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Effects of fluctuation on mean-field $\alpha - \Omega$ dynamos

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We analyse the role of a fluctuating α-effect in $\alpha - \Omega$ dynamo models. Numerical experiments show that if the alpha-effect is calculated from direct simulation of the governing MHD equations there are typically large fluctuations compared to the mean, even if the mean is zero. Earlier work has suggested that these fluctuations alone, in concert with the Omega-effect (differential rotation) can lead to dynamo action. Much of this other work has concentrated on reduced versions of the governing equations, but did not address various questions such as the effect of the new mechanism on the speed of dynamo waves and cycle times for models in realistic geometries. Previous developments involving the spatial distribution of the fluctuations were also unnecessarily cumbersome.

By means of a simple ansatz we show that there can be a mechanism for magnetic field generation, valid at large scale separation, deriving from the interaction between a mean shear and a fluctuating α-effect. An equivalent term can arise from the 'shear current effect'. The resulting equations, including the new term representing the mean effect of the fluctuations, are investigated in planar and spherical geometries. We show that the new effect can act as a dynamo even in the absence of a mean alpha-effect, and that the time-scale for dynamo waves is strongly affected by the presence of fluctuations, with the largest values of the fluctuations leading to steady dynamo action.

Introduction The mean field dynamo ansatz has been used for many years to produce tractable models of the solar dynamo. The outcome of the theory is a mean emf $\mathcal{E} = \boldsymbol{\alpha} : \boldsymbol{B}$, where $\boldsymbol{B}$ is the mean magnetic field, and $\boldsymbol{\alpha}$ is a pseudo tensor, which changes sign under reflection. The effect is commonly modelled as arising due to the helicity in small-scale turbulence, and in any case is not present when the turbulence has reflection-symmetric statistics. When incorporated into the induction equation for the mean field, the resulting new term (if well-defined) is guaranteed to lead to dynamo action on large enough length and time scales as it has fewer derivatives than the diffusive term. There are other more complicated mean field effects but these are not guaranteed to lead to dynamo action as the new term has the same number of derivatives as the diffusive term.

However it is very difficult to fit numerical calculations into the α-effect (mean field) formalism. When the magnetic Reynolds number on the small scale is large, as in astrophysical applications, we find that there are large fluctuations in the emf - even for non-helical flows. Recent calculations highlight the difficulties. Simulations of convection in a plane layer of Boussinesq fluid, rotating about a vertical axis, as in [4], might be expected to yield a large mean emf according to conventional ideas. But the mean emf in the top half of the layer is tiny compared to the rms value, as seen in Figure 1. Thus it makes sense to investigate whether fluctuations in the mean emf can act to promote dynamo action. Early work by [6] (see also [7], Ch. 7), considered an α^2 dynamo, with no mean flow, in which the α-effect exhibited fluctuations with zero mean, but the results were inconclusive. [13] showed that fluctuations of α in a simple $\alpha - \Omega$ dynamo model can lead to dynamo action, but did not attempt a systematic survey. An approach close to that of the present paper was adopted by [12]. After very lengthy and detailed analysis it was concluded that the effect of fluctuations was a term of α-effect type, derived from spatial variations of the fluctuations. However there is another effect (sometimes called the 'incoherent dynamo' as in [13]), which does not depend on such inhomogeneity. It may or may not act to enhance dynamo action, depending on the space-time spectrum of the fluctuations in the α-effect. A recent paper ([5]) indicates that, using first order smoothing, the new term enhances dynamo action only in special cases, but the general situation is unknown. A recent paper [1] demonstrates the existence of the effect in a numerical simulation. When it does enhance the dynamo process, it has an important effect upon the temporal period of the solutions in a finite geometry. In what follows we present a derivation of the new mean term, following [8] and [10], and give preliminary results concerning finite geometries, following [10], [3].

1. Derivation of model equations We start with a one-dimensional dynamo wave model, originally proposed by Parker and adapted by ([9]); let $\boldsymbol{B} = B(x,t)\boldsymbol{e}_y + \boldsymbol{\nabla} \times [A(x,t)\boldsymbol{e}_y]$ in a Cartesian geometry, where y represents the azimuthal direction, and x the North-South direction. We first assume that the domain has infinite extent in x. The governing equations can be written

$$A_t = \alpha B + \eta \left(A_{xx} - \ell^2 A\right) \tag{1}$$

$$B_t = \Omega' A_x + \eta \left(B_{xx} - \ell^2 B\right) \tag{2}$$

where η is the magnetic diffusivity, ℓ is an inverse lengthscale, and the subscripts x and t denote differentiation with respect to x and t, respectively. The $\Omega' A_x$ term represents the effect of large-scale shear, producing toroidal from poloidal field.

We let α be a sum of its mean and fluctuating parts, where the fluctuations vary on scales slower than the averaging process defined in the derivation of mean-field theory, but faster than the time and space scales for mean field evolution. To enforce the fact that the fluctuations are large, we introduce a small parameter ε so that

$$\alpha = \alpha_0 + \varepsilon^{-1}\alpha_1\left(\tau, \xi\right)$$

where the subscripts 0 and 1 represent the mean and fluctuating parts respectively, τ is the intermediate timescale, and ξ is the intermediate spatial scale. (We suppress any dependence on the global lengthscales x, t). To obtain the correct scalings, we set $\partial_t \to \partial_t + \varepsilon^{-1}\partial_\tau$, $\partial_x \to \partial_x + \varepsilon^{-\frac{1}{2}}\partial_\xi$. These fluctuations induce corrections to A and B and we write $A \to A_0 + A_1(\tau,\xi)$, $B \to B_0 + \varepsilon B_1(\tau, \xi)$. Substituting into 1,2) we obtain at leading order in ε

$$A_{1\tau} = \alpha_1 B_0 + \eta A_{1_{\xi\xi}} \tag{3}$$

$$B_{1\tau} = \Omega'(A_{1_x} + \underline{\varepsilon^{-\frac{1}{2}} A_{1\xi}}) + \eta B_{1_{\xi\xi}} \tag{4}$$

Though the underlined term appears to violate the scaling it can be shown that its net effect on the emf (after averaging over the short scale) is to produce an order unity contribution to the α-effect term. Similarly, any effect of the spatial variation of α_1 on the global scale x will also lead to a modification of the α-effect, as noted by [12] and so we also ignore such variations.

The equations (3,4) are solved by Fourier transforming in ξ and τ, denoting the transform of α_1 by $\tilde{\alpha}(k,\omega)$, and similarly for the other variables. Then we find (ignoring the underlined term and any x dependence of α_1)

$$\tilde{A} = \frac{\tilde{\alpha} B_0}{\mathrm{i}\omega + \eta k^2}, \quad \tilde{B} = \frac{\Omega' B_{0x}\tilde{A}}{\mathrm{i}\omega + \eta k^2} = \frac{\Omega' B_{0x}\tilde{\alpha}}{(\mathrm{i}\omega + \eta k^2)^2} \tag{5}$$

Finally, taking the average of (1) over the short length and time scales, and denoting this average by $\langle\cdot\rangle$, we obtain a new term of the form $\langle \alpha_1 B_1\rangle$, and the poloidal equation becomes

$$A_{0t} = \alpha_0 B_0 - G\Omega' B_{0x} + \eta\left(A_{xx} - \ell^2 A\right), \text{ where } G = \left\langle \frac{|\hat{\alpha}(k,\omega)|^2(\omega^2 - \eta^2 k^4)}{(\omega^2 + \eta^2 k^4)^2} \right\rangle \quad (6)$$

In the equation for G the averages are now taken over Fourier space. The new term in G can plainly take either sign, depending on the spectrum of α_1. Dynamo action is only enhanced when $G > 0$. This term is the same in form as can arise from off-diagonal terms in the turbulent diffusivity, otherwise known as the shear-current effect [11].

We can also apply very similar analysis to mean-field dynamos in a spherical geometry (see [8] for more details). Consider an axisymmetric mean field $\boldsymbol{B} = B(r,\theta)\mathbf{e}_\phi + \boldsymbol{B}_p$ (where $\boldsymbol{B}_p \equiv \boldsymbol{\nabla}\times[A(r,\theta)\mathbf{e}_\phi]$) in spherical polar coordinates (r,θ,ϕ). The only mean flow is that of zonal shear with differential rotation $\Omega(r,\theta)$. There is an α-effect proportional to B, of the form $\alpha(r,\theta,t)B$. The induction equation then takes the form for an $\alpha\Omega$ dynamo,

$$\frac{\partial A}{\partial t} = \alpha B + \eta D^2 A, \quad (7)$$

$$\frac{\partial B}{\partial t} = r\sin\theta \boldsymbol{B}_p \cdot \nabla\Omega + \eta D^2 B, \quad (8)$$

where $D^2 = \nabla^2 - 1/r^2\sin^2\theta$. Applying exactly analogous methods as for the Cartesian case, we obtain a modified set of equations with (8) unchanged, and (7) replaced by (dropping the suffix 0):

$$\frac{\partial A}{\partial t} = \alpha B + \langle\alpha_1 B_1\rangle + \eta D^2 A, \text{ where} \quad (9)$$

$$\langle\alpha_1 B_1\rangle = -(G\nabla(r\sin\theta B_0)\times\mathbf{e}_\phi + \frac{1}{2}\nabla G \times r\sin\theta B_0\mathbf{e}_\phi)\cdot\nabla\Omega \quad (10)$$

2. Marginal Dynamo Waves In an infinite domain, equations (6,2) can now be solved to find marginal solutions of the form $A, B \propto \exp i(kx+\omega t)$ [8],[10]. Substituting this ansatz into equations (6,2) yields the dispersion relation

$$\left(i\omega + \eta(k^2+\ell^2)\right)^2 = G\Omega'^2 k^2 + i\Omega'\alpha_0 k. \quad (11)$$

The imaginary part is $\omega = \Omega'\alpha_0 k/2\eta(k^2+\ell^2)$ (so that the dynamo waves travel if $\Omega'\alpha_0 \neq 0$), while the real part can be written in dimensionless form. If we write $k = \ell m$, and define

$$\mathcal{D} = \frac{\Omega'\alpha_0}{\eta^2\ell^3}, \quad \mathcal{Q} = \frac{G\Omega'^2}{\eta^2\ell^2}, \quad (12)$$

then $\omega = \eta\ell^2\mathcal{D}m/2(1+m^2)$ and $\mathcal{D}$, $\mathcal{Q}$ and m are related by

$$\mathcal{Q}m^2 + \mathcal{D}^2\frac{m^2}{4(1+m^2)^2} = (1+m^2)^2. \quad (13)$$

The envelope of this family gives the marginal stability boundary; we can show that this is given parametrically by $\mathcal{D}^2 = 2m^{-4}(1+m^2)^4(1-m^2)$, $\mathcal{Q} = \frac{1}{2}m^{-4}(1+m^2)^2(3m^2-1)$. When $\mathcal{Q} = 0$ we have the usual $\alpha\Omega$ dynamo. Dynamo action is possible when $|\mathcal{D}| \geq 32/3\sqrt{3}$, with equality when $m = 1/\sqrt{3}$. Even when $\mathcal{D} = 0$, we can find dynamo action when $\mathcal{Q} \geq 4$, with equality when $m = 1$. Although in this case $\omega = 0$ so that the wave does not travel, the example shows that the fluctuations in α alone can lead to growing magnetic fields, even when there is no mean emf. The stability boundary is shown in Figure 2.

3. Numerical results for waves in a finite domain In order to understand how a finite geometry will affect the time dependence of non-linear solutions, consider a simple one-dimensional model. This is related to the Parker model above, but now we take our x-domain to be $0 < x < l$, so that $x = l/2$ represents the 'equator'. We take α to be antisymmetric about the equator, while we take the new term (which on physical grounds should be even about the equator) to be uniform. A and B obey zero boundary conditions at $x = 0, l$, and we set $\Omega' = \ell = \eta = 1$. Nonlinear effects are represented by a simple quenching term. Our equations then take the form (with r, d positive constants)

$$\frac{\partial A}{\partial t} = -\frac{rB_x + d\sin(2\pi x/l)B}{1+B^2} + A_{xx} - A, \qquad (14)$$

$$\frac{\partial B}{\partial t} = A_x + B_{xx} - B. \qquad (15)$$

It was shown in [8] that for large enough r, solutions for any d become steady in this geometry. A careful numerical investigation by [10] for solutions with dipole parity (with A even and B odd about $x = l/2$) gives the regions of r, d space where steady and oscillatory solutions are stable (no aperiodic solutions were found for the particular value $l = 10$). The results are shown in Figure 3.

It is interesting that dipolar solutions are not always selected. In fact depending on the parameters either dipole, quadrupole or mixed mode solutions may be stable. Here we show some preliminary results: more details are given in [10]. In Figure 4 are shown the regions of r, d space where different types of solution may be found for $l = 10$, and blow-up of the regions marked $(a), (b)$ in the left panel. Further work is needed to pin down the very complex boundaries between different behaviours, and to investigate the dependence on l.

4. Numerical simulations in spherical geometry Finally we give some results from fully nonlinear mean-field dynamo simulations in a spherical shell. The model used is that developed by Bushby [2] (in which full details may be found) and features a fully resolved axisymmetric dynamo calculation, with a prescribed differential rotation mimicking flow in the tachocline. The model is augmented by the addition of the new term given by equation (10), and there is a simple quenching term on the lines of the nonlinearity in (14), rather than a dynamical interaction with the shear as used in [2]. Here we give some preliminary results from the model; full details will be given in [3]. Two sets of plots are given here: in the first set the fluctuation term, G, is taken to be proportional to the magnitude of α_0^2 (so that the ratio of the mean α to the rms α is uniform). In Figure 5 are shown greyscale plots of the toroidal field as a function of latitude and time at the base of the convection zone, for various values of the fluctuation amplitude. The results resemble those for the one-dimensional model given above: the period lengthens and if the fluctuations are sufficiently large a steady dynamo results. In Figure 6 are shown runs for fluctuations that are proportional to the magnetic diffusivity η (which in the model falls off rapidly at the base of the convection zone) but independent of the size of α_0, Now as the amplitude is increased we first see variations in the butterfly diagram that are predominantly restricted to high latitudes (where α_0 is small but the shear $d\Omega/dr$ is large). In the 0.002 case, there are clear longer period oscillations at high latitudes, and there is a small increase in period at low latitudes. The steady mode appears abruptly in this case, due to the long-period high latitude oscillations becoming dominant when the fluctuation parameter reaches 0.003. Further runs are clearly needed to determine more carefully the reason for these two different approaches to steady dynamo action.

5. Conclusions In this paper we have shown how incorporating the effects of fluctuations of the α-effect in the presence of shear can lead to a new type of mean field term that can act to promote dynamo action. This term can also arise as a consequence of the shear-current effect [11]. We also present calculations for three different models to show the effect of the new term. A principal conclusion is that the new effect can have profound consequences for the period of any cyclic dynamo, and thus that any estimates of cycle periods derived by assuming no fluctuations in α must be regarded with caution.

Further work is in progress to fully explore the consequences of the new term in realistic mean field models, and will appear in the forthcoming papers [10],[3].

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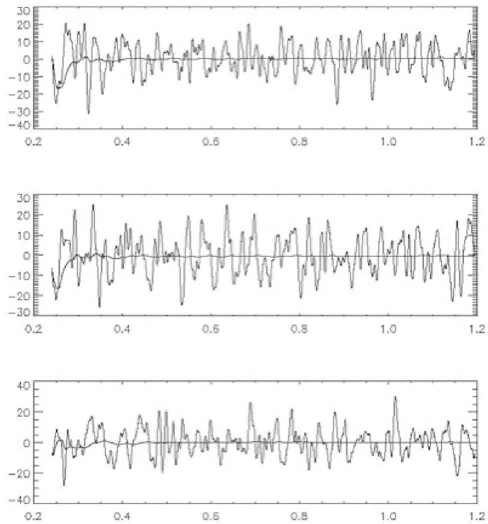

Figure 1: Time series showing the behaviour of the mean emf (averaged in the horizontal and over the top half of the layer) for the rotating convection simulation of Cattaneo & Hughes 2006). The three plots show the measured emf in each of the three coordinate directions (z is vertical). The heavier line shows the running time-average. The imposed magnetic field is in the x direction.

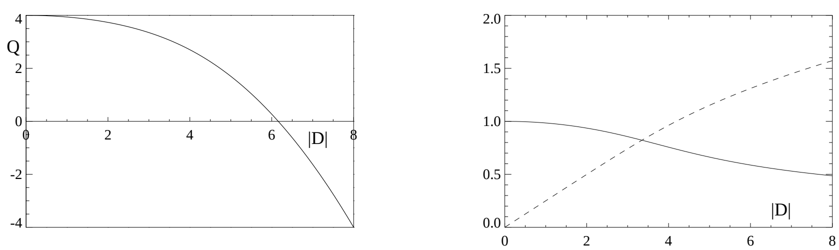

Figure 2: Left panel: relation between $|\mathcal{D}|$ and $\mathcal{Q}$ for marginal stability. Dynamo action is possible above the curved line. Right panel: variation of spatial wavenumber m and scaled frequency $\omega/\eta\ell^2$ (dashed) as a function of $|\mathcal{D}|$ along the marginal curve.

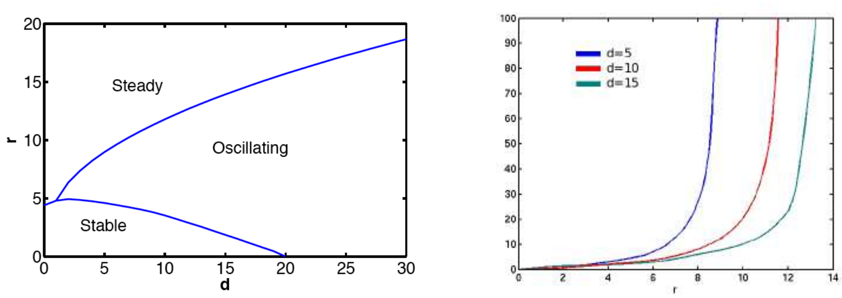

Figure 3: Left panel: regions of r, d space where various forms of solutions to (14,15) can be found, with $l = 10$. Right panel: variation of the periods of the solutions with r for various values of d.

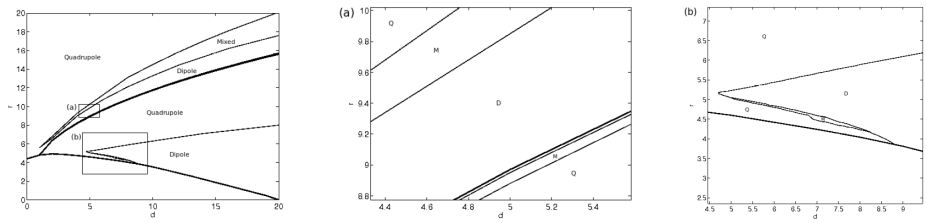

Figure 4: Left panel: regions of r, d space where dipole, quadrupole and mixed mode solutions are stable, for $l = 10$. Centre and right panels: blowups of the region (a) and (b) in the left hand figure.

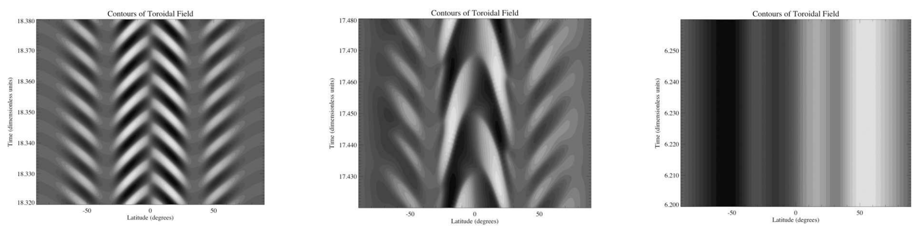

Figure 5: Greyscale plots of the toroidal field at the base of the convection zone for the spherical model with $G = \beta\alpha_0^2$. From left to right:$\beta = 0.005, 0.015, 0.0175$.

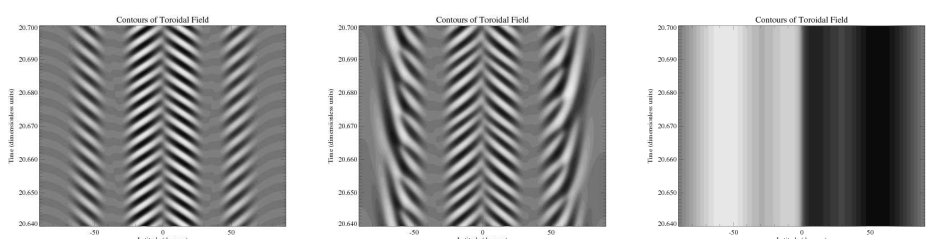

Figure 6: Greyscale plots of the toroidal field at the base of the convection zone for the spherical model with G proportional to the r-dependent magnetic diffusivity η. From left to right:$\max G = 0.001, 0.002, 0.003$.