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The structure of Taylor's constraint in three dimensions

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In a 1963 edition of *Proc. R. Soc. A*, J. B. Taylor (Taylor 1963 *Proc. R. Soc. A* **9**, 274–283) proved a necessary condition for dynamo action in a rapidly rotating electrically conducting fluid in which viscosity and inertia are negligible. He demonstrated that the azimuthal component of the Lorentz force must have zero average over any geostrophic contour (i.e. a fluid cylinder coaxial with the rotation axis). The resulting dynamical balance, termed a Taylor state, is believed to hold in the Earth's core, hence placing constraints on the class of permissible fields in the geodynamo. Such states have proven difficult to realize, apart from highly restricted examples. In particular, it has not yet been shown how to enforce the Taylor condition exactly in a general way, seeming to require an infinite number of constraints. In this work, we derive the analytic form for the averaged azimuthal component of the Lorentz force in three dimensions after expanding the magnetic field in a truncated spherical harmonic basis chosen to be regular at the origin. As the result is proportional to a polynomial of modest degree (simply related to the order of the spectral expansion), it can be made to vanish identically on every geostrophic contour by simply equating each of its coefficients to zero. We extend the discussion to allow for the presence of an inner core, which partitions the geostrophic contours into three distinct regions.

Keywords: Taylor's constraint; magnetostrophic balance; Taylor state

1. Introduction

The Earth's magnetic field is sustained by complex motions of electrically conducting fluid in the core, where the action of the flow to stretch and amplify the field competes against its natural tendency to decay. Despite many advances in observation and computer modelling, the details of this dynamo mechanism remain elusive (Kono & Roberts 2002; Olsen *et al.* 2007). Observations suffer from the fundamental problem that it is only possible to observe the magnetic field on the outer boundary of the core, and not within; in addition, even this restricted view is tempered by the superposition of the crustal field structure that effectively screens out all small-scale features. Geodynamo models are constructed to solve the equations governing the evolution of the coupled magnetic field and fluid flow but, of necessity, they operate in a parameter regime

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far from that appropriate to the core. Nonetheless, they have been surprisingly successful at reproducing many of the features that we associate with the terrestrial field, notably including aperiodic reversals. It is unclear, however, whether the dynamical balances responsible for producing these features in numerical models are substantially the same as those operating in the Earth. The main point of contention is the dominant force balance in the core. The equations describing the dynamics in the core are the Navier–Stokes and magnetic induction equations which, on adopting the Boussinesq approximation, are

$$R_o\left(\frac{\partial \boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\right)+\hat{\boldsymbol{z}}\times\boldsymbol{u}=-\nabla\Pi+R_a qT\boldsymbol{r}+E\nabla^2\boldsymbol{u}+[\nabla\times\boldsymbol{B}]\times\boldsymbol{B} \tag{1.1}$$

and

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\nabla^2\boldsymbol{B}, \tag{1.2}$$

along with $\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{B}=0$, following Fearn (1998), where $\boldsymbol{u}$ denotes the core flow; Π the modified pressure; $\boldsymbol{B}$ the magnetic field; and T the temperature. For the purposes of this paper, the key parameters are the Rossby number R_o and the Ekman number E, which for the Earth have estimated values of 10^{-9} and 10^{-15}, respectively. Of great importance to the geodynamo is the term driving the convection, which is multiplied by the Rayleigh number R_a and the Roberts number q. Although thermally driven convection is a standard core model, we could equally well consider compositional convection, driven by buoyant material released at the inner-core boundary. It transpires that our interest will lie only in the azimuthal component of (1.1) and so the physical basis of the convection, so long as it is driven by a force in the radial direction, is not a factor in our considerations.

Crucially, that both R_o and E are so small suggests that the core is in a magnetostrophic balance, where the Coriolis force, pressure, buoyancy and the Lorentz force are in quasi-equilibrium. But a direct test of whether such a balance obtains in this parameter regime is not possible in numerical models, principally due to limitations on the feasible range for E. In common with virtually all large-scale geophysical flows, the required dynamic range both temporally and spatially far exceeds available processor speed and memory. State-of-the-art simulations use typical values of $E=\mathcal{O}(10^{-6})$ (Kono & Roberts 2002), which is still $\mathcal{O}(10^9)$ larger than the approximate geophysical value, so the best that can be hoped for is that solutions of equations (1.1) and (1.2) rapidly achieve an asymptotic scaling.

Over 40 years ago, Taylor (1963) proved a necessary condition for a magnetic field generated by a system in a magnetostrophic balance. By considering an average over geostrophic surfaces in a rotating homogeneous incompressible fluid, namely cylinders concentric with the rotation axis and bounded by a spherical shell (figure 1), he was able to show that the ϕ component of equation (1.1) results in a requirement that

$$\mathcal{T}(s)\equiv\int_{C(s)}([\nabla\times\boldsymbol{B}]\times\boldsymbol{B})_\phi s\,\mathrm{d}\phi\,\mathrm{d}z=0, \tag{1.3}$$

where (s, ϕ, z) are cylindrical polar coordinates. This follows from noting that there is nothing to balance the cylindrical average of the azimuthal Lorentz force, the buoyancy has no component in the azimuthal direction and the other terms have zero average over the geostrophic contours. In fact, Taylor's condition is also satisfied in the wider setting of fluids that are compressible but

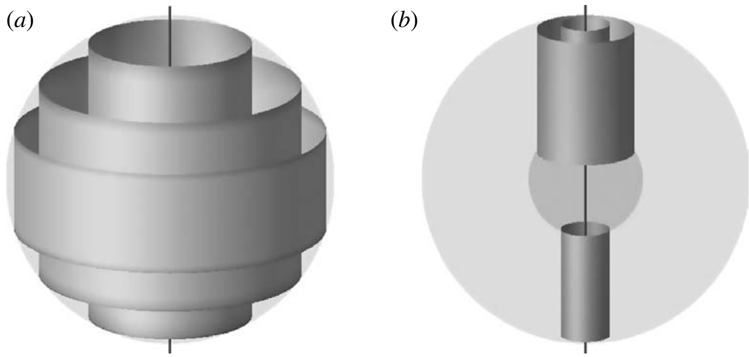

Figure 1. An illustration of cylinders over which Taylor's constraint is defined: (*a*) in the bulk of the core and (*b*) inside the tangent cylinder. The outer-core and inner-core spherical boundaries are shown in light grey.

stratified (of which the Boussinesq approximation is a special case), as described by the anelastic approximation (Smylie & Rochester 1984). In the presence of a solid inner core of radius r_i, the cylinders $C(s)$ with $s \leq r_i$ partition into two sets: those above and below the inner core.

Taylor's condition is an idealization that describes the dominant balance of terms in the core. Small departures from a Taylor state, i.e. a regime where Taylor's condition is identically satisfied, require us to reinstate one of the terms from (1.1) that does not appear in the magnetostrophic balance. Of the three possibilities, viscosity, Reynolds stresses or flow acceleration, in his 1963 paper, Taylor showed that the latter (at least when cylindrically averaged) would act to keep the average magnetic field in a Taylor state by means of damped torsional oscillations having a period of decades (Braginsky 1970; Dumberry & Bloxham 2003). Indeed, there is strong observational evidence of this theory stemming from the inferred structure of flow at the outer edge of the core and its consistency with observed changes in the length of day (Jault *et al.* 1988; Jackson 1997).

This reinforces the view that, to a leading order in E, the geomagnetic field is in a Taylor state and, since the 1970s, much effort has been expended in order to find at least one example. The approach adopted was to solve the mean-field (or averaged) dynamo equations at small E, with the expectation that the magnetic field would converge towards a Taylor state and become independent of viscosity as $E \rightarrow 0$, a scenario proposed by Malkus & Proctor (1975). A systematic development of this idea was first carried out by Soward & Jones (1983), where it was shown that, beyond a critical level of forcing of a planar axisymmetric mean-field dynamo, an exact Taylor state becomes energetically possible. Their model exhibits a variety of such nonlinear solutions, only some of which bifurcate from zero amplitude. Details of the subsequent post-critical equilibration were explored by Hollerbach & Ierley (1991) with a spherical axisymmetric model, in which the transition from viscous to inviscid solutions was explicitly documented. In an alternative approach, Fearn & Proctor (1987) solved the steady axisymmetric kinematic problem, defined by (1.2) alone (and ignoring (1.1)), and attempted to find the geostrophic component of $\boldsymbol{u}$ that minimized the departure of the generated magnetic field from a Taylor state. Relative to the exploration here, it should be noted that both Fearn & Proctor (1987) and Hollerbach & Ierley (1991) enforced the vanishing of $\mathcal{T}(s)$ only in a certain projected sense.[1] Jault (1995) showed that reinstating the inertial terms, giving rise to torsional oscillations, led to an alternative scheme that was able to evaluate the limit of vanishingly small viscosity and the approach to a Taylor state.

Although mean-field models offer a certain simplicity, it would be too much to expect anything more than a suggestive relation to large-scale numerical models (Kono & Roberts 2002). The limitations are many, perhaps the most obvious of which is the restriction to axisymmetry. It is unclear from these models how exact Taylor states can be produced more generally. Subsequently, only limited progress has been made in finding Taylor states in three dimensions. By balancing residual electromagnetic torque by inertia (in contrast to viscosity), Jault & Cardin (1999) showed that such an algorithm was a viable approach to the Taylor regime. They also pointed out that it is possible to produce magnetic fields that *exactly* satisfy the Taylor constraint through kinematic dynamo analysis. However, this procedure relies on the prescribed core-flow model possessing a certain rotational symmetry and so fails to establish a general prescription.

Some of the most recent numerical dynamo models have adopted normalized measures of $\mathcal{T}(s)$ as an indication of how closely a geodynamo model approaches a Taylor state (Ierley 1985); specifically, interest lies in the scaling of this measure with E since (1.3) is only intended as a statement about the leading-order balance (Rotvig & Jones 2002). When E is sufficiently small, then $\mathcal{T}(s)$ through the interior of the fluid is expected to vanish as E^{μ}, with $\mu>0$ determined by the boundary conditions on the velocity field.

The discovery of general Taylor states, of which the geomagnetic field is believed to be one, is not straightforward. From a numerical perspective, the computation of $\mathcal{T}(s)$ itself is problematic, as $\boldsymbol{B}$ has a natural expansion in spherical basis functions, while the Taylor integrals require that we compute in cylindrical coordinates, generally leading to interpolation on a grid (Walker & Barenghi 1998). More importantly, the existing literature does not address the expected analytic form of $\mathcal{T}(s)$, leaving the question of how many constraints are required to enforce the vanishing of $\mathcal{T}(s)$ uncertain. Indeed, although not explicitly stated, it appears widely believed that an infinite number of conditions are required, one for each choice of cylindrical radius s.

In this paper, we examine the structure of the Taylor integral, $\mathcal{T}(s)$, and its implications for studies of magnetic fields in the magnetostrophic regime. Our main result may be stated as follows. If $\boldsymbol{B}$ is expanded in a suitably regular basis set, then, in the absence of an inner core,

$$\mathcal{T}(s) = s^2\sqrt{1-s^2}\,Q_N(s^2),$$

for some polynomial Q_N, of maximum degree N related to the truncation adopted for $\boldsymbol{B}$. This remarkable result has the implication that the seeming infinity of the Taylor constraints is reduced to just $N+1$; it is readily seen that if $\mathcal{T}(s_i)=0$ for distinct cylindrical radii $s_1, s_2, \ldots, s_{N+1}$, then $\mathcal{T}(s)\equiv 0$.[2] For most of our analysis, we

[1] In the case of Hollerbach & Ierley (1991), only the shearing action of the geostrophic flow (related to $\mathcal{T}(s)$) on the poloidal field is zero. As Hollerbach (1990) demonstrated, the associated $\mathcal{T}(s)$ itself is not pointwise zero and, indeed, has a structure that is not simply related. Fearn & Proctor (1987) minimized the mean-squared value of $\mathcal{T}(s)$, as evaluated on an equally spaced grid in s; $\mathcal{T}(s)$ was not enforced to be pointwise zero.

[2] In fact, for reasons we discuss, the number of constraints can be reduced to N.

shall work in a full sphere, that is, ignoring the inner core; this is principally to keep things as simple as possible, but also because it exposes all the key issues. Later, we address the modifications required to accommodate the presence of an inner core. These are minor in the sense that the essential algebraic character of the problem is unchanged, but that is not at all to say the same of the resulting structure of the field and its associated *dynamics* (Hollerbach & Jones 1993).

We begin, in §2, by setting out various preliminaries necessary for developing the theory. In §3, we discuss the implications of symmetry and, in §4, we present the main result of this paper: the form of the Taylor integral. In §§ 5–7, we obtain further results concerning the structure of the integral, modifications required in the presence of an inner core and the structure of torsional oscillations. We include three appendices, which are referenced in the text.

2. Taylor interactions

We begin our discussion of Taylor integrals by writing the divergence-free magnetic field in the standard vector spherical harmonic decomposition

$$\boldsymbol{B} = \sum \boldsymbol{S}_l^{m\ s/c} + \boldsymbol{T}_l^{m\ s/c}, \tag{2.1}$$

where

$$\left.\begin{aligned} \boldsymbol{S}_l^{m\ s/c} &= \nabla \times \nabla \times [Y_l^{m\ s/c}(\theta,\phi) S_l^{m\ s/c}(r)\hat{\boldsymbol{r}}] \\ \text{and}\qquad \boldsymbol{T}_l^{m\ s/c} &= \nabla \times [Y_l^{m\ s/c}(\theta,\phi)\, T_l^{m\ s/c}(r)\hat{\boldsymbol{r}}], \end{aligned}\right\} \tag{2.2}$$

in spherical polar coordinates (r, θ, ϕ) and with $\hat{\boldsymbol{r}}$ denoting the unit position vector. The notation $Y_l^{m\ s/c}$ represents a spherical harmonic of degree l and order m and a particular azimuthal dependence

$$Y_l^{m\ s/c} = P_l^m(\cos\theta)\begin{Bmatrix} \sin m\phi \\ \cos m\phi \end{Bmatrix}, \tag{2.3}$$

where P_l^m is an associated Legendre function of degree l and order m. As is common in geomagnetism, we adopt the Schmidt quasi-normalization, such that

$$\int_0^{2\pi}\int_0^{\pi} \left[Y_l^{m\ s/c}\right]^2 \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = \frac{4\pi}{2l+1}.$$

We may therefore write the Taylor integral (1.3) as

$$\mathcal{T}(s) = \sum_{\alpha<\beta}([\boldsymbol{T}_\alpha, \boldsymbol{T}_\beta] + [\boldsymbol{S}_\alpha, \boldsymbol{S}_\beta] + [\boldsymbol{T}_\alpha, \boldsymbol{S}_\beta]),$$

where α and β index the spherical harmonic components with $\alpha<\beta$ to avoid double counting the symmetric quantity $[\boldsymbol{B}_\alpha, \boldsymbol{B}_\beta]$ (defined below). We show, in §3, that $[\boldsymbol{B}_\alpha, \boldsymbol{B}_\alpha]=0$.

$$\begin{aligned}[\boldsymbol{B}_\alpha, \boldsymbol{B}_\beta] &= \int_{C(s)} \left([\nabla\times\boldsymbol{B}_\alpha]\times\boldsymbol{B}_\beta + [\nabla\times\boldsymbol{B}_\beta]\times\boldsymbol{B}_\alpha\right)_\phi s\,\mathrm{d}\phi\,\mathrm{d}z \\ &= \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}}\int_0^{2\pi} \left([\nabla\times\boldsymbol{B}_\alpha]\times\boldsymbol{B}_\beta + [\nabla\times\boldsymbol{B}_\beta]\times\boldsymbol{B}_\alpha\right)_\phi s\,\mathrm{d}\phi\,\mathrm{d}z, \end{aligned} \tag{2.4}$$

where $\boldsymbol{B}_\alpha$ and $\boldsymbol{B}_\beta$ denote any two vector spherical harmonics and $C(s)$ denotes the cylinder at cylindrical radius s lying within the core. We recall that the core has a non-dimensional spherical radius of 1 and so the cylinders have end surfaces at $z=\pm\sqrt{1-s^2}$.

We term each contribution to $\mathcal{T}(s)$, $[\boldsymbol{B}_\alpha, \boldsymbol{B}_\beta]$, the 'interaction' between spherical harmonics $\boldsymbol{B}_\alpha$ and $\boldsymbol{B}_\beta$. We say that two harmonics 'interact' if their interaction is non-zero. Thus, the Taylor condition is equivalent to requiring that the sum of all vector spherical harmonic interactions is zero.

The components of the vector spherical harmonics are given in spherical polar coordinates as

$$\left.\begin{aligned}\nabla\times\nabla\times(S_l^m Y_l^m \hat{\boldsymbol{r}}) &= \left(\frac{l(l+1)S_l^m Y_l^m}{r^2}, \frac{1}{r}\frac{\mathrm{d}S_l^m}{\mathrm{d}r}\frac{\partial Y_l^m}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\mathrm{d}S_l^m}{\mathrm{d}r}\frac{\partial Y_l^m}{\partial\phi}\right)\\ \text{and}\qquad \nabla\times(T_l^m Y_l^m \hat{\boldsymbol{r}}) &= \left(0, \frac{T_l^m}{r\sin\theta}\frac{\partial Y_l^m}{\partial\phi}, -\frac{T_l^m}{r}\frac{\partial Y_l^m}{\partial\theta}\right),\end{aligned}\right\} \tag{2.5}$$

where the superscripts signifying azimuthal dependence have been suppressed for clarity. Noting that the curl of a poloidal vector is toroidal,

$$\nabla\times\nabla\times\nabla\times(f(r)Y_l^m(\theta,\phi)\hat{\boldsymbol{r}}) = \nabla\times\left(-\nabla_l^2 f(r)Y_l^m(\theta,\phi)\hat{\boldsymbol{r}}\right), \tag{2.6}$$

where $\nabla_l^2=\mathrm{d}^2/\mathrm{d}r^2-l(l+1)/r^2$, the various forms of the Taylor interaction can be expressed as

$$[\boldsymbol{T}_l^m, \boldsymbol{T}_n^m] = \int_{C(s)} \frac{l(l+1)T_l^m(r)T_n^m(r)}{r^3\sin\theta}\left(Y_l^m\frac{\partial Y_n^m}{\partial\phi}\right)s\,\mathrm{d}z\,\mathrm{d}\phi + \mathrm{sc}, \tag{2.7}$$

$$[\boldsymbol{S}_l^m, \boldsymbol{S}_n^m] = \int_{C(s)} \frac{l(l+1)S_l^m\nabla_n^2 S_n^m}{r^3\sin\theta}\left(Y_l^m\frac{\partial Y_n^m}{\partial\phi}\right)s\,\mathrm{d}z\,\mathrm{d}\phi + \mathrm{sc} \tag{2.8}$$

and

$$\begin{aligned}[\boldsymbol{T}_l^m, \boldsymbol{S}_n^m] = \int_{C(s)} \frac{1}{r^3}\Bigg(&l(l+1)T_l^m\frac{\mathrm{d}S_n^m}{\mathrm{d}r}Y_l^m\frac{\partial Y_n^m}{\partial\theta}\\ &-n(n+1)S_n^m\frac{\mathrm{d}T_l^m}{\mathrm{d}r}Y_n^m\frac{\partial Y_l^m}{\partial\theta}\Bigg)s\,\mathrm{d}z\,\mathrm{d}\phi,\end{aligned} \tag{2.9}$$

where 'sc' denotes *symmetric counterpart*, obtained by interchanging the vector harmonics. There is no counterpart for the last interaction because, recalling that the curl of a poloidal vector is toroidal, $(\boldsymbol{T}_1\times\boldsymbol{T}_2)_\phi=0$ for any two toroidal vectors $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$. Only the interactions involving identical wavenumber m have been considered in (2.7)–(2.9) since, by virtue of a symmetry property discussed in §3, all other terms are zero.

Lastly, we state an elementary, though important, consequence of the analytic form of equation (2.8). Suppose each poloidal scalar satisfies the relation $\nabla_n^2 S_n^m = \alpha n(n+1)S_n^m/r^2$ for some arbitrary α that only depends, at most, on m and r. By adding in the symmetric counterpart, the angular integrand becomes the exact differential $(\partial/\partial\phi)(Y_n^m Y_l^m)$ and integrates to zero in ϕ. Thus, this particularly simple family of purely poloidal fields identically satisfies Taylor's condition, although they fail to be regular at the origin (see §4). Further discussion of these solutions is taken up in §6.

3. Symmetry

Since the Taylor integrals involve averages over cylindrical surfaces $C(s)$, and owing to the symmetry inherent in spherical harmonics, it turns out that many of the Taylor interactions are identically zero. In particular, it is useful to consider three types of symmetry: (i) azimuthal rotations, (ii) reflections in the equator, and (iii) rotational symmetry of π radians about the x-axis (defined by the line $\phi=0$, $\theta=\pi/2$).

First, any integrand appearing in equation (2.4) must have a component that is axisymmetric for it to yield a non-zero interaction and all other components average to zero over $0\le\phi\le 2\pi$. Since such a term depends on the product of two trigonometric functions with wavenumbers m and m', it follows that, unless $m=m'$, the interaction vanishes. Even with the two wavenumbers being identical, there is clearly also the issue of their relative phase possibly rendering the integral identically zero; this will be discussed in the light of symmetry (iii).

In addition, as described in Gubbins & Zhang (1993), every vector spherical harmonic is either equatorially symmetric (E^S) or equatorially antisymmetric (E^A), as defined by the symmetric (S) and antisymmetric (A) vectors

$$[S_r, S_\theta, S_\phi](r, \pi-\theta, \phi) = [S_r, -S_\theta, S_\phi](r, \theta, \phi) \tag{3.1}$$

and

$$[A_r, A_\theta, A_\phi](r, \pi-\theta, \phi) = [-A_r, A_\theta, -A_\phi](r, \theta, \phi). \tag{3.2}$$

The ϕ component of an $E^S(E^A)$ vector is clearly also $E^S(E^A)$. By using the facts that the $\nabla\times$ operator reverses the equatorial symmetry of a vector and $\boldsymbol{u}\times\boldsymbol{v}$ is E^S if the equatorial symmetries of $\boldsymbol{u}$ and $\boldsymbol{v}$ differ, it is clear that $([\nabla\times\boldsymbol{B}_\alpha]\times\boldsymbol{B}_\beta)_\phi$ is E^S if and only if $\boldsymbol{B}_\alpha$ and $\boldsymbol{B}_\beta$ belong to the same equatorial symmetry class. Since $\mathcal{T}(s)$ is defined over an equatorially symmetric cylinder $C(s)$, it is immediate that, unless this condition is fulfilled, the interaction will be zero. Within the space of vector spherical harmonics, the symmetry classes are defined as

$$E^S = \left\{\boldsymbol{T}_l^{m\ s/c} : l-m\ \text{odd}\right\} \cup \left\{\boldsymbol{S}_l^{m\ s/c} : l-m\ \text{even}\right\} \tag{3.3}$$

and

$$E^A = \left\{\boldsymbol{T}_l^{m\ s/c} : l-m\ \text{even}\right\} \cup \left\{\boldsymbol{S}_l^{m\ s/c} : l-m\ \text{odd}\right\}. \tag{3.4}$$

As noted above, we may also consider the symmetry of a rotation of π about the x-axis. As shown in appendix A, in addition to their definite equatorial symmetry, vector spherical harmonics are either rotationally symmetric (R^S) or antisymmetric (R^A) with respect to this operation as defined by

$$[S_r, S_\theta, S_\phi](r, \pi-\theta, 2\pi-\phi) = [S_r, -S_\theta, -S_\phi](r, \theta, \phi) \tag{3.5}$$

and

$$[A_r, A_\theta, A_\phi](r, \pi-\theta, 2\pi-\phi) = [-A_r, A_\theta, A_\phi](r, \theta, \phi). \tag{3.6}$$

We note that the ϕ component of a vector that is R^S is R^A (and vice versa). It is also shown in appendix A that $\boldsymbol{u}\times\boldsymbol{v}$ is R^S (R^A) if $\boldsymbol{u}$ and $\boldsymbol{v}$ have the same (different) rotational symmetry and that $\nabla\times\boldsymbol{u}$ has the same rotational symmetry as $\boldsymbol{u}$. It therefore follows that the interaction is identically zero unless $([\nabla\times\boldsymbol{B}_\alpha]\times\boldsymbol{B}_\beta)_\phi$

is R^S, which requires $[\nabla\times\boldsymbol{B}_\alpha]\times\boldsymbol{B}_\beta$ to be R^A and so $\boldsymbol{B}_\alpha$ and $\boldsymbol{B}_\beta$ must belong to different rotational symmetry classes. The rotational symmetry classes are given by

$$R^S = \{\boldsymbol{T}_l^{mc} : l-m\,\text{even}\}\cup\{\boldsymbol{T}_l^{ms} : l-m\,\text{odd}\} \tag{3.7}$$

and

$$R^A = \{\boldsymbol{T}_l^{ms} : l-m\,\text{even}\}\cup\{\boldsymbol{T}_l^{mc} : l-m\,\text{odd}\}, \tag{3.8}$$

where the poloidal harmonics follow precisely the same symmetry as the corresponding toroidal harmonics. Note that the symmetries depend on the azimuthal phase of the harmonics, in contrast to the equatorial symmetry. We also make the crucial observation that, by rotational symmetry, the interaction of any vector harmonic with itself is zero. Thus, a magnetic field defined by any single vector harmonic is a Taylor state.

The case where we have an interaction of axisymmetric harmonics requires special attention. Since $\nabla\times\boldsymbol{S}_l$ (where the absence of a superscript indicates axisymmetry) is an axisymmetric toroidal harmonic, it only has a component in the ϕ direction. In the same vein, $\boldsymbol{T}_l$ only has a ϕ component; these two observations imply that $[\boldsymbol{S}_l, \boldsymbol{S}_n] = [\boldsymbol{T}_l, \boldsymbol{T}_n] = 0$ (as can also be deduced directly from (2.7) and (2.8)). Therefore, only one type of interaction, $[\boldsymbol{T}_l, \boldsymbol{S}_n]$, is non-zero. We may write the axisymmetric magnetic field in the alternate form

$$\boldsymbol{B} = \nabla\times A\hat{\boldsymbol{\phi}} + B\hat{\boldsymbol{\phi}},$$

for some axisymmetric scalars A and B, which, in the representation of (2.5), are

$$B = -\sum_l \frac{T_l(r)}{r}\frac{\partial Y_l(\theta)}{\partial\theta} \quad\text{and}\quad A = -\sum_n \frac{S_n(r)}{r}\frac{\partial Y_n(\theta)}{\partial\theta}. \tag{3.9}$$

On making the additional assumption of electrically insulating boundary conditions (rendering $B_\phi = 0$ on $r=1$), it is possible to write the Taylor interaction (Fearn 1994) as

$$[\boldsymbol{T}_l, \boldsymbol{S}_n] = -\frac{2\pi}{s}\frac{\mathrm{d}}{\mathrm{d}s}s^2\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} B\frac{\partial A}{\partial z}\mathrm{d}z. \tag{3.10}$$

Following the analysis of Rochester (1962), this relation can be deduced from the identity

$$\int_0^s s'\mathcal{T}(s')\mathrm{d}s' = \int_{C(s)} s([\nabla\times\boldsymbol{B}]\times\boldsymbol{B})_\phi\,\mathrm{d}V = \int_{C(s)+\mathcal{N}+\mathcal{S}} sB_\phi\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}, \tag{3.11}$$

where the middle integral is taken over the volume of the cylinder $C(s)$ and $\mathcal{N}$ and $\mathcal{S}$ are, respectively, the north and south polar caps on $r=1$ of the cylinder $C(s)$ that complete a surface of element $\mathrm{d}\boldsymbol{S}$. If $\boldsymbol{B}$ is axisymmetric and in contact with an electrical insulator, then $B_\phi = 0$ on $\mathcal{N}$ and $\mathcal{S}$ so that the integrals over the caps vanish. Since $B_s = -\partial A/\partial z$ and by differentiating with respect to s, equation (3.10) follows at once.

In the spirit of Bullard & Gellman (1954), we can derive a set of selection rules (1–5) that are necessary conditions for a non-zero Taylor interaction. Rules 1–3 follow from the symmetry properties described above, rules 4 and 5 from further properties that are derived below. The interaction between two vector spherical

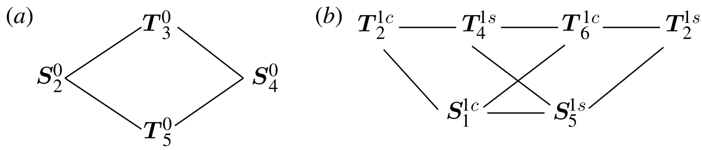

Figure 2. The interactions, indicated by the solid lines, within illustrative sets of (*a*) $m=0$ and (*b*) $m=1$ harmonics. As a specific example, the harmonics $\boldsymbol{T}_2^{1c}$ and $\boldsymbol{T}_4^{1s}$ interact by rules 1, 2 and 4; rules 3 and 5 do not apply. The harmonics $\boldsymbol{T}_2^{1s}$ and $\boldsymbol{T}_2^{1c}$ do not interact by rule 4.

harmonics of wavenumbers m_1 and m_2 and of degrees l_1 and l_2 is identically zero unless all the following conditions, where they apply, are true:

(1) S–S, T–T or T–S: $m_1=m_2$,
(2) S–S or T–T: $m_1\neq 0$, $l_1-l_2=0 \pmod 2$; not both sine or cosine,
(3) T–S: $l_1-l_2=1 \pmod 2$; both sine or both cosine,
(4) $T-T$: $l_1\neq l_2$, and
(5) S–S, $l_1=l_2$: $S_l^{ms}(r)/S_l^{mc}(r)$ not constant.

Primarily owing to rule 1, the number of non-zero interactions of harmonics of maximum degree $L_{\max}$ is vastly less than N_H^2, where $N_H=2L_{\max}(L_{\max}+2)$, the total number of vector harmonics. A straightforward counting argument shows that the number of interactions is $\mathcal{O}(N_H/L_{\max})$.

The extra selection rules (4 and 5) listed above follow from additional identities arising from the explicit form of the Taylor integrals (equations (2.7)–(2.9)). To derive rule 4, consider the interaction of two toroidal harmonics of degree l, order m (rule 1) and of a different azimuthal phase (rule 2),

$$[\boldsymbol{T}_l^{ms},\boldsymbol{T}_l^{mc}]=\int\frac{l(l+1)\,T_l^{ms}\,T_l^{mc}}{r^3\sin\theta}\,\frac{\partial}{\partial\phi}(\,Y_l^{ms}\,Y_l^{mc})s\,\mathrm{d}z\,\mathrm{d}\phi=0. \tag{3.12}$$

Thus, the interaction of two toroidal harmonics that have identical spherical harmonic degrees and orders (but may differ in radial behaviour) is zero.

Rule 5 follows from a similar but weaker result for the interaction between two poloidal harmonics of identical degree, order and radial dependence. If $S_l^{mc}(r)=AS_l^{ms}(r)$ for some constant A, then

$$[\boldsymbol{S}_l^{ms},\boldsymbol{S}_l^{mc}]=A\int_{C(s)}\frac{l(l+1)S_l^{ms}\nabla_l^2 S_l^{ms}}{r^3\sin\theta}\,\frac{\partial}{\partial\phi}(\,Y_l^{ms}\,Y_l^{mc})s\,\mathrm{d}z\,\mathrm{d}\phi=0. \tag{3.13}$$

We illustrate these selection rules in figure 2, which shows all interactions (solid lines) within two sets of harmonics. Owing to the structure of the selection rules, it is not possible to construct a triad of harmonics that maximally interact. That is, if two harmonics both interact with another harmonic, then they cannot interact with each other.

4. The form of the Taylor integral

Having dealt with the crucial consequences of symmetry, we proceed to the main result of this paper that the Taylor integral $\mathcal{T}(s)$ takes on a certain analytic form if the radial expansion is chosen appropriately.

(a) Regularity

Let us write the poloidal and toroidal scalar functions in terms of finite-degree polynomials in radius of the following regular form:

$$S_l^{m\ s/c}(r) = r^{l+1}\sum_{j=0}^{N_{\max}} a_j r^{2j} \quad \text{and} \quad T_l^{m\ s/c}(r) = r^{l+1}\sum_{j=0}^{N_{\max}} b_j r^{2j}, \tag{4.1}$$

where a_j and b_j are real coefficients and $N_{\max}$ is given.

This representation in radius that is intimately tied to the angular dependence, involving a prefactor of r^{l+1} and a power series in r^2, is crucial in proving the main result of this paper, as it allows us to make a connection between smooth vector representations in different coordinate systems. By exploiting the fact that $r^{l+2j}Y_l^m$, where $j\geq 0$, can be written as a multinomial in (x,y,z) of finite degree (Backus *et al.* 1996; Boyd 2001), we are guaranteed that the vector spherical harmonic can be written in a Cartesian coordinate system with components that are multinomials.[3] The connection to the Cartesian system is important as, in contrast to spherical polar coordinates or cylindrical coordinates, the unit vectors are smooth, i.e. continuously differentiable including at the origin. If we additionally adopt the truncation $0\leq m\leq l\leq L_{\max}$ in solid angle, then it follows that both $\boldsymbol{B}$ and $\nabla\times\boldsymbol{B}$ are smooth and possess Cartesian components that are finite-degree multinomials in (x,y,z).

In order to compute $\mathcal{T}(s)$, we need only concern ourselves with those terms in $([\nabla\times\boldsymbol{B}]\times\boldsymbol{B})_\phi$ that are axisymmetric, all others vanish on integrating with respect to ϕ. We now appeal to the known result (Lewis & Bellan 1990) that infinite differentiability of an arbitrary axisymmetric smooth vector field $\boldsymbol{F}$ implies that

$$F_\phi = s f(s^2, z), \tag{4.2}$$

for some function f, thus the same must hold for $([\nabla\times\boldsymbol{B}]\times\boldsymbol{B})_\phi$. Additionally, in view of the identities

$$F_\phi = -\sin\phi F_x + \cos\phi F_y, \qquad x = s\cos\phi \quad \text{and} \quad y = s\sin\phi, \tag{4.3}$$

we are led to the inescapable conclusion that f must be a multinomial in s^2 and z of finite degree if $\boldsymbol{B}$ is truncated as described above.

In fact, we can do better than this by noting that only the equatorially symmetric component of $([\nabla\times\boldsymbol{B}]\times\boldsymbol{B})_\phi$ contributes to $\mathcal{T}(s)$, in which case f must depend only on z^2 rather than z.

Thus, we may restrict attention to the expansion

$$([\nabla\times\boldsymbol{B}]\times\boldsymbol{B})_\phi = s\sum_{j,k} c_{jk} s^{2j} z^{2k}, \tag{4.4}$$

where $0\leq j+k$ is bounded from above.

It remains to insert this expression into equation (1.3),

$$\begin{aligned}\mathcal{T}(s) &= s\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \sum_{j,k} c_{jk} s^{2j+1} z^{2k}\,\mathrm{d}z = s^2\sqrt{1-s^2}\sum_{j=0}^{N} d_j s^{2j} \\ &= s^2\sqrt{1-s^2}\,Q_N(s^2),\end{aligned} \tag{4.5}$$

[3] Note that since the position vector $r\hat{\boldsymbol{r}}=(x,y,z)$, but not $\hat{\boldsymbol{r}}$, is everywhere smooth, we require a prefactor of r^{l+1} rather than r^l.

Table 1. The greatest exponent of s, $2N$, occurring in $Q_N(s^2)$ in the analytic form of $\mathcal{T}(s)$ for the interaction of two spherical harmonics of identical order m, degrees l_1 and l_2 and radial truncations N_1 and N_2. (An upper bound is given in column 2, which is optimal if the selection rules given in §3 are followed. The maximum possible value of $2N$ under the restrictions $l_1,\ l_2 \leq L_{\max}$ and $N_1,\ N_2 \leq N_{\max}$ is given in column 3.)

interaction	$2N$	maximum value for $2N$
$[\boldsymbol{T},\boldsymbol{T}]$	$l_1+l_2+2N_1+2N_2-2$	$2L_{\max}+4N_{\max}-4$
$[\boldsymbol{T},\boldsymbol{S}]$	$l_1+l_2+2N_1+2N_2-3$	$2L_{\max}+4N_{\max}-4$
$[\boldsymbol{S},\boldsymbol{S}]$	$l_1+l_2+2N_1+2N_2-4$	$2L_{\max}+4N_{\max}-6$

for some coefficients d_j and a general (unknown) polynomial Q_N of degree N. In what follows, it will be important to note that $Q_N(s^2)$ has a highest degree of $2N$ in s.

We can derive an upper bound on $2N$ by tracking the greatest exponent of the dimension of length through changes in the coordinate systems and the application of derivatives of the various poloidal and toroidal scalars to equation (1.3). Any regular scalar quantity $f(r,\theta,\phi)$, written in spherical polar coordinates, where r appears in powers of at most r^n, is also expressible in cylindrical polar coordinates (s,ϕ,z), where s and z appear only as $s^j z^k$, with $j+k \leq n$, and likewise in Cartesian coordinates, where x, y, z appear only in triplets $x^i y^j z^k$, with $i+j+k \leq n$.

Let us take a specific example of a general interaction between two toroidal vector harmonics of identical wavenumber m, of spherical harmonic degrees l_1 and l_2 and whose toroidal scalars have respective radial truncations N_1 and N_2 rather than both being of truncation $N_{\max}$. It is apparent that the total radial exponent in $([\nabla\times\boldsymbol{T}]\times\boldsymbol{T})_\phi$ is $l_1+l_2+2(N_1+N_2)+2-3$ since three curl operations are involved. We then must integrate in z, raising the degree by 1, add in an extra factor of s (from the definition of $\mathcal{T}(s)$) and then remove a factor of $s^2\sqrt{1-s^2}$ from the resulting expression. We therefore obtain an even polynomial, of maximum degree $l_1+l_2+2(N_1+N_2)+2-3+1+1-3 = l_1+l_2+2(N_1+N_2)-2$, in s. Noting the selection rules of §3, we see that, under the restrictions $l_1, l_2 \leq L_{\max}$ and $N_1, N_2 \leq N_{\max}$, the maximum possible degree is $2N = 2L_{\max}+4N_{\max}-4$. A similar argument applied to all types of interaction gives values for $2N$ as shown in table 1. In particular, we see that

$$\mathcal{T}(s) = s^2\sqrt{1-s^2}\,Q_{L_{\max}+2N_{\max}-2}(s^2). \tag{4.6}$$

There are ostensibly $L_{\max}+2N_{\max}-1$ degrees of freedom in such an analytic form (namely the polynomial coefficients, including s^0). However, as we show in §5a, these coefficients are not independent and satisfy a certain linear relation. Thus, the continuous form of the Taylor constraint is reduced to just $L_{\max}+2N_{\max}-2$ conditions.

More broadly, equation (4.6) gives a precise statement of how dense Taylor states are among all possible magnetic fields in a given space. It is clear that such states are ubiquitous among three-dimensional fields, as within the set of $2(N_{\max}+1)L_{\max}(L_{\max}+2)$ magnetic field coefficients, one has only to arrange that $L_{\max}+2N_{\max}-2$ constraints (albeit quadratic) are satisfied.

Table 2. Taylor interactions between low-degree vector spherical harmonic basis functions. (For typographic purposes, we have scaled each expression to have a prefactor of unity.)

$\boldsymbol{B}_1$	$\boldsymbol{B}_2$	$[\boldsymbol{B}_1, \boldsymbol{B}_2]$
${}_2\boldsymbol{T}_1^{1s}$	${}_2\boldsymbol{T}_3^{1c}$	$s^2(1-s^2)^{5/2}\,(6720s^6-8368s^4+3143s^2-340)$
${}_2\boldsymbol{T}_2^{2s}$	${}_1\boldsymbol{T}_4^{2c}$	$s^4(1-s^2)^{5/2}\,(120s^4-103s^2+18)$
${}_2\boldsymbol{T}_3^{3c}$	${}_2\boldsymbol{T}_5^{3s}$	$s^6(1-s^2)^{5/2}\,(76\,608s^6-125\,632s^4+65\,495s^2-10\,696)$
${}_1\boldsymbol{S}_1^{1c}$	${}_2\boldsymbol{S}_1^{1s}$	$s^2\sqrt{1-s^2}(27s^4-74s^2+32)$
${}_2\boldsymbol{S}_3^{3c}$	${}_2\boldsymbol{S}_5^{3s}$	$s^6\sqrt{1-s^2}(87\,894\,080s^8-263\,939\,720s^6+287\,361\,837s^4-133\,727\,634s^2+22\,226\,112)$
${}_1\boldsymbol{T}_1^{1s}$	${}_2\boldsymbol{S}_2^{1s}$	$s^2(1-s^2)^{3/2}\,(5984s^6-7719s^2+1980)$
${}_2\boldsymbol{T}_2^{2s}$	${}_2\boldsymbol{S}_3^{2s}$	$s^4(1-s^2)^{3/2}\,(205\,440s^6-368\,672s^4+196\,085s^2-29\,178)$

Although the analysis in this section is strongly based on regularity, in fact, this condition is not necessary for the reduction of Taylor's condition to a finite set of constraints, as we show in appendix C. Indeed, on some level, such a conclusion may be anticipated given that the magnetic field has finite truncation and $\mathcal{T}(s)$ is therefore necessarily of some finite algebraic form. However, what is far from obvious is whether or not the number of spectral coefficients representing the magnetic field is less than, or greater than, the number of resulting constraints. In contrast to the regular polynomials, some choices of radial basis functions yield more constraints than available degrees of freedom, and thus, in such a representation, exact Taylor states do not exist (except in special cases, for instance, a magnetic field of single spherical harmonic dependence). A fuller discussion of this point is provided in §8.

(*b*) *Examples*

We now illustrate the theory given in §4*a* by some explicit examples of the Taylor interactions between various vector spherical harmonics. We restrict the generalized radial expansion of (4.1) to one that satisfies electrically insulating boundary conditions at $r=1$, appropriate for the geodynamo. As shown by, for example, Hollerbach (2000), these take the form

$$T_l^m|_{r=1} = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}S_l^m}{\mathrm{d}r} + lS_l^m\right|_{r=1} = 0,$$

for each harmonic, where we have again suppressed azimuthal dependence. We construct a radial basis set where element n has maximum radial degree r^{l+2n+1}, $n=1,2,\ldots$, by recombining suitable Jacobi polynomials to give an optimal basis expansion that, in addition to satisfying the boundary conditions, is orthogonal, regular and quasi-equal-ripple (Livermore in preparation). For the purposes of the present discussion, however, it is sufficient to note that we now define the scalar functions (along with their vector spherical harmonic representation)

$${}_nS_l^{m\ s/c} = Y_l^{m\ s/c} f_n^l(r) \quad \text{and} \quad {}_nT_l^{m\ s/c} = Y_l^{m\ s/c} g_n^l(r),$$

where $f_n^l(r)$ and $g_n^l(r)$ are polynomial basis functions of maximum degree $l+2n+1$, $n\geq 1$.

The illustrative interactions are given in table 2; note that they are all of form (4.5) with polynomial degree given by table 1.

5. Degeneracies of the Taylor integral

We have shown that the Taylor integral may be written in the form (4.6), a prefactor multiplying a polynomial $Q_N(s^2)$ of finite degree N. However, as we demonstrate in §5*a*, the coefficients of Q_N are linearly dependent if electrically insulating boundary conditions are satisfied. The restriction of this case to a purely toroidal field is explored in §5*b*, where we show that Q_N has a factor of $(1-s^2)^2$. Lastly, a somewhat related issue is addressed in §5*c*, where we prove that, for a magnetic field of single azimuthal wavenumber m, $\mathcal{T}(s)$ is modulated by s^{2k}, where $k=\min(1,m)$.

(*a*) *Electromagnetic torque degeneracy*

We consider a constraint on the form of the Taylor integrals that is related to the electromagnetic torque that the magnetic field supplies to the core. One may argue physically that, if a magnetic field is internally generated, then its net electromagnetic torque must vanish. This is because the flow cannot, by itself and excluding any external forces, alter its own net angular momentum, and we regard the magnetic field as part of (and existing only due to the action of) the flow. However, it is clear that such an argument depends on the boundary conditions one adopts for the magnetic field (and thus defining the concept of 'self-generation').

We define the volume-averaged electromagnetic torque as

$$\Gamma = \int_V \boldsymbol{r} \times ([\nabla \times \boldsymbol{B}] \times \boldsymbol{B}) \mathrm{d}V, \tag{5.1}$$

where V denotes the core. Writing $\boldsymbol{r} = s\hat{\boldsymbol{s}} + z\hat{\boldsymbol{z}}$, the z component of this gives

$$\Gamma_z = \int_V s([\nabla \times \boldsymbol{B}] \times \boldsymbol{B})_\phi \, \mathrm{d}V = \int_0^1 s\mathcal{T}(s)\mathrm{d}s, \tag{5.2}$$

since we may split up the volume integration into two sequential integrations: first, over all cylinders coaxial with $\hat{\boldsymbol{z}}$ and, second, over s. Following Rochester (1962), by using the Maxwell stress and the divergence theorem, the volume integral may be replaced by

$$r\int_{\partial V} B_r B_\phi \sin\theta \, \mathrm{d}S, \tag{5.3}$$

where ∂V denotes the boundary of V and $\mathrm{d}S$ is the surface element. If the boundary conditions are such that $B_r=0$, then it is clear that (5.3) vanishes. Additionally, if electrically insulating boundary conditions are assumed in the exterior to V, $\hat{V}$, then by using continuity of $\boldsymbol{B}$ across ∂V and by reversing the above argument,

$$\int_0^1 s\mathcal{T}(s)\mathrm{d}s = -\int_{\hat{V}} s([\nabla \times \boldsymbol{B}] \times \boldsymbol{B})_\phi \, \mathrm{d}V = 0, \tag{5.4}$$

where we have made use of the facts that the boundary surface at infinity makes no contribution and $\nabla \times \boldsymbol{B} = 0$ in $\hat{V}$.

The requirement that

$$\int_0^1 s\mathcal{T}(s)\mathrm{d}s = 0$$

further constrains the form of $\mathcal{T}(s) = s^2\sqrt{1-s^2}\,Q(s^2)$, where Q is a polynomial.

If $Q(s^2)=\sum d_j s^{2j}$, then the coefficients d_j satisfy the degeneracy

$$\sum_j d_j\zeta_j = 0, \qquad \zeta_j = \frac{15}{2}\int_0^1 s^{3+2j}\sqrt{1-s^2}\mathrm{d}s = \frac{15\sqrt{\pi}}{8}\frac{\Gamma(2+j)}{\Gamma(7/2+j)},$$

where, without loss of generality, we have multiplied this homogeneous expression through by 15/2 to arrange that $\zeta_0=1$.

The coefficients ζ_j have a generating function

$$\frac{15\sqrt{\pi}}{8}\sum_{j=0}^{\infty} s^{2j}\frac{\Gamma(2+j)}{\Gamma(7/2+j)} = -\frac{5}{2}\frac{1}{s^5}\left(s^3-3s+3\sqrt{1-s^2}\sin^{-1} s\right)$$
$$= {}_2F_1(1,2;7/2;s^2), \tag{5.5}$$

where F is a hypergeometric function, and take values

$$\zeta_j = \frac{15\sqrt{\pi}}{8}\frac{\Gamma(2+j)}{\Gamma(7/2+j)} = \left(1,\frac{4}{7},\frac{8}{21},\frac{64}{231},\frac{640}{3003},\ldots\right). \tag{5.6}$$

(*b*) *Toroidal degeneracy*

In the particular case of a purely toroidal field that satisfies zero boundary conditions at $r=1$, $\mathcal{T}(s)$ takes on extra zeros at $s=\pm 1$, resulting in a common factor of $(1-s^2)^{5/2}$ rather than $\sqrt{1-s^2}$. To show this, consider the interaction of two toroidal vector harmonics, which, on consulting equation (2.7) for given l and l', is proportional to

$$\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \frac{P_l^m(\cos\theta)P_{l'}^m(\cos\theta)\,T_l^m(r)\,T_{l'}^m(r)}{r^3\sin\theta}\,s\,\mathrm{d}z, \tag{5.7}$$

where we have assumed that the azimuthal dependence is such that the average over ϕ is non-zero. We will show that the double zero in r at ± 1 translates to a double zero in s at ± 1 in the resulting form. Note that $m\geq 1$ (by rule 2 of §3, the interaction vanishes if $m=0$), so the ostensibly troublesome factor of $\sin\theta$, in fact, cancels with a factor of $\sin\theta$ from P_l^m in the numerator. By inspecting the equivalent forms of equation (4.4) and the integrand in equation (5.7), and using the fact that each toroidal scalar vanishes at $r=1$, we see that the interaction must be of the form

$$\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} s^2\sum_{jk} c_{jk}s^{2j}z^{2k}(1-s^2-z^2)^2\,\mathrm{d}z, \tag{5.8}$$

for some coefficients c_{jk}; $j,k\geq 0$, $j+k$ bounded from above. Each individual term is of the form

$$\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} s^{2j+2}z^{2k}(1-s^2-z^2)^2\,\mathrm{d}z$$
$$= s^{2j+2}\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} z^{2k+4}-2(1-s^2)z^{2k+2}+(1-s^2)^2z^{2k}\,\mathrm{d}z$$
$$= 2s^{2j+2}(1-s^2)^{k+2}\sqrt{1-s^2}\left(\frac{1}{2k+5}-\frac{2}{2k+3}+\frac{1}{2k+1}\right). \tag{5.9}$$

Since $j,k\geq 0$ are arbitrary, the interaction is hence of the form

$$\mathcal{T}(s) = s^2(1-s^2)^{5/2}Q_{N-2}(s^2),$$

for some polynomial Q_{N-2}, of degree 2 less than before. An alternative analysis is shown in appendix B, where we integrate (5.7) directly and obtain an explicit (though complex) form for the interaction.

We additionally point out that for the same boundary conditions on T_l^m, as can be identified from table 2, the interaction $[\boldsymbol{T}_l^m, \boldsymbol{S}_n^m]$ has a common factor of $s^2(1-s^2)^{3/2}$. We do not supply a proof of this result here.

(*c*) *Modulation close to* $s=0$

Finally, consider any individual interaction given by equations (2.7)–(2.9). We now show that

$$\mathcal{T}(s) = s^{2k}\sqrt{1-s^2}\,Q_{N-k+1}(s^2),$$

where $k=\min(1,m)$. That is, for $m\geq 1$, the Taylor interactions are modulated by $\mathcal{O}(s^{2m})$ behaviour and so, for azimuthally small-scale fields (large m), $\mathcal{T}(s)\approx 0$ in the bulk of the domain where $s<1$. Thus, Taylor's constraint is approximately satisfied for these interactions, except near the equator.

The proof is straightforward, and relies on the observation that the integrand from equations (2.7) and (2.8) and that of equation (2.9) are respectively of the form

$$f_1(r)\frac{P_l^m P_n^m}{\sin\theta} \quad \text{and} \quad f_1(r)P_l^m\frac{\mathrm{d}P_n^m}{\mathrm{d}\theta} + f_2(r)P_n^m\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta}, \tag{5.10}$$

for some functions $f_1(r)$ and $f_2(r)$. We now argue that we simply need to track the common factor of $(\sin\theta)^n$ in these expressions, since on substituting

$$\sin\theta = s/\sqrt{s^2+z^2}, \qquad \cos\theta = z/\sqrt{s^2+z^2} \quad \text{and} \quad r^2 = s^2+z^2,$$

we obtain a common factor of s^n. The latitudinal part of each spherical harmonic may be written as

$$P_l^m(\cos\theta) = (\sin\theta)^m Q(\cos\theta),$$

where Q is some polynomial of degree $l-m$. It is immediately apparent, at least in the first case, that the resulting expression has a common factor of s^{2m-1} that may be taken outside the integration with respect to z. On using the extra factor of s from the definition of $\mathcal{T}(s)$, we obtain the factor s^{2m}. In the second case, we note that

$$\begin{aligned}\frac{\mathrm{d}P_l^m}{\mathrm{d}\theta} &= m(\sin\theta)^{m-1}\cos\theta\, Q(\cos\theta) - (\sin\theta)^{m+1}Q'(\cos\theta)\\ &= (\sin\theta)^{m-1}W(\cos\theta),\end{aligned} \tag{5.11}$$

where W is a polynomial of degree $l-m+1$ and the same common factor results.

This result is entirely caused by the modulation $(\sin\theta)^m$ of spherical harmonics of order m close to either geographical pole; the fact that they are small there requires that the Taylor integrals, at least for $s<1$, also inherit this property.

6. The effect of an inner core

As noted in §1, the existence of an inner core splits up the geostrophic contours (the cylinders over which we enforce Taylor's constraint) into three distinct regions: those outside the tangent cylinder and those inside the tangent cylinder above and below the inner core. Figure 3 illustrates the different regions; refer to figure 1 for an example of geostrophic contours.

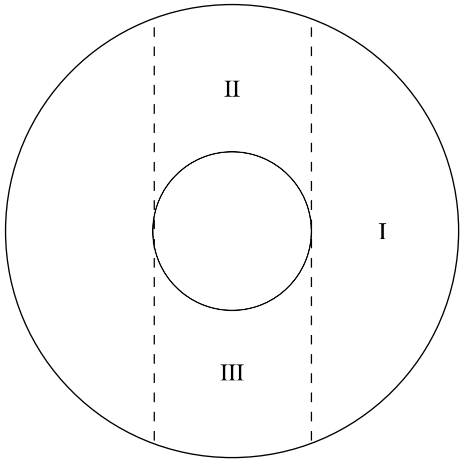

Figure 3. Illustration of the fluid core showing the three different regions in which geostrophic contours are defined: outside the tangent cylinder (I) and inside the tangent cylinder above (II) and below (III) the inner core. See figure 1 for an example of cylindrical contours.

One important consequence of the fact that contours inside the tangent cylinder are non-equatorially symmetric is that symmetry considerations no longer play such an important role in determining which interactions are non-zero. Thus, the full set of selection rules detailed in §3 no longer holds, as neither reflectional equatorial symmetry nor rotational symmetry about the x-axis are consequential. However, azimuthal symmetry remains important; it is clear that two interacting vector harmonics must have the same wavenumber and their relative azimuthal phase (i.e. either sine or cosine dependence) must satisfy a constraint that can be gleaned from equations (2.7)–(2.9) and turns out to be identical to those set out in rules 2 and 3 of §3.

We proceed by imposing the same form of magnetic field solutions in $r \leq 1$, that of (4.1), as previously adopted for the full sphere. That is, we will impose that the field is everywhere smooth, regular and of finite truncation, in particular, not allowing for discontinuities of any derivative of the field across the inner-core boundary. In the most general setting, a magnetic field will not satisfy this prescription as it need only be continuous at the inner-core boundary and its expansion outside the tangent cylinder may be of non-regular form. The analysis we pursue here therefore is clearly a severely constrained subset of the general case; however, by assuming the same expansion for the field as before, it enables us to show in the simplest manner how the theory is modified by the presence of the inner core.

Outside the tangent cylinder, the form of the Taylor integral remains unchanged, that is,

$$\mathcal{T}(s) = s^2\sqrt{1-s^2}\sum_{j=0}^{N} d_j s^{2j}, \tag{6.1}$$

for some integer N and some coefficients d_j.

Inside the tangent cylinder, the form of the Taylor integral is markedly different. If the inner-core radius is denoted r_i, then in either region II or III, from equation (4.4) for $s \le r_i$ (with geophysical value $r_i \approx 7/20$),

$$\begin{aligned}\mathcal{T}(s) &= s\int_{\sqrt{r_i^2-s^2}}^{\sqrt{1-s^2}} \sum_{j,k} c_{jk} s^{2j+1} z^k \, \mathrm{d}z \\ &= \sum_{j,k} d_{jk} s^{2j+2}(1-s^2)^{(k+1)/2} + e_{jk} s^{2j+2}(r_i^2-s^2)^{(k+1)/2},\end{aligned} \tag{6.2}$$

where d_{jk} and e_{jk} are some coefficients, and we note that k can take on any integer value as equatorial symmetry is no longer assumed. This becomes

$$\mathcal{T}(s) = s^2\left(\sqrt{1-s^2}\sum_{j=0}^{j=N} f_j s^{2j} + \sqrt{r_i^2-s^2}\sum_{j=0}^{j=N} g_j s^{2j} + \sum_{j=0}^{j=N} h_j s^{2j}\right), \tag{6.3}$$

for some integer N and some coefficients f_j, g_j and h_j. That the prefactors of these expressions, $\sqrt{r_i^2-s^2}$, $\sqrt{1-s^2}$ and 1, are not linearly related means that there are $3N$ unknown coefficients. In addition, we note that f_j and g_j stem from the integrand that is equatorially symmetric; thus, these coefficients must be identical in both regions II and III. By contrast, h_j stem from the integrands that are equatorially antisymmetric and therefore must exhibit a change of sign between these two regions. Crucially, however, if f_j, g_j and h_j are zero in either region II or III, then they are zero in the other, and $3N$ constraints suffice within the tangent cylinder. In fact, since the magnetic field is smooth, it has the same functional dependence both inside and outside the tangent cylinder; inspecting (6.3) and (6.1), we see that $d_j = f_j$. Therefore, $\mathcal{T}(s)$ automatically vanishes in region I if it does in regions II and III, and hence $3N$ constraints suffice for regions I, II and III.

Although the above provides a general construction for a smooth magnetic field obeying Taylor's constraint, it is worth remarking that it is possible to engineer a particularly simple family of continuous (although not smooth) solutions that matches any profile of radial field on $r=1$ with electrically insulating boundary conditions. We make use of the observation noted at the end of §2 that any purely poloidal field that satisfies $\nabla_n^2 S_n^m = \alpha n(n+1) S_n^m / r^2$, where α is some arbitrary function depending only on, at most, m and r, satisfies Taylor's constraint automatically. Assuming the simplest case with no r dependence and by defining $\beta = \alpha + 1$, this requires

$$\frac{d^2 S_n^m}{\mathrm{d}r^2} = \frac{\beta n(n+1) S_n^m}{r^2}, \tag{6.4}$$

which has solutions

$$S_n^m = A r^{\gamma_+} + B r^{\gamma_-}, \qquad \gamma_\pm = \frac{1 \pm \sqrt{1+4\beta n(n+1)}}{2}. \tag{6.5}$$

In particular, we note that $\beta = 0$ defines a linear profile for S_n^m, and $\beta = 1$ a potential field solution for $\boldsymbol{B}$ that is associated with zero current density in the outer core. The two free parameters of each harmonic, A and B, can be tailored to fit the two required conditions at $r=1$, and we can complete the solution by matching to a regular inner-core representation by enforcing continuity at $r = r_i$.

With regard to the electromagnetic torque discussed in §5*a*, it is no longer the case that $\mathcal{T}(s)$ is necessarily degenerate when assuming an electrically insulating mantle and a conducting inner core. The analysis is modified to

$$\int_{C_{\mathrm{I,II,III}}(s)} s\mathcal{T}(s)\mathrm{d}s = \int_{r=1} B_r B_\phi \sin\theta \,\mathrm{d}\Omega - r_i \int_{r=r_i} B_r B_\phi \sin\theta \,\mathrm{d}\Omega, \tag{6.6}$$

where $C_{\mathrm{I,II,III}}$ denote the different regions over which $\mathcal{T}(s)$ is defined and $\mathrm{d}\Omega$ is the element of solid angle. Although the boundary term at $r=1$ is zero (as before), we are not guaranteed that the boundary term at $r=r_i$ is zero for arbitrary $\boldsymbol{B}$. However, if we choose boundary conditions at $r=r_i$ that render this term zero (such as electrically insulating or perfectly conducting conditions), then

$$\int_{C_{\mathrm{I,II,III}}(s)} s\mathcal{T}(s)\mathrm{d}s = 0,$$

and the coefficients in the expansion for $\mathcal{T}(s)$ become linearly degenerate for arbitrary $\boldsymbol{B}$. Thus, the number of sufficient conditions required to impose Taylor's constraint reduces to $3N-1$.

Lastly, we remark that if $\boldsymbol{B}$ is chosen so that $\mathcal{T}(s)$ is zero on every geostrophic contour and we impose electrically insulating conditions on $r=1$, it follows that the boundary term at $r=r_i$ is also zero. Thus, the vanishing of the electromagnetic torque on the inner core follows automatically.

7. The form of cylindrically averaged torsional motion

As discussed in §1, departures from the Taylor state are thought to be corrected by means of coaxial cylindrical motions. Here, we discuss the form that cylindrical averages of such motions must take, arguing much as we have already done for the Lorentz force. The dynamical balance that governs such torsional oscillations is

$$R_{\mathrm{o}} \frac{\mathrm{d}}{\mathrm{d}t} \int_{C(s)} u_\phi s \,\mathrm{d}\phi \,\mathrm{d}z = \mathcal{T}(s), \tag{7.1}$$

and it is useful to define

$$\mathcal{U}(s) = \int_{C(s)} u_\phi s \,\mathrm{d}z \,\mathrm{d}\phi$$

as a representation of the fluid response. We assume here that the flow $\boldsymbol{u}$ is incompressible and thus represented in poloidal–toroidal decomposition, whose scalar fields are expanded in spherical harmonics and polynomials in radius, just as (2.1) and (4.1). By inspecting equation (2.5), it is clear that the only components to survive the cylindrical average are $\boldsymbol{T}_l^0$, where l is odd, and so

$$\mathcal{U}(s) = -2\pi s \sum_{l \text{ odd}} \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} \frac{T_l^0(r)}{r} \frac{\mathrm{d}P_l^0}{\mathrm{d}\theta} \mathrm{d}z.$$

Each l component of $\mathcal{U}(s)$ has the form $\mathcal{U}_l(s) = s^2\sqrt{1-s^2}\,Q_{N_l}(s^2)$, where Q_{N_l} is a polynomial of degree $N_l = (l-1)/2 + N_{\max}$.

This follows by appealing to regularity just as in §4; we can write

$$\mathcal{U}(s) = -2\pi s^2 \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} f(s^2, z^2)\mathrm{d}z,$$

where f is some multinomial. Arguing as before, the form of $\mathcal{U}(s)$ follows immediately; by counting up the dimensions of length, we see that $Q_{N_l}(s^2)$ has the greatest exponent of s of at most $2N_l = l + 2N_{\max} - 1$.

Lastly, we note that, since viscosity has been discarded, the only remaining physical choice of boundary conditions is stress free. With electrically insulating boundary conditions, the vanishing electromagnetic torque is consistent with the zero rate of change of net angular momentum.

8. Discussion

We have derived the form of the Taylor integral on the assumption of a particular truncated expansion of the magnetic field, regular at the origin. The result can be made to vanish on *every* geostrophic contour (here, cylinders coaxial with the rotation axis) by imposing a finite sequence of quadratic constraints, whose number is simply related to the degree of the truncation and constitutes a small fraction of the available degrees of freedom.

One might think to compare Taylor's constraint (1.3) with the solenoidal condition,

$$\int_C \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} = 0,$$

which holds over any surface C. But the distinctions that C is arbitrary, rather than a geostrophic contour, and that the integrand involves all three components of $\boldsymbol{B}$, rather than the ϕ component alone, are crucial. The consequence of solenoidality, namely that $\nabla\cdot\boldsymbol{B}=0$, means that the vector field $\boldsymbol{B}$ has not three independent components but two: the toroidal and poloidal scalar fields. The solenoidal condition, which holds pointwise, thus provides an infinite number of constraints on the magnetic field in contrast to the finite number imposed by the Taylor condition. A more apt analogy is that between the Taylor condition and homogeneous boundary conditions on a specified spherical surface. The latter would typically result in $L_{\max}(L_{\max}+2)$ constraints on each of the toroidal and poloidal scalar functions, one for each spherical harmonic component.

As we set out in appendix C, in view of the ultimate aim to collapse the continuum of the Taylor constraints onto a finite number, the adoption of a regular basis expansion for the magnetic field is expedient, though not necessary. In fact, we show in general terms that any choice of radial basis functions is likely to result in $\mathcal{T}(s)$ that is confined to a space of dimension N_T, for some N_T whose dependence on the truncation is determined by the particular choice of basis. Defining N_B as the number of spectral magnetic field coefficients, it is clear that only if $N_B > N_T$ can a Taylor state be found. This issue has significant bearing on the choice of numerical schemes that might be employed in order to solve the Navier–Stokes equations (1.1), or any associated reduced form, with vanishingly small E in order to find an exact Taylor state in the full sphere. As the analysis shows, basis sets based on Bessel functions have $N_T \gg N_B$, and thus, barring special cases, no solutions exist and any search for an exact Taylor state will fail. A non-regular expansion in terms of Chebyshev polynomials could theoretically produce a Taylor state since $N_T \ll N_B$, although care would have to be taken in handling the singular behaviour of the resulting algebraic form at the origin. However, the best way to proceed is to use the regular polynomial form assumed in this paper that evidently leads to the most elementary form of $\mathcal{T}(s)$ and, we conjecture, is optimal in the sense that it minimizes N_T/N_B.

Lastly, we remark on the construction of Taylor states in the absence (or presence) of an inner core. It is clear that, since $N_B \gg N_T$ (at least for the regular polynomial basis set) that simply on the grounds that the system is vastly underdetermined, Taylor states are ubiquitous. Rather than attempt to characterize a general Taylor state, in a forthcoming paper, we consider the simpler question as to whether it is possible to construct a Taylor state for a given poloidal field by choosing an appropriate toroidal field (or vice versa). Even this reduced problem is not entirely straightforward as, in general, the system comprises coupled quadratic equations for the unknown toroidal coefficients.

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Appendix A. Rotational symmetry

In this section, we discuss the symmetry of vector spherical harmonics about the x-axis. This axis is singled among all possible choices because it coincides with $\phi=0$, built into the coordinate system, about which both $\sin m\phi$ and $\cos m\phi$ enjoy simple symmetry. A rotation about the x-axis is defined to be the mapping

$$(r,\theta,\phi) \to (r,\pi-\theta,2\pi-\phi).$$

Let us define the classes of rotationally symmetric vectors R^S and the rotationally antisymmetric vectors R^A by the property of each $S \in R^S$ and $A \in R^A$,

$$[S_r, S_\theta, S_\phi](r,\pi-\theta,2\pi-\phi) = [S_r, -S_\theta, -S_\phi](r,\theta,\phi) \tag{A 1}$$

and

$$[A_r, A_\theta, A_\phi](r,\pi-\theta,2\pi-\phi) = [-A_r, A_\theta, A_\phi](r,\theta,\phi). \tag{A 2}$$

We now show that any vector spherical harmonic

$$\left.\begin{aligned} \boldsymbol{S}_l^{m\ s/c} &= \nabla\times\nabla\times[Y_l^{m\ s/c}(\theta,\phi)S_l^{m\ s/c}(r)\hat{\boldsymbol{r}}] \\ \text{and} \qquad \boldsymbol{T}_l^{m\ s/c} &= \nabla\times[Y_l^{m\ s/c}(\theta,\phi)T_l^{m\ s/c}(r)\hat{\boldsymbol{r}}], \end{aligned}\right\} \tag{A 3}$$

with

$$Y_l^{m\ s/c} = P_l^m(\cos\theta)\left\{\begin{matrix}\sin m\phi \\ \cos m\phi\end{matrix}\right\}, \tag{A 4}$$

has a definite rotational symmetry.

First, we note that the unit position vector $\hat{\boldsymbol{r}}$ remains unchanged by the mapping, thus $(1,0,0) \in R^S$. Second, the associated Legendre function has the property that

$$P_l^m(\cos(\pi-\theta)) = (-1)^{l-m} P_l^m(\cos\theta),$$

and clearly $\sin m\phi$ is antisymmetric under $\phi \to 2\pi - \phi$, with $\cos m\phi$ being symmetric. Hence, $Y_l^{ms/c}\hat{\boldsymbol{r}} \in R^S$ if either $l-m$ is even and the azimuthal dependence is cosine, or $l-m$ is odd and the azimuthal dependence is sine; it is a member of R^A otherwise.

Next, we show that if $\boldsymbol{a}$ and $\boldsymbol{b}$ are both R^S or R^A, then $\boldsymbol{a}\times\boldsymbol{b} \in R^S$; if $\boldsymbol{a}$ and $\boldsymbol{b}$ differ in symmetry, then $\boldsymbol{a}\times\boldsymbol{b} \in R^A$. This result follows immediately from the identity

$$\boldsymbol{a}\times\boldsymbol{b} = (a_\theta b_\phi - a_\phi b_\theta, a_\phi b_r - a_r b_\phi, a_r b_\theta - a_\theta b_r),$$

and by using equation (A 1), we see that if $\boldsymbol{a}$ and $\boldsymbol{b}$ belong to the same rotational symmetry class, then the radial component is sign invariant and the other two components must change sign; thus, $\boldsymbol{a}\times\boldsymbol{b} \in R^S$. Conversely, if they belong to different symmetry classes, then each component has a further sign change and $\boldsymbol{a}\times\boldsymbol{b} \in R^A$. Lastly, the gradient vector can be written as

$$\nabla = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) \in R^S,$$

so that $\nabla\times\boldsymbol{u}$ takes on the same symmetry as $\boldsymbol{u}$.

It follows that if $Y_l^{ms/c}\hat{\boldsymbol{r}} \in R^S$, then $\boldsymbol{S}_l^{ms/c} \in R^S$ and likewise for the R^A symmetry. Thus,

$$R^S = \{\boldsymbol{S}_l^{mc} \text{ or } \boldsymbol{T}_l^{mc} : l-m \ \text{ even}\} \cup \{\boldsymbol{S}_l^{ms} \text{ or } \boldsymbol{T}_l^{ms} : l-m \ \text{ odd}\} \tag{A 5}$$

and

$$R^A = \{\boldsymbol{S}_l^{ms} \text{ or } \boldsymbol{T}_l^{ms} : l-m \ \text{ even}\} \cup \{\boldsymbol{S}_l^{mc} \ \text{ or } \ \boldsymbol{T}_l^{mc} : l-m \ \text{ odd}\}. \tag{A 6}$$

Lastly, we make a remark about the solutions of the (non-dimensional) magnetic induction equation,

$$\frac{\partial\boldsymbol{B}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{B}) + \nabla^2\boldsymbol{B}.$$

If $\boldsymbol{u} \in R^S$, then all the operators $\partial/\partial t$, $\nabla\times(\boldsymbol{u}\times\cdot)$ and ∇^2 preserve definite rotational symmetry of $\boldsymbol{B}$. It follows that the family of solutions of the induction equation separates into the two independent classes of rotational symmetry.

Appendix B. An analytic form for the toroidal–toroidal interaction

In this section, we derive the analytic form for the interaction between two toroidal vector spherical harmonics.

According to equation (2.7), the interaction $[\boldsymbol{T}_{l,m}, \boldsymbol{T}_{l',m}]$ is proportional to

$$\int \frac{P_l^m(\cos\theta)P_{l'}^m(\cos\theta)\,T_l^m(r)\,T_{l'}^m(r)}{r^3\sin\theta}\, s\,\mathrm{d}z, \tag{B 1}$$

where we have already integrated with respect to ϕ (and assume that the result is non-zero). Since $T_l^m(r)=\sum_{j=0} a_j r^{l+1+2j}$, we consider the contribution

$$G(s;l,l',m,p)=\int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}}\frac{P_l^m(\cos\theta)P_{l'}^m(\cos\theta)r^{l+l'+2p-1}}{\sin\theta}\,s\,\mathrm{d}z \qquad (\mathrm{B}\,2)$$

to equation (B 1), where $p\geq 0$. We assume that the symmetry selection rules, detailed in §3, are obeyed, which are necessary for the interaction to be non-zero; this means that $l\neq l'$, $l-l'=0(\mathrm{mod}\ 2)$ and $m\geq 1$. Since the a_j are arbitrary, it is clear that any identity that applies to (B 1), at least in the absence of boundary conditions, must also apply to $G(s;l,l',m,p)$, and vice versa.

We write the integral in terms of the co-latitude θ (but keeping s constant), by writing $r=s/\sin\theta$ and $\cos\theta=z/\sqrt{z^2+s^2}$, so that $\mathrm{d}z=-s/\sin^2\theta\,\mathrm{d}\theta$ and, on using equatorial symmetry, the integral becomes

$$-2s^{l+l'+2p+1}\int_{\sin^{-1}s}^{\pi/2}\frac{P_l^m(\cos\theta)P_{l'}^m(\cos\theta)}{(\sin\theta)^{l+l'+2p+2}}\mathrm{d}\theta. \qquad (\mathrm{B}\,3)$$

Up to a normalization, the associated Legendre functions can be written as

$$\left.\begin{aligned}P_l^m(\cos\theta)&=(\sin\theta)^m\sum_{k=0}^{\lfloor\frac{l-m}{2}\rfloor}\mu_{lmk}(\cos\theta)^{l-m-2k}\\ \text{and}\qquad \mu_{lmk}&=\frac{(2l-2k)!(-1)^k}{(l-m-2k)!k!(l-k)!},\end{aligned}\right\} \qquad (\mathrm{B}\,4)$$

where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x, and so

$$G(s;l,l',m,p)=As^{l+l'+2p+1}\sum_{k=0}^{\lfloor\frac{l-m}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{l'-m}{2}\rfloor}\mu_{lmk}\mu_{l'mj}\int_{\sin^{-1}s}^{\pi/2}\frac{(\cos\theta)^{l+l'-2m-2j-2k}}{(\sin\theta)^{l+l'+2p-2m+2}}\mathrm{d}\theta, \qquad (\mathrm{B}\,5)$$

for some constant A. The integral appearing in the expression (B 5) can be written exactly in terms of a hypergeometric function,

$$\int_{\sin^{-1}s}^{\pi/2}\frac{(\cos\theta)^b}{(\sin\theta)^{b+c}}\mathrm{d}\theta=\frac{1}{b+c-1}\frac{(1-s^2)^{(b+1)/2}}{s^{b+c-1}}\,{}_2F_1\left(1,1-\frac{c}{2};\frac{3-b-c}{2};s^2\right), \qquad (\mathrm{B}\,6)$$

subject to restrictions on b and c (satisfied here) that are tedious to state and more readily apprehended as plotted in figure 4.

So we can now write, up to a multiplicative factor,

$$\begin{aligned}G(s;l,l',m,p)=s^{2m}\sqrt{1-s^2}\sum_{k=0}^{\lfloor\frac{l-m}{2}\rfloor}\sum_{j=0}^{\lfloor\frac{l'-m}{2}\rfloor}(1-s^2)^{(l+l')/2-m-j-k}\frac{\mu_{lmk}\mu_{l'mj}}{l+l'-2m+2p+1}\\ \times{}_2F_1\left(1,-(p+j+k);\frac{3-l-l'+2m-2p-2}{2};s^2\right),\end{aligned} \qquad (\mathrm{B}\,7)$$

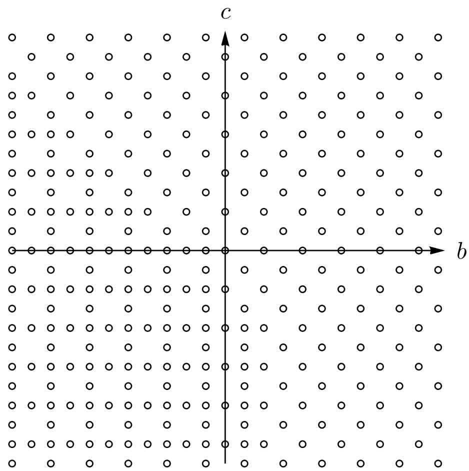

Figure 4. Allowed integer values of b and c for which (B 6) applies. (Circles on the left-half b-axis are at unit spacing.)

which here takes the form of finite-degree polynomials in s^2. We note that this closed expression for $G(s;l,l',m,p)$ (and therefore $[\boldsymbol{T}_{l,m}, \boldsymbol{T}_{l',m}]$) is consistent with the structure derived in §§4 and 5c. However, it is not easily proven that such a term satisfies the condition of a zero average electromagnetic torque from §5a,

$$\int_0^1 sG(s;l,l',m,p)\mathrm{d}s = 0.$$

Reordering the sums to isolate the coefficient of $\sqrt{1-s^2}s^{2(j+m)}$ leads to the necessary vanishing of a relation involving ${}_4F_0$ and ${}_4F_7$ that seems not in the form of a known identity for generalized hypergeometric functions. Yet, the result must hold and indeed is straightforwardly verified to do so for any specific choice of parameters (l, l', m, p).

Appendix C. Finiteness with a non-regular basis

We now return to the extent to which regularity of the basis expansion for the magnetic field bears on the main result of this paper, i.e. whether or not the continuum of the Taylor constraints still collapses to a finite number N_T of conditions. In particular, we now examine the effect of relaxing the functional form of (4.1) by, for example, expanding the poloidal and toroidal scalars in Chebyshev polynomials or Bessel functions, both with argument r.

The key to the whole issue is the determination of the existence of a finite algebra in which $\mathcal{T}(s)$ must lie or, in other words, finding a finite-dimensional space to which the analytic form of any interaction belongs. In the case we have considered in the bulk of this paper, this space is merely that of polynomials of a finite degree (with an additional prefactor of $\sqrt{1-s^2}$). Here, we do not give a proof of the general case, but rather indicate the variety and dimension of finite algebras that can arise for any choice of radial basis, regular or not. Of particular note is the relative size of N_T in comparison with the total number of spectral coefficients, $N_B = 2L_{\max}(L_{\max}+2)N_{\max}$, used to define the magnetic field. Three regimes are important. First, if $N_T \ll N_B$ (as in §4), then it is likely that many non-trivial Taylor states exist within the specified truncation as the problem is vastly underdetermined. Second, if $N_T \approx N_B$, then Taylor states exist so long as the system of quadratic equations is soluble. Finally, if $N_T \gg N_B$, then, barring exceptional circumstances, no exact Taylor states can be found. The reader may wonder how the single statement of Taylor's condition can lead to more than one collection of constraints; this is due to the fact that any two truncated expansions, in different radial basis sets, are not in general equivalent. For instance, it is not possible to write a finite sum of trigonometric functions as a finite sum of polynomials; different algebras result and significant simplification in structure (with a subsequent reduction in the number of constraints) may occur in some cases, but not others.

We begin by considering a radial expansion of the toroidal and poloidal scalars in terms of the natural polynomial set $1, r, r^2, \ldots$ (or any linear combination thereof, e.g. including the commonly used Chebyshev polynomials). As an illustration of the effect of introducing non-regular behaviour, the toroidal–toroidal interaction results in three general terms. These assume the form

$$\mathcal{T}(s) = \frac{\sin^{-1} s}{s}\,\tilde{Q}_1(s^2) + \sqrt{1-s^2}\,\tilde{Q}_2(s^2) + s^2 \log\left(\frac{1}{s} - \frac{\sqrt{1-s^2}}{s}\right)\tilde{Q}_3(s^2),$$

where each of $\tilde{Q}_1$, $\tilde{Q}_2$ and $\tilde{Q}_3$ is a polynomial of finite degree. There is no reason to expect that pointwise vanishing of *this* $\mathcal{T}(s)$ bears a close relation to the formulation in (4.5); the two sets of constraints exist in different spaces, as do the fields represented. Most significant is that individual interactions computed with a non-regular basis set (though which satisfies electrically insulating boundary conditions) are not guaranteed to satisfy the requirement that

$$\int_0^1 s\mathcal{T}(s)\mathrm{d}s = 0$$

as an identity.

The above discussion might be thought restricted exclusively to the case of a polynomial expansion in r, but this is not the case. By way of example, note that if the lowest order radial expansion is taken as

$$T_l = r^{l+1}(\mathrm{e}^{1-r^2} - 1) \quad \text{and} \quad S_l = r^{l+1}\left(\frac{2l+1}{2l-1}\mathrm{e}^{1-r^2} - 1\right) \tag{C 1}$$

for the toroidal and poloidal fields, respectively, the general form of $\mathcal{T}(s)$, allowing for $[T, T]$, $[T, S]$ and $[S, S]$ interactions, is

$$\begin{aligned}\mathcal{T}(s) = {} & s^2\sqrt{1-s^2}\,\tilde{Q}_1(s^2) + s^2\mathrm{e}^{-s^2}\mathrm{erf}(\sqrt{1-s^2})\,\tilde{Q}_2(s^2) \\ & + s^2\,\mathrm{e}^{-2s^2}\,\mathrm{erf}(\sqrt{2(1-s^2)})\,\tilde{Q}_3(s^2),\end{aligned} \tag{C 2}$$

where each of the $\tilde{Q}_k$ is a polynomial in s^2 of maximal degree determined by a scheme similar to that in table 1.

Taking $L_{\max}$ for the poloidal field and $L_{\max}=4$ for the toroidal (and $N_{\max}=1$), from table 1, we find the resulting $Q_N(s^2)$ is of degree 8. Hence there are five constraint equations, however, owing to degeneracy, only four of these are independent. By comparison, for the expansion here, $\tilde{Q}_k$ are of degrees [4,6,6], respectively, yielding 11 possible constraint equations. Accounting for degeneracy drops one equation from each set, for a net of eight constraint equations. The exact enumeration of constraints is, *inter alia*, a question in the theory of integration in finite terms. But, guided by the examples above, we can anticipate four general cases based simply on multiplicative properties of the radial basis functions, which we here denote as $\chi_{l,n}(r)$, not needing, for the present purpose, to distinguish the poloidal from the toroidal expansion.

In the main body of this paper, we focus on a regular expansion with $\chi_{l,n}(r)$ given as a polynomial in r. The key to the result is that exponents add and that the indices l and n enter on the same footing; hence, the final result can only depend upon the sum $l_1+l_2+2n_1+2n_2$, rather than any of these separately. It follows that vanishing of $\mathcal{T}(s)$, with or without an inner core, must require $N_T=\mathcal{O}(L_{\max}+2N_{\max})$ constraints. We can even, as briefly outlined above, drop the requirement of regularity and, although N_T is increased by some constant independent of $L_{\max}$ and $N_{\max}$, the same form applies.

The form of the second result for $\mathcal{T}(s)$ above reflects that exponents of exponentials also add; however, as these constitute a separate class of functions from the polynomial dependence on l, the final result must depend on l_1+l_2, and $2n_1+2n_2$ separately, and so purely as an exercise in counting, vanishing of $\mathcal{T}(s)$ now requires $N_T=\mathcal{O}(L_{\max}N_{\max})\ll N_B$ constraints. This result extends to trigonometric functions as well, where the product depends on sums and differences of n_1 and n_2. While the number of constraints is a considerable increase over the polynomial case, there are still many more degrees of freedom than constraints.

If $\chi_{l,n}(r)$ is chosen in the product form $r^{l+1}\psi_n(r)$ where products of the $\psi_n(r)$ have no reduction, then the form for $\mathcal{T}(s)$ will depend on n_1 and n_2 individually and so the number of constraints is then $N_T=\mathcal{O}(L_{\max}N_{\max}^2)$ (assuming expansions of equal degree for the poloidal and toroidal fields). The existence of a general Taylor state now depends on the choice of $L_{\max}$ relative to $N_{\max}$.

Lastly, we have the case where $\chi_{l,n}(r)$ is not of the product form but rather a choice such as $j_l(\mu_n r)$, the lth-order spherical Bessel function. In this case, the result now also depends separately on the indices l_1 and l_2 and so the number of constraints saturates at $N_T=\mathcal{O}(L_{\max}^2N_{\max}^2)\gg N_B$. In general, no exact Taylor state is possible. However, there are exceptional cases, such as that of a single pure decay mode, since it does not self-interact. There is no contradiction here; it is simply that a counting argument to establish the order does not incorporate selection rules based on symmetries, and hence special solutions remain possible.

In summary, a closed set of equations to determine fields such that $\mathcal{T}(s)$ vanishes identically does *not* require purely polynomial forms in r, although these certainly lead to the most elementary construction of general three-dimensional Taylor states and, we conjecture, the one with the fewest constraints in relation to the available degrees of freedom.

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