The magneto-hydrodynamics of a rotating fluid and the earth's dynamo problem

By J. B. Taylor

U.K.A.E.A., Culham Laboratory, Abingdon, Berkshire

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This paper discusses a rotating, incompressible fluid enclosed within a rigid boundary which is a surface of revolution. It is shown that if viscous forces are negligible, then, in the presence of magnetic fields, the fluid can execute slow, steady relative motions only if the magnetic force satisfies a constraint. In cylindrical polar co-ordinates this constraint can be written

$$\int_{r=r_0} (\mathbf{j}\times\mathbf{B})_\phi \, \mathrm{d}\phi \, \mathrm{d}z = 0;$$

that is, the couple exerted by the magnetic forces on any cylinder of fluid coaxial with the axis of rotation must vanish.

Furthermore, subject to certain restrictions on the shape of the container (which, for example, are fulfilled by a sphere but not by a cylinder), it is shown that if the field satisfies the above condition then the fluid velocity is completely determined by the instantaneous value of the magnetic field (together with that of the density if buoyancy forces are important). This velocity is such that the necessary conditions on the field will continue to be satisfied. An algorithm for the determination of the velocity is given and its application to the earth's dynamo problem is indicated.

1. Introduction

This paper is concerned with the influence of magnetic forces on the behaviour of a rotating, incompressible, conducting fluid. It is well known (Proudman 1916; Taylor 1921) that in the absence of external forces the slow relative motions of a rotating, inviscid fluid are confined to planes at right angles to the axis of rotation. Bullard & Gellman (1954) have pointed out that if the rotating fluid is contained in a rigid spherical container then the restriction on the motion is even more severe; in fact the only slow free motions in a rotating fluid contained in a rigid spherical envelope are those in which cylindrical shells rotate like rigid bodies about the axis of the main rotation.

The present discussion concerns theorems of a similar nature to that of Proudman and Taylor, but which refer to the influence of external forces, such as the Lorentz ($\mathbf{j}\times\mathbf{B}$) force in a conducting fluid. Specifically the present results will be derived in the context of the earth's dynamo problem since this is an application in which they may be particularly useful.

(*a*) *The dynamo problem*

It is widely believed that the magnetic field of the earth is produced by a self-excited dynamo action resulting from convection in a rotating, conducting fluid such as the earth's core. (See, for example, the recent review by Hide & Roberts (1961) which contains references to earlier work, in particular that of Bullard and of Elsasser.)

One approach (Bullard & Gellman 1954) to the problem of calculating this dynamo action is to take the fluid velocity, $\mathbf{v}$, as given and to compute from it the magnetic field, $\mathbf{B}$, by the equation

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{B}) + \frac{\eta}{4\pi}\nabla^2\mathbf{B}, \tag{1·1}$$

where η is the resistivity of the core. In this approach the dynamics of the motion is entirely ignored.

A more complete procedure would be to solve equation (1·1), together with the equations of motion, continuity and heat generation, for $\mathbf{v}$ and $\mathbf{B}$ simultaneously. It is customary to use a model in which the changes in fluid density depend only on temperature and the problem can then be reduced to one of incompressible flow by the Boussinesq approximation (Chandrasekhar 1961; Jeffreys 1930; Spiegel & Veronis 1960). In a co-ordinate system rotating with the earth at angular speed $\mathbf{\Omega}$ the equations can then be written

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t} = -\nabla p' + (\mathbf{j}\times\mathbf{B}) + \rho'\nabla\phi - 2\rho(\mathbf{\Omega}\times\mathbf{v}) + \rho\nu\nabla^2\mathbf{v}, \tag{1·2}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{1·3}$$

$$\frac{\partial \rho'}{\partial t} = -(\mathbf{v}\,.\,\nabla)\,\rho' - S + \kappa\nabla^2\rho', \tag{1·4}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{B}) + \frac{\eta}{4\pi}\nabla^2\mathbf{B}, \tag{1·5}$$

where $\mathrm{D}/\mathrm{D}t$ denotes $\partial/\partial t + (\mathbf{v}\,.\,\nabla)$.

In these equations $\rho'\nabla\phi$ is the buoyancy force due to changes, ρ', in fluid density and the potential ϕ includes both gravitational and centrifugal forces

$$\nabla\phi = \mathbf{g} + \nabla[\tfrac{1}{2}(\mathbf{r}\times\mathbf{\Omega})^2]. \tag{1·6}$$

Equation (1·4) is derived from the equation for temperature; S corresponds to a source of heat and κ is the heat diffusivity.

The problem of the earth's dynamo, therefore, is one incentive for a study of the motion of incompressible, rotating fluids under electromagnetic and buoyancy forces and we now turn to a discussion of this problem as defined by equations (1·2) to (1·5).

(*b*) *Slow motions*

The solution of the full equations (1·2) to (1·5) is a formidable task, even for a modern computer, because they admit the possibility of rapid oscillatory motion. For many purposes, such as the dynamo problem, these rapid oscillations are unimportant and one therefore tries to set up equations which adequately represent the slow, long term, behaviour of the fluid but which eliminate rapid oscillations. Fortunately in the case of motion in the earth's core the inertial $\rho\,\mathrm{D}\mathbf{v}/\mathrm{D}t$ and viscous force $\nu\nabla^2\mathbf{v}$ are small (Bullard 1949*a*; Hide 1956) and may be neglected. This is the approximation of 'slow' motion which is also made in the Proudman–Taylor theorem.

Such 'slow' relative motion is governed by the equations

$$2\rho(\mathbf{\Omega}\times\mathbf{v}) = (\mathbf{j}\times\mathbf{B}) - \nabla p' + \rho'\nabla\phi, \tag{1·7}$$

$$\operatorname{div}\mathbf{v} = 0, \tag{1·8}$$

$$\frac{\partial\mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{B}) + \frac{\eta}{4\pi}\nabla^2\mathbf{B}, \tag{1·9}$$

$$\frac{\partial\rho'}{\partial t} = -(\mathbf{v}\,.\,\nabla)\rho' + S + \kappa\nabla^2\rho'. \tag{1·10}$$

The question now arises: under what conditions can these equations have physically meaningful solutions? We shall prove two theorems relating to this question; the second provides an algorithm for the solution of (1·7) to (1·10) and might form the basis for a more realistic attack on the dynamo problem than any hitherto made.

THEOREM 1. *If the rotating fluid is contained in a rigid envelope in the form of a surface of revolution then for there to exist a velocity* $\mathbf{v}$ *compatible with our equations it is necessary for the magnetic field to satisfy a constraint which can be written, in cylindrical polar co-ordinates* (r, ϕ, z):

$$\int_{r=\text{constant}} (\mathbf{j}\times\mathbf{B})_\phi\,\mathrm{d}\phi\,\mathrm{d}z = 0, \tag{1·11}$$

i.e. the couple on any annular cylinder parallel to the axis of rotation must vanish.

If this constraint is not satisfied then rapid motions occur (in which the acceleration term $\mathrm{D}\mathbf{v}/\mathrm{D}t$ cannot be neglected) until such time as the couples producing this motion satisfy (1·11). The possibility of the magnetic field producing rapid motions was noted by Bullard (1949*a*, *b*).

THEOREM 2. *The constraint* (1·11) *is not only necessary for the existence of* $\mathbf{v}$ *but is also sufficient, at least if the bounding surface is a sphere or any surface of revolution in which the normal directions at each end of a line parallel to the axis are not themselves parallel.* (*This restriction excludes a cylindrical container.*)

By sufficient we mean that if one is given a density ρ' and a magnetic field which satisfies (1·11) then the fluid velocity is completely determined by our approximate equations (1·7) to (1·10); also this velocity ensures that (1·11) will continue to be satisfied. In demonstrating this we will be providing an algorithm for the determination of $\mathbf{v}$ in terms of $\mathbf{B}$ and ρ', showing that this involves only space quadrature and the solution of an ordinary differential equation. For the dynamo problem the importance of this result lies in the fact that, once $\mathbf{v}$ is expressed in terms of $\mathbf{B}$ and ρ', the time dependence of the problem is then contained only in the simpler equations (1·9) and (1·10).

2. NECESSARY CONDITION FOR SOLUTION

To derive the necessary condition (1·11) we first abbreviate equations (1·7) and (1·8) to

$$2\rho(\mathbf{\Omega}\times\mathbf{v}) = \mathbf{A} - \nabla p', \tag{2·1}$$

where $\mathbf{A} \equiv (\mathbf{j}\times\mathbf{B}) + \rho'\nabla\phi$, and $\operatorname{div}\mathbf{v} = 0.$ (2·2)

Introduce cylindrical polar co-ordinates (r, ϕ, z), the z-axis being the axis of rotation and consider a cylinder of radius r_0 co-axial with the axis of rotation (figure 1).

Let us integrate the ϕ-component of equation (2·1) over that part of the cylinder which lies within the fluid (for simplicity we consider the fluid to be enclosed in a sphere but it will be apparent that the arguments of this section can be extended to apply to any surface of revolution). Then

$$2\rho \int_{r=r_0} (\mathbf{\Omega} \times \mathbf{v})_\phi \, r \, \mathrm{d}\phi \, \mathrm{d}z = \int_{r=r_0} r A_\phi \, \mathrm{d}\phi \, \mathrm{d}z \tag{2·3}$$

or

$$2\rho\Omega \int_{r=r_0} r v_r \, \mathrm{d}\phi \, \mathrm{d}z = \int_{r=r} r A_\phi \, \mathrm{d}\phi \, \mathrm{d}z, \tag{2·4}$$

which can be written

$$2\rho\Omega \int_{r=r_0} \mathbf{v} . \mathbf{dS} = \int r A_\phi \, \mathrm{d}\phi \, \mathrm{d}z. \tag{2·5}$$

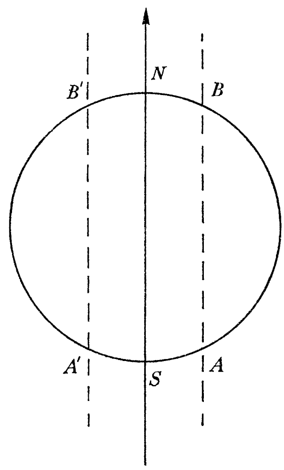

FIGURE 1.

At this stage the integral on the left of this equation is only over the cylindrical surface, but since the normal component of $\mathbf{v}$ is zero over the spherical caps $B'NB$ and $A'SA$ we can extend the integral to cover the *closed* surface formed by the cylinder plus the polar caps: then using Gauss's theorem,

$$2\rho\Omega \int (\operatorname{div} \mathbf{v}) \, \mathrm{d}\tau = \int_{r=r_0} r A_\phi \, \mathrm{d}\phi \, \mathrm{d}z \tag{2·6}$$

and since $\operatorname{div} \mathbf{v} = 0$ we obtain a necessary condition for a solution of (2·1) and (2·2), namely

$$\int_{r=r_0} A_\phi \, \mathrm{d}\phi \, \mathrm{d}z \equiv 0, \tag{2·7}$$

or finally,

$$\int_{r=r_0} (\mathbf{j} \times \mathbf{B})_\phi \, \mathrm{d}\phi \, \mathrm{d}z = 0, \tag{2·8}$$

since the buoyancy force has no ϕ-component.

This condition can conveniently be interpreted as stating that the couple on any annular cylinder co-axial with the axis of rotation must be zero.

3. Sufficient conditions for solution

We now show that if the condition (2·8) *is* satisfied then the equations

$$2\rho(\mathbf{\Omega}\times\mathbf{v}) = -\nabla p' + (\mathbf{j}\times\mathbf{B}) + \rho'\nabla\phi \tag{3·1}$$

and

$$\operatorname{div}\mathbf{v} = 0, \tag{3·2}$$

together with the boundary condition $(\mathbf{v}.\mathbf{n}) = 0$ on the surface of a sphere, are just sufficient to determine $\mathbf{v}$ when ρ' and $\mathbf{B}$ are given. We prove this by giving an explicit demonstration of the manner in which $\mathbf{v}$ might be calculated.

Taking the curl of equation (3·1) we obtain

$$\mathrm{d}\mathbf{v}/\mathrm{d}z = \operatorname{curl}\mathbf{A}, \tag{3·3}$$

where a constant factor $-2\rho\Omega$ has now been absorbed into the definition of $\mathbf{A}$ $(-2\rho\Omega\mathbf{A} \equiv \mathbf{j}\times\mathbf{B}+\rho'\nabla\phi)$. It will now be shown that this equation possesses a unique solution when $\mathbf{A}$ is given and condition (2·8) is satisfied.

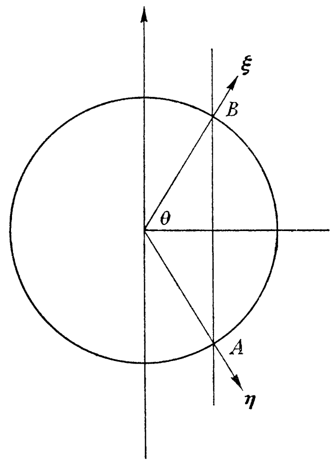

Figure 2.

Consider a line parallel to the axis of rotation, intersecting the fluid boundary at A, B (figure 2). Introduce as axes the normal directions at B and A (by defining unit vectors $\boldsymbol{\xi}, \boldsymbol{\eta}$), then we can resolve any vector into components in the $\boldsymbol{\xi}, \boldsymbol{\eta}$ directions, thus

$$v_\xi = v_r\cos\theta + v_z\sin\theta, \tag{3·4}$$

$$v_\eta = v_r\cos\theta - v_z\sin\theta, \tag{3·5}$$

where $+\theta$ and $-\theta$ are the latitudes of B and A, respectively.

Then we can write

$$v_r = \frac{v_\xi+v_\eta}{2\cos\theta}, \quad v_z = \frac{v_\xi-v_\eta}{2\sin\theta}, \tag{3·6}$$

and the boundary conditions are $v_\xi = 0$ at B; $v_\eta = 0$ at A. By using (3·3) v_ξ and v_η may be expressed as integrals along AB from B and A, respectively, then from (3·6) one finds

$$\left.\begin{aligned} v_r &= \frac{1}{2\cos\theta}\left\{\int_B^z (\operatorname{curl}\mathbf{A})_\xi\,\mathrm{d}z + \int_A^z (\operatorname{curl}\mathbf{A})_\eta\,\mathrm{d}z\right\}, \\ v_z &= \frac{1}{2\sin\theta}\left\{\int_B^z (\operatorname{curl}\mathbf{A})_\xi\,\mathrm{d}z - \int_A^z (\operatorname{curl}\mathbf{A})_\eta\,\mathrm{d}z\right\}. \end{aligned}\right\} \tag{3·7}$$

Since $\mathbf{A}$ is given, these equations clearly define v_r and v_z along AB and the construction may be repeated until v_r and v_z are known everywhere. This fact does not depend on a spherical container; a similar but more complicated construction would be possible if the container were any surface of revolution *provided that the directions* $\boldsymbol{\xi}, \boldsymbol{\eta}$ *were not parallel*. It is only necessary, therefore, for the container to be a surface of revolution in which the normals at B, A are not parallel. Thus any surface which is everywhere convex is permissible but a cylinder (which has plane ends) is excluded. In a spherical container the axis is a particular line on which $\boldsymbol{\xi}, \boldsymbol{\eta}$ are parallel but this causes no difficulty as the construction can be continued arbitrarily closely to the axis; it can also be shown that the formulae (3·8) and (3·9) below lead to properly behaved velocities as $r \to 0$.

We can express $(\text{curl}\,\mathbf{A})_\xi$ and $(\text{curl}\,\mathbf{A})_\eta$ in terms of cylindrical co-ordinates by means of equations similar to (3·4), (3·5); then

$$v_r = \frac{1}{2}\left\{\int_A^z + \int_B^z (\text{curl}\,\mathbf{A})_r\,\mathrm{d}z - \tan\theta \int_A^B (\text{curl}\,\mathbf{A})_z\,\mathrm{d}z\right\}, \tag{3·8}$$

with similar expression for v_z,

$$v_z = \frac{1}{2}\left\{\int_A^z + \int_B^z (\text{curl}\,\mathbf{A})_z\,\mathrm{d}z - \cot\theta \int_A^B (\text{curl}\,\mathbf{A})_r\,\mathrm{d}z\right\}. \tag{3·9}$$

It might appear at this point that v_r is not well behaved as $r \to 0$ (and thus $\tan\theta \to \infty$). However, by considering the necessary condition (2·7) as $r \to 0$ it can be shown that it leads to

$$\int_A^B (\text{curl}\,\mathbf{A})_z\,\mathrm{d}z \to 0$$

as $r \to 0$, in just such a way that v_r has the correct behaviour as $r \to 0$.

By using equation (3·3) and the boundary conditions, then, we have found v_r and v_z; however, we cannot find v_ϕ in a similar way because the boundary conditions do not involve v_ϕ. Instead we attempt to find v_ϕ from the equation $\text{div}\,\mathbf{v} = 0$, that is

$$\frac{1}{r}\frac{\partial}{\partial\phi}v_\phi = -\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial}{\partial z}v_z\right\}. \tag{3·10}$$

We have already found v_r, v_z so the right-hand side is now known, but (3·10) will possess a single-valued solution v_ϕ *if and only if*

$$\oint\left\{\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{\partial v_z}{\partial z}\right\}\mathrm{d}\phi = 0, \tag{3·11}$$

where v_r, v_z are to be considered as given by (3·8), (3·9). It is convenient to rewrite (3·11) as

$$Q \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\oint v_r\,\mathrm{d}\phi\right) + \oint\frac{\partial v_z}{\partial z}\,\mathrm{d}\phi = 0 \tag{3·12}$$

and equation (3·3) (which v_z satisfies by definition) gives

$$\oint\frac{\partial v_z}{\partial z}\,\mathrm{d}\phi = \oint(\text{curl}\,\mathbf{A})_z\,\mathrm{d}\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma) \tag{3·13}$$

where we have introduced $\qquad \Gamma \equiv \oint A_\phi\,\mathrm{d}\phi.$

Furthermore, from equation (3·8) we obtain

$$\oint v_r\,\mathrm{d}\phi = \frac{1}{2}\left\{\int_A^z + \int_B^z -\frac{\partial\Gamma}{\partial z}\,\mathrm{d}z - \tan\theta\int_A^B \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma)\,\mathrm{d}z\right\} \tag{3·14}$$

and therefore from (3·12) to (3·14),

$$Q = \frac{1}{r}\frac{\partial}{\partial r}r\left[\frac{\Gamma_A+\Gamma_B}{2} - \tan\theta\int_A^B \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma)\,\mathrm{d}z\right]. \tag{3·15}$$

We now observe that

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_A^B r\Gamma\,\mathrm{d}z = \int_A^B \frac{1}{r}\frac{\partial}{\partial r}(r\Gamma)\,\mathrm{d}z + \Gamma_B\left(\frac{\partial z}{\partial r}\right)_B - \Gamma_A\left(\frac{\partial z}{\partial r}\right)_A, \tag{3·16}$$

where the additional terms on the right come from variation of the length of the path AB with r; also

$$\left(\frac{\partial z}{\partial r}\right)_B = -\cot\theta, \quad \left(\frac{\partial z}{\partial r}\right)_A = +\cot\theta$$

so that finally

$$Q = -\frac{1}{r}\frac{\partial}{\partial r}\left\{r\left[\tan\theta\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\int_A^B r\Gamma\,\mathrm{d}z\right]\right\}. \tag{3·17}$$

However, by definition,

$$\int_A^B r\Gamma\,\mathrm{d}z \equiv \int_{r=\text{constant}} rA_\phi\,\mathrm{d}\phi\,\mathrm{d}z \tag{3·18}$$

and we have previously shown that this quantity *necessarily* vanishes. We see now that the vanishing of (3·18) is also a *sufficient* condition for Q to vanish and therefore for equation (3·10) to have a single-valued solution v_ϕ.

We have thus far shown, then, that if (2·8) is satisfied we can solve equation (3·3) for v_r and v_z and we can solve (3·10) for v_ϕ. The solution of this last equation contains an arbitrary function $u_\phi(r, z)$, but (3·3) determines the z-dependence of v_ϕ so we are left only with an indeterminate function $u_\phi(r)$. At this stage therefore we have

$$\mathbf{v} = (v_r, v_\phi + u_\phi(r), v_z), \tag{3·19}$$

where v_r, v_ϕ, v_z are now explicitly known, but $u_\phi(r)$ is still to be determined.

4. Determination of $u_\phi(r)$

In this section we complete the determination of $\mathbf{v}$ by determining $u_\phi(r)$, the only remaining unknown. In fact we show that u_ϕ is determined when we require that (2·8) shall be satisfied not only at a given instant but for all time, that is we now demand that as $\mathbf{B}$ develops according to equation (1·9) the necessary constraint (2·8) shall continue to apply. For this it is sufficient that

$$\frac{\partial}{\partial t}\int_{r=r_0} (\mathbf{j}\times\mathbf{B})_\phi\,\mathrm{d}\phi\,\mathrm{d}z = 0, \tag{4·1}$$

or

$$\int\left[\left(\operatorname{curl}\frac{\partial \mathbf{B}}{\partial t}\right)\times\mathbf{B}\right]_\phi \mathrm{d}\phi\,\mathrm{d}z + \int\left[(\operatorname{curl}\mathbf{B})\times\frac{\partial \mathbf{B}}{\partial t}\right]_\phi \mathrm{d}\phi\,\mathrm{d}z = 0, \tag{4·2}$$

where

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{B}) + \frac{\eta}{4\pi}\nabla^2\mathbf{B}. \tag{4·3}$$

Now $\mathbf{v}$ consists of a known vector $\mathbf{v}^*(=v_r, v_\phi, v_z)$ and an unknown part $(0, u_\phi(r), 0)$ which we will denote by $\mathbf{u}$. It is obvious that if we insert $\mathbf{v} = \mathbf{v}^* + \mathbf{u}$ into equation (4·3) to obtain $\partial\mathbf{B}/\partial t$ and then substitute this into (4·2) the resulting equation will be of the form

$$\int[(\mathrm{curl}^2(\mathbf{u}\times\mathbf{B}))\times\mathbf{B}]_\phi\,\mathrm{d}\phi\,\mathrm{d}z + \int[(\mathrm{curl}\,\mathbf{B})\times\mathrm{curl}\,(\mathbf{u}\times\mathbf{B})]_\phi\,\mathrm{d}\phi\,\mathrm{d}z = \int G(r,\phi,z)\,\mathrm{d}\phi\,\mathrm{d}z, \tag{4·4}$$

where $G(r, \phi, z)$ is a function which depends only on the known part $\mathbf{v}^*$ of $\mathbf{v}$ and on $\mathbf{B}$, and is therefore itself a known function. At this stage in the argument, then, the right-hand side of equation (4·4) is a known function of r, say $\bar{G}(r)$. Furthermore, since $\mathbf{u}$ has only a ϕ-component and varies only with r, the left-hand side of (4·4) can be evaluated and the equation then becomes

$$\alpha(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{u_\phi}{r}\right) + \beta(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_\phi}{r}\right) = \bar{G}(r), \tag{4·5}$$

where
$$\alpha(r) = \int rB_r^2\,\mathrm{d}\phi\,\mathrm{d}z, \tag{4·6}$$

$$\beta(r) = \int\left(\mathbf{B}\,.\,\nabla B_r + 2\frac{B_r^2}{r}\right)\mathrm{d}\phi\,\mathrm{d}z. \tag{4·7}$$

We have now shown that $u_\phi(r)$ must satisfy a second-order, inhomogeneous, ordinary differential equation with known coefficients. Thus we have finally determined $u_\phi(r)$ apart from the two scalar constants which must appear in the solution of (4·5).

It is clear that one solution of the homogeneous part of (4·5) is just

$$u_\phi(r) = r\Omega^*, \tag{4·8}$$

where Ω^* is an arbitrary constant. One of the arbitrary constants in the full solution of (4·5) therefore corresponds to the arbitrariness in our original choice of the velocity of rotation Ω of our co-ordinate system. In practice it would be fixed by (say) the requirement that the angular momentum be zero in the rotating co-ordinate system. The final arbitrary constant is fixed by the condition that $v_\phi(r)/r$ be finite as $r \to 0$.

We have now shown that if (2·8) is satisfied, and continues to be satisfied, the velocity $\mathbf{v}$ is completely determined by equations (3·1) and (3·2) and the boundary conditions.

5. Discussion

It has been shown that a rotating, conducting fluid can execute slow relative motions only if the condition

$$\int_{r=\text{constant}} (\mathbf{j}\times\mathbf{B})_\phi\,\mathrm{d}\phi\,\mathrm{d}z = 0 \tag{5·1}$$

is satisfied, i.e. the couple exerted by the magnetic forces on any cylinder of fluid co-axial with the axis of rotation must vanish.

Furthermore, once this condition has been satisfied then the instantaneous fluid velocity is specified by the instantaneous values of magnetic field $\mathbf{B}$ and density ρ', together with the boundary conditions of vanishing normal velocity over a sphere (or other surface of revolution satisfying the conditions in theorem 2), apart from an arbitrary azimuthal velocity $u_\phi(r)$. This azimuthal velocity is itself determined by the requirement that (5·1) continue to be satisfied as $\mathbf{B}$ develops according to the appropriate electromagnetic equation (1·9), for then it is found that $u_\phi(r)$ must satisfy the equation

$$\alpha(r)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(\frac{u_\phi}{r}\right)+\beta(r)\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{u_\phi}{r}\right)=\bar{G}(r), \tag{5·2}$$

where $\alpha(r)$, $\beta(r)$ and $G(r)$ are expressible in terms of the magnetic field.

In the course of demonstrating that (5·1) is, indeed, a sufficient condition for the solution of our basic equations (1·7) to (1·10) we also provided an algorithm for their solution; it would appear that this algorithm should provide a basis for a new calculation of the earth's dynamo problem in which most of the important dynamical effects would be included.

The physical meaning of these results is illuminated by returning to the full equation of motion

$$\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{D}t}=-\boldsymbol{\nabla}p'+(\mathbf{j}\times\mathbf{B})-2\rho(\boldsymbol{\Omega}\times\mathbf{v})+\rho'\boldsymbol{\nabla}\phi. \tag{5·3}$$

Although for slow motions the Coriolis term is so much larger than the inertial $\mathrm{D}\mathbf{v}/\mathrm{D}t$ term that the latter can usually be neglected, there is a class of motions for which the Coriolis force is ineffective. These are just the motions described by $u_\phi(r)$; for these motions we can integrate the ϕ-component of (5·3) over a cylinder to obtain

$$\rho\frac{\partial u_\phi}{\partial t}\int_{r=r_0}\mathrm{d}z\,\mathrm{d}\phi=\int(\mathbf{j}\times\mathbf{B})_\phi\,\mathrm{d}\phi\,\mathrm{d}z \tag{5·4}$$

and condition (5·1) is seen to be necessary to prevent large accelerations in the ϕ-direction. If (5·1) were not satisfied, rapid torsional motion would be set up in which each concentric cylindrical annulus rotated as a rigid body. The adjacent annuli are coupled together, as if by elastic strings, through the magnetic field B_r. Because of this linkage, the torsional motion would modify the fields until a state was reached in which (5·1) was satisfied. That this view is correct is confirmed by the form of equation (5·2) which finally determines $u_\phi(r)$, for it will be observed that the coefficients of this equation α, β, vanish when, but only when, B_r is zero everywhere over one of the cylindrical annuli. When this occurs that annulus is free to rotate independently of its neighbours.

We conclude that the equations discussed, together with the algorithm for calculating $\mathbf{v}$, do indeed give a consistent description of the slow relative motions of a conducting rotating fluid such as the earth's core and might form the basis of a method for calculating the 'dynamo problem'. The two theorems we have enunciated here can clearly be regarded as extensions of the Proudman–Taylor theorem and a modification of it given by Bullard & Gellman (1954). The Proudman–Taylor result showed that rotation restricts slow motions to a plane at right angles to the axis, while Bullard & Gellman showed that in a spherical (or generally a convex) container the restriction is still more severe; in fact the only slow, free motion possible is that in which cylindrical shells move rigidly about the axis of rotation. It is only to be expected, therefore, that when external forces are present they should have to satisfy a constraint such as we have found necessary and that when this constraint is satisfied the motion should be specified by the forces acting.

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References

Bullard, E. C. 1949*a* *Proc. Roy. Soc.* A, **197**, 433–453.
Bullard, E. C. 1949*b* *Proc. Roy. Soc.* A, **199**, 413–443.
Bullard, E. C. & Gellman, H. 1954 *Phil. Trans.* A, **247**, 213–278.
Chandrasekhar, S. 1961 *Hydrodynamic and hydromagnetic stability.* Oxford University Press.
Hide, R. 1956 *Physics and chemistry of the earth,* vol. 1 (Eds.: Ahrens, Rankama & Runcorn). London: Pergamon Press.
Hide, R. & Roberts, P. H. 1961 *Physics and chemistry of the earth,* vol. 4 (Eds.: Ahrens, Rankama & Runcorn). London: Pergamon Press.
Jeffreys, H. 1930 *Proc. Camb. Phil. Soc.* **26**, 170–172.
Proudman, J. 1916 *Proc. Roy. Soc.* A, **92**, 408–424.
Spiegel, E. A. & Veronis, G. 1960 *Astrophys. J.* **131**, 442–447.
Taylor, G. I. 1921 *Proc. Roy. Soc.* A, **100**, 114–121.
Taylor, G. I. 1923 *Proc. Roy. Soc.* A, **104**, 213–218.