter. When k is of the order of magnitude of ν, the length (6·3) is of the order of the length-scale of the vorticity. In the case in which θ

† Obukhoff (1949) has come to a different conclusion. He assumes that k, ϵ and the rate at which $\overline{\theta^2}$ decreases are the only parameters determining the length-scale of **G**, so that on dimensional grounds he finds it is of the order $(k^3/\epsilon)^{\frac{1}{4}}$. The present author's opinion is that a dimensional argument is adequate but that the mean-square rate of strain, or $\overline{\omega^2}$, $= \epsilon/\nu$, should be used as the parameter representing the effect of the turbulence (for reasons that have been made clear in the body of this paper), with the result (6·3).

represents temperature, we have $\nu/k \approx 0{\cdot}8$ for air, so that the length-scales of vorticity and temperature gradient should (ultimately) be about equal (equivalently, energy dissipation and thermal dissipation should occur in the same range of wave-numbers), whereas for water we have $\nu/k \approx 9$ so that the length-scale of vorticity should here be about three times that of temperature gradient.

Unfortunately it does not seem possible by the same kind of argument to determine the equilibrium value of $\overline{\mathrm{G}^2}$ (the scale of θ is arbitrary, in view of the linearity of (5·1), but we should like to be able to determine the ratio $\overline{\mathrm{G}^2}/\overline{\theta^2}$).

References

Batchelor, G. K. 1949 *Aust. J. Sci. Res.* **2**, 437.
Batchelor, G. K. 1950 *Proc. Roy. Soc.* A, **201**, 405.
Batchelor, G. K. 1952 *Proc. Camb. Phil. Soc.* **48**, 345.
Obukhoff, A. M. 1949 *Bull. Acad. Sci. U.R.S.S.*, Geogr. Geofiz, **13**, 58.
Prim, R. & Truesdell, C. 1950 *Proc. Amer. Math. Soc.* **1**, 32.

Spectra of flames supported by free atoms

By A. G. Gaydon and H. G. Wolfhard
Chemical Engineering Department, Imperial College, London, S.W. 7

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[Plates 7 and 8]

The spectra of the 'atomic flames' of organic compounds with the products of discharges through oxygen, hydrogen and water vapour have been examined. Hydrocarbons give bright flames with water products but not, contrary to previous reports, with dry atomic hydrogen. Carbon tetrachloride and other organic halides and also carbon disulphide do give bright flames with hydrogen. The rotational intensity distribution in the OH band in most atomic flames corresponds to a very high temperature, around 8000° K; it is high for methyl alcohol with water products, but only around 2500° K for methyl alcohol with atomic oxygen. The flame of acetylene with products from heavy water, D_2O, gives mainly OD and CD rather than OH and CH bands. The mechanism of formation and excitation of CH and OH in flames is discussed. When iron carbonyl is introduced into atomic flames very high electronic excitation of iron atoms is observed, as in the reaction zones of ordinary premixed flames; this high excitation is also observed for the flame of carbon tetrachloride with atomic hydrogen.

Introduction

There are many unexplained effects which occur in the reaction zones of normal combustion processes. The method of formation of active species such as CH and C_2 is unknown, and the cause of the strong excitation of OH radicals in organic flames and the abnormally high effective rotational temperature which they show has not been definitely attributed to any simple process. The reaction zones also show a generally high level of excitation to high electronic states, and there is some evidence that the amount of ionization of the gases greatly exceeds that to be expected for thermal equilibrium. Many of these effects have been brought