Fast Dynamos:
Large- and Small-Scale Dynamos at High Rm

David Hughes
Department of Applied Mathematics
University of Leeds

Classification of dynamos

Dynamos can be classified into various categories:

- Kinematic or dynamic
- Slow or fast
- Large- or small-scale

Kinematic vs. dynamic

A *kinematic* dynamo is one for which the magnetic field evolves under the induction equation, with a prescribed velocity field. It is characterised by exponential growth of the magnetic energy.

The magnetic field has no influence on the velocity field.

A *dynamic* or *magnetohydrodynamic* dynamo is one for which the magnetic and velocity field are treated self-consistently, via solution of both the momentum equation (the Navier-Stokes equation) and the induction equation. The magnetic energy is maintained at a non-zero average value.

Here the magnetic field influences the velocity field via the Lorentz force.

Slow vs. fast

This is a means of distinguishing kinematic dynamos, by consideration of their growth rates *s* as the magnetic Reynolds Number Rm _ ∞.

If Re(s) > 0 as Rm _ ∞ then the dynamo is said to be *fast.*

If not then the dynamo is said to be *slow*.

For dynamic dynamos it is, in general, not possible to classify them as slow or fast. (Although one could do this with a knowledge of the fastness of the underlying kinematic dynamo.)

If a dynamic dynamo is oscillatory with reversals on a fast (non-diffusive) time scale (the solar dynamo e.g.) then it seems reasonable to designate it as *fast*.

Large- vs. small-scale

Most clear cut definition of a *large-scale* dynamo is one that can only operate in a domain that is large compared with a typical turbulent velocity scale; i.e. reducing the size of the box containing the dynamo will turn it off.

This makes sense in Cartesian geometry, less so for dynamos in spheres.

More generally, a large-scale dynamo is one with significant energy on large scales.

A small-scale dynamo has magnetic energy predominantly on scales comparable with those of the velocity.

The mechanisms invoked for small-scale (high Rm) and large-scale dynamos are very different.

Dynamos in periodic domains

Many studies have been performed on dynamos in periodic domains: e.g. the G.O. Roberts flow

$$\mathbf{u} = \nabla \times \psi \hat{\mathbf{z}} + \psi \hat{\mathbf{z}}, \qquad \psi = \cos x + \cos y$$

or the ABC flow

$$\mathbf{u} = A(0, \sin x, \cos x) + B(\cos y, 0, \sin y) + C(\sin z, \cos z, 0)$$

Assume the magnetic field has the same periodicity as the flow. Are the dynamos driven by these flows small-scale or large-scale? (They are often studied using the ideas of mean field theory.)

Magnetic field for the Roberts flow

For the flow

$$\mathbf{u} = (-\sin y, \sin x, \cos x + \cos y)$$

the field has the Fourier decomposition:

$$\mathbf{B} = e^{ikz+st} \sum_{l,m=0}^{\infty} \mathbf{B}_{lm} e^{i(lx+my)}$$

So the field is on a scale comparable with and smaller than that of the velocity, but also has a "large-scale" component with _ = m = 0.

So there is a mean field ***B***(z,t), but this is also just part of the small-scale eigenfunction.

Large-scale dynamos: scale separation

Starting point is the magnetic induction equation of MHD:

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B},}$$

where **B** is the magnetic field, **u** is the fluid velocity and _ is the magnetic diffusivity (here assumed constant for simplicity).

Assume scale separation between large- and small-scale field and flow:

$$\boxed{\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \quad \mathbf{U} = \mathbf{U}_0 + \mathbf{u},}$$

where **B** and **U** vary on some large length scale L, and **u** and **b** vary on a much smaller scale l.

$$\boxed{\langle \mathbf{B} \rangle = \mathbf{B}_0, \quad \langle \mathbf{U} \rangle = \mathbf{U}_0,}$$

where averages are taken over some intermediate scale $l \ll a \ll L$.

Large-scale dynamos: the mean induction equation

Induction equation for mean field:

$$\boxed{\frac{\partial \mathbf{B}_0}{\partial t} = \nabla\times(\mathbf{U}_0\times\mathbf{B}_0) + \nabla\times\mathsf{E} + \eta\nabla^2\mathbf{B}_0,}$$

where mean emf is $\boxed{\mathsf{E} = \langle \mathbf{u}\times\mathbf{b}\rangle}$

This equation is exact, but is only useful if we can relate E to B_0.

Large-scale dynamos: fluctuating field

Consider the induction equation for the fluctuating field:

$$\boxed{\frac{\partial \mathbf{b}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{B}_0) + \nabla\times(\mathbf{U}_0\times\mathbf{b}) + \nabla\times\mathbf{G} + \eta\nabla^2\mathbf{b},}$$

where $\mathsf{G} = \mathsf{u}\times\mathsf{b} - \langle \mathsf{u}\times\mathsf{b}\rangle$.

Traditional approach is to assume that the fluctuating field is driven solely by the large-scale magnetic field.

i.e. in the absence of $\boldsymbol{B}_0$ the fluctuating field decays. ***No small-scale dynamo.***

Under this assumption, the relation between $\mathbf{b}$ and $\mathbf{B}_0$ (and hence between E and $\mathbf{B}_0$) is linear and homogeneous.

Expression for the emf

Postulate an expansion of the form:

$$\mathsf{E}_i = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + \ldots$$

where _$_{ij}$ and _$_{ijk}$ are *pseudo*-tensors.

Simplest case is that of isotropic turbulence, for which _$_{ij}$ = __$_{ij}$ and _$_{ijk}$ = __$_{ijk}$.
Then mean induction equation becomes:

$$\frac{\partial \mathbf{B}_0}{\partial t} = \nabla \times (\mathbf{U}_0 \times \mathbf{B}_0) + \nabla \times (\alpha \mathbf{B}_0) + (\eta + \beta)\nabla^2 \mathbf{B}_0.$$

_: regenerative term, responsible for large-scale dynamo action.
Since **E** is a polar vector whereas $\mathbf{B}_0$ is an axial vector then _ can be non-zero only for turbulence lacking reflexional symmetry (i.e. possessing handedness).

_: turbulent diffusivity.

Determination of _

_ can be rigorously determined analytically only if min(*Rm*, *St*) << 1
(*Rm* = magnetic Reynolds number = *UL*/_,
St = Strouhal number = ratio of correlation to turnover times).

The ***G*** term can then be neglected – so-called *first order smoothing approximation*.
Not satisfied astrophysically.

For small correlation times _:

$$\alpha = -\frac{\tau}{3}\langle \mathbf{u} \cdot \nabla \times \mathbf{u} \rangle$$

(Krause & Rädler)

i.e. _ is intimately related to the helicity of the flow.

There are no rigorous results for the astrophysically relevant regime of *Rm* >> 1 and *S* = *O*(1).

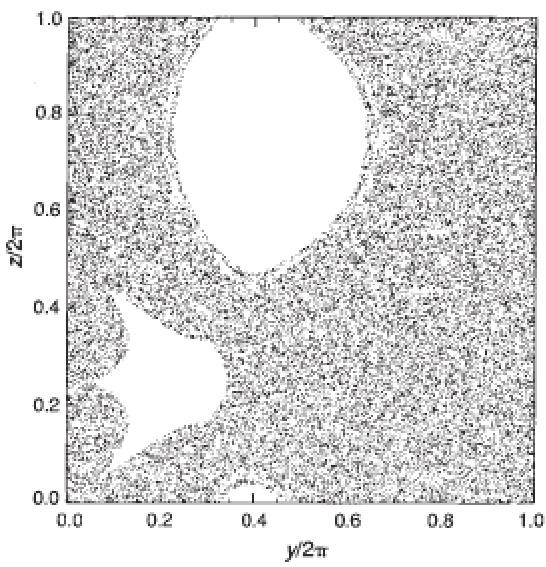
Generation mechanisms

Large-scale (mean field) dynamos rely on a lack of reflectional symmetry, e.g. as provided by helical flows. Though the relationship between _ and helicity is not, in general, straightforward.

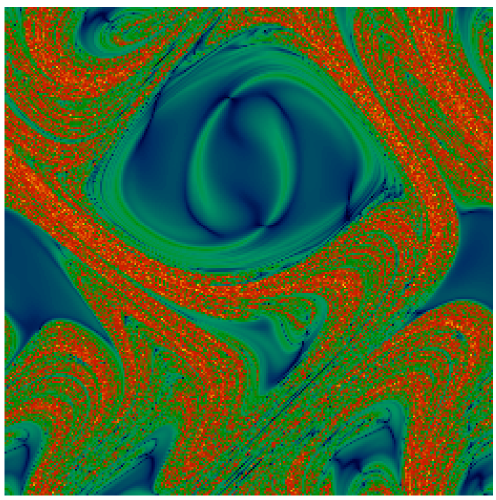
Small-scale dynamos have mainly been studied in the context of fast dynamos. Fast dynamos require Lagrangian chaos in the flow (Vishic, Klapper & Young).

In general, astrophysical flows will be both helical and chaotic.

Chaos for GP flow

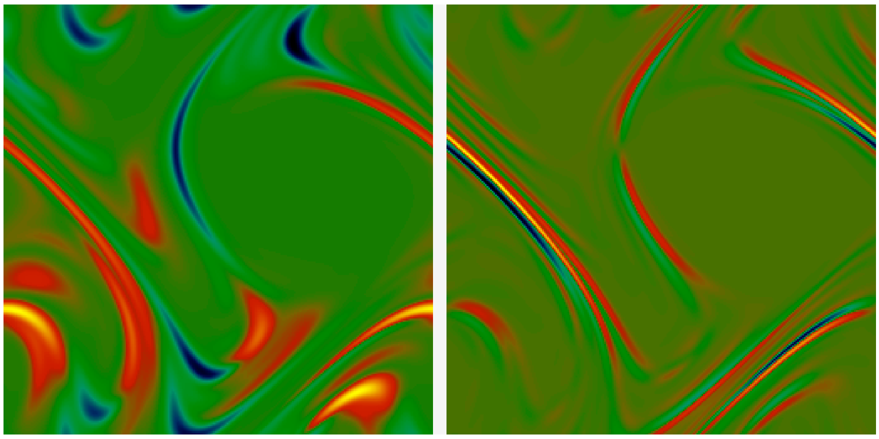


Poincaré map



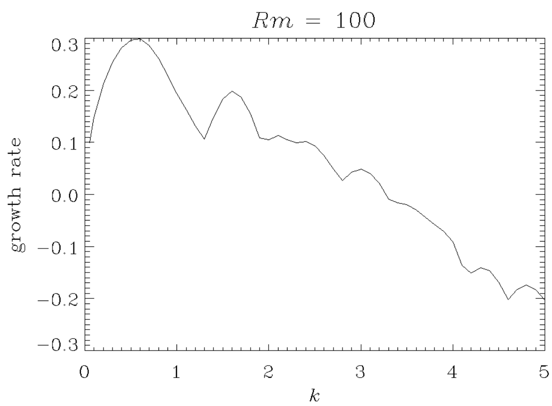
Lyapunov exponents

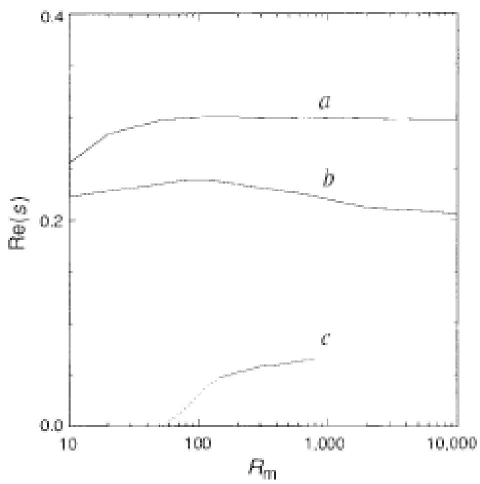
Eigenfunctions: Rm=100



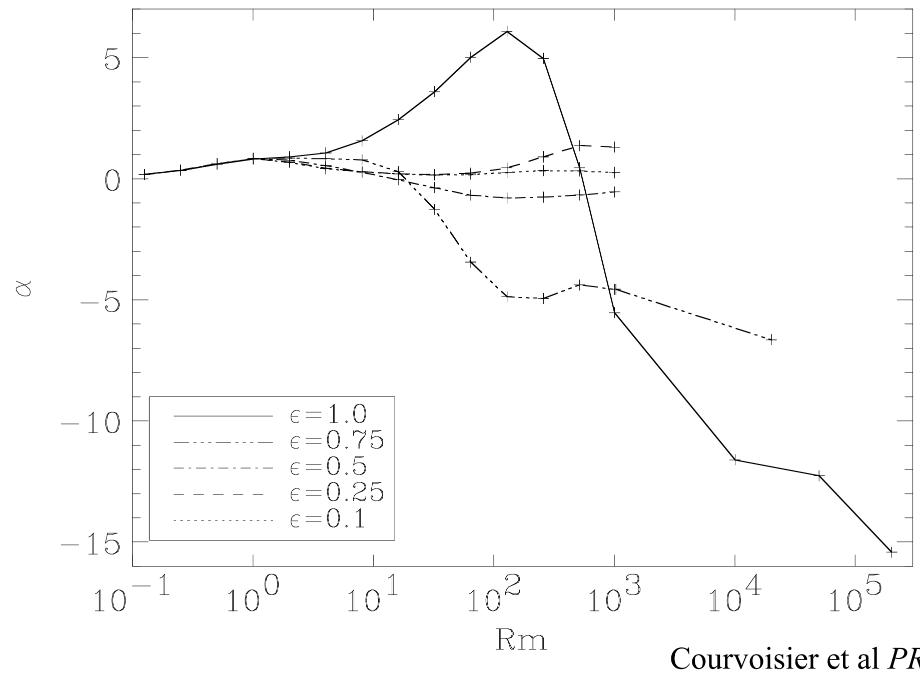
B_z J_z

Growth rates





_-effect for GP flow



Courvoisier et al *PRL* (2006)

The kinematic _-effect

Consider the measurement and interpretation of the kinematic _-effect.

This is typically done by measuring the mean emf resulting from the imposition of a uniform magnetic field (or different fields in order to pick up the different components).

How big a domain is needed for sensible averaging?

Consider the relation for the standard error of the mean:

$$N \sim \left(\frac{\sigma}{\varepsilon}\right)^2$$

Here _ is the desired uncertainty in the mean of *N* independent samples, each with standard deviation _ – thus, given _ and _, *N* follows.

N represents the number of patches of size _ needed to achieve required accuracy.

Measuring the kinematic _-effect

Make the assumption (well verified at large *Rm*) that the mean emf is small in comparison with the fluctuations. Then

$$\sigma \sim \frac{|\mathbf{u}\times\mathbf{b}|}{|\langle\mathbf{B}\rangle|} \sim |\mathbf{u}|\frac{|\mathbf{b}|}{|\langle\mathbf{B}\rangle|}$$

Two cases to consider:

1. No small-scale dynamo. Then $\frac{|\mathbf{b}|}{|\langle\mathbf{B}\rangle|} \sim Rm^{\gamma_1} \qquad (1/2 < \gamma_1 < 1)$

Cattaneo & Hughes 2009

Hence $L \sim \ell\left(\frac{|\mathbf{u}|\,Rm^{\gamma_1}}{\varepsilon}\right)^{2/3}$

2. Small-scale dynamo. Then $|\mathbf{u}\times\mathbf{b}|$ grows exponentially at the small-scale growth rate *s*.

Hence $L \sim \ell\left(\frac{|\mathbf{u}|\,Rm^{\gamma_2}e^{st}}{\varepsilon}\right)^{2/3}$

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The kinematic _-effect: interpretation

In the absence of small-scale dynamo action, knowledge of _ (and _) provides information about the growth of a field on a large scale ~ 1/*k:*

$$s \sim \alpha k - \beta k^2 \qquad (*)$$

Increasing the domain size from $L \sim$ _ will allow dynamo action to set in when $L =$ _/_.

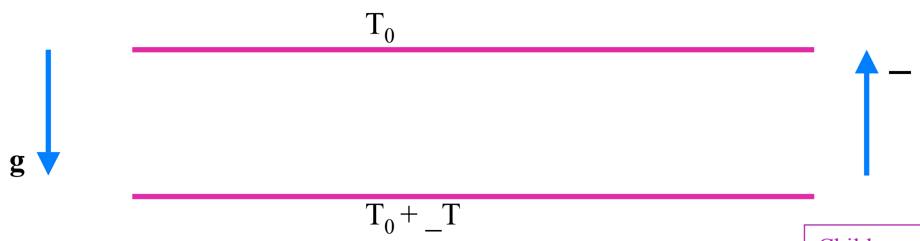
However, if there is small-scale dynamo action then, by definition, there is dynamo action when $L \sim$ _.
This will be essentially independent of domain size *L*.
Any average of the magnetic field at intermediate scales will grow with the same growth rate.
The growth of the magnetic field on large scales has nothing to do with that predicted by equation (*).

A specific example

The ideas discussed above can be illustrated by consideration of a specific model – plane layer rotating Boussinesq convection.

In particular it is possible to show the difficulties in determining an _ coefficient, the convergence to the (small) mean being extremely slow, and also to see that the field that emerges is not a large-scale dynamo, even in the presence of helicity.

Dynamo action driven by rotating convection



Boussinesq convection.
Boundary conditions: impermeable, stress-free, fixed temperature, perfect electrical conductor.

Childress & Soward 1972
Soward 1974
St Pierre 1993
Jones & Roberts 2000
Rotvig & Jones 2002
Stellmach & Hansen 2004

Taylor number, $Ta = 4\_^2 d^4/\_^2 = 5 \text{ x } 10^5$.
Prandtl number $Pr = \_/\_ = 1$.
Magnetic Prandtl number $Pm = \_/\_ = 5$.
Critical Rayleigh number for onset of convection = 59 008.
Critical Rayleigh number for onset of dynamo action ≈ 170 000.

Cattaneo & Hughes 2006 *JFM*
Hughes & Cattaneo 2008 *JFM*

Rotating Boussinesq convection: averaging

Averaging is taken over horizontal planes, so the mean magnetic field takes the form $(B_x(z), B_y(z), 0)$.

$$\frac{\partial\langle\mathbf{B}\rangle}{\partial t} = \mathbf{e}_z \times \frac{\partial \mathsf{E}}{\partial z} + \eta \frac{\partial^2 \langle\mathbf{B}\rangle}{\partial z^2}$$

For Boussinesq convection, E_x and E_y are anti-symmetric about the mid-plane.

Thus a meaningful average is one over the upper (or lower) half of the domain.

Dynamo close to marginal convection

Problem first analysed by Soward (1974) (following Childress & Soward 1972).

Assumed $E = Ta^{-1/2} << 1$, $Pr = O(1)$ and considered mildly supercritical convection with

$$\frac{Ra - Ra_c}{Ra_c} = O(E^{1/3})$$

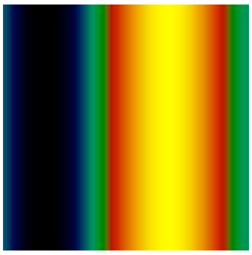
Then, under first order smoothing:

$$\alpha_{ij} = \frac{\pi}{E^{1/2}\eta} \sum_{\mathbf{k}} \frac{k_i k_j}{k^6} q(\mathbf{k}, t) \sin 2\pi z$$

where $q(\mathbf{k}, t) = |w(\mathbf{k}, t)|^2$.

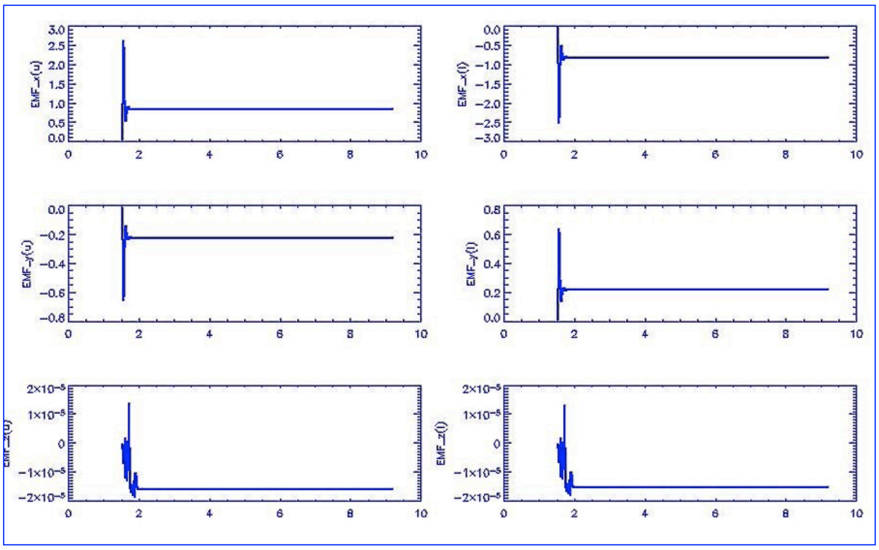
Calculation of _ for small box, laminar convection

***Ra* = 80 000**



Temperature contours:
aspect ratio = 0.5

$\langle \mathbf{u}^2 \rangle = 330$



3 components of e.m.f. *vs* time, calculated over upper and lower half-spaces.

$\_{xx} \approx 8.5$

Mean field theory: philosophy

The hope of mean field dynamo theory is to obtain information on _, _, etc. either analytically or via “small” numerical simulations, and then to apply this information to say something about dynamo action on a large scale.

Changing the aspect ratio

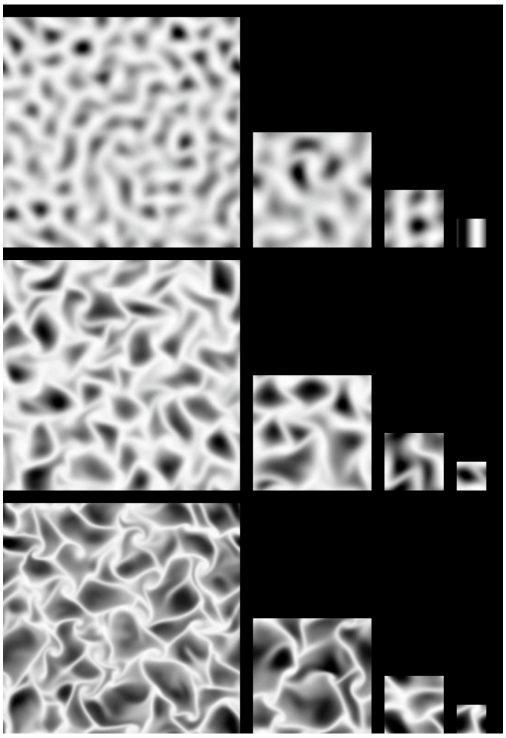
_ = 5, _ = 2, _ = 1, _ = 0.5

Convective Planforms:
No field

Ra = 80,000

Ra = 150,000

Ra = 500,000



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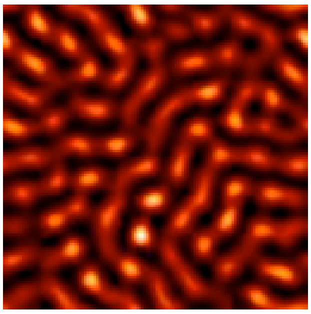
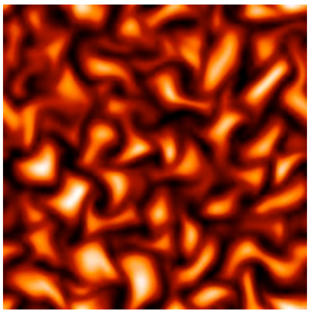
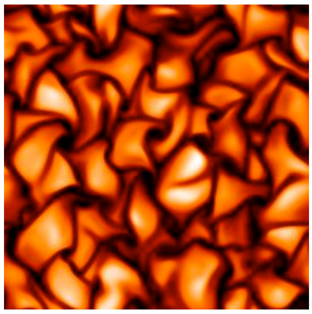
Rotating convection: large aspect ratio



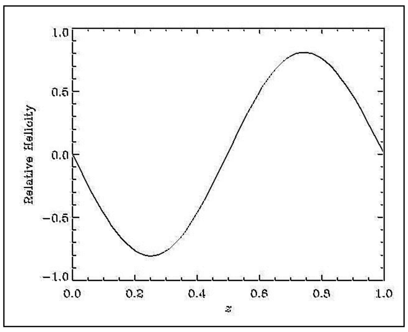
$Ra = 70{,}000$

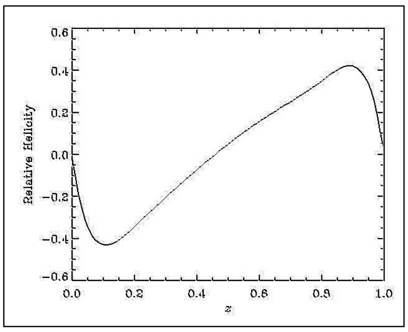


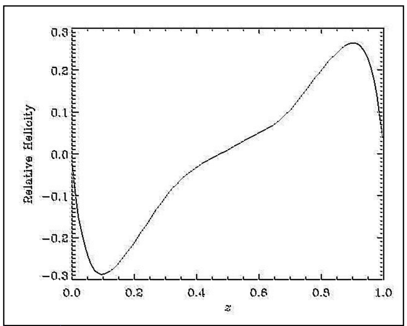
$Ra = 150{,}000$



$Ra = 500{,}000$







Relative Helicity $= \dfrac{\langle \mathbf{u}\cdot\grave{\mathbf{u}} \rangle}{\langle u^2\rangle^{1/2}\langle \omega^2\rangle^{1/2}}$

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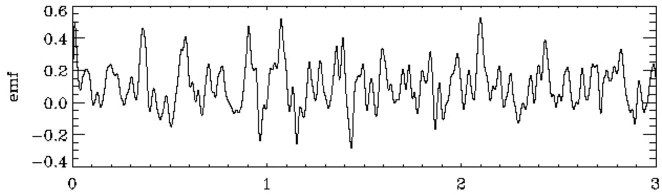
Longitudinal emf versus time

Aspect ratio = 5

$B_0 = 0.1$

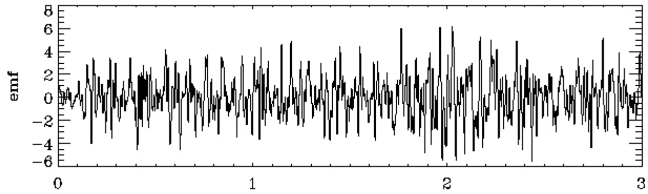
Ra = 80,000

$\langle u^2 \rangle \approx 300$



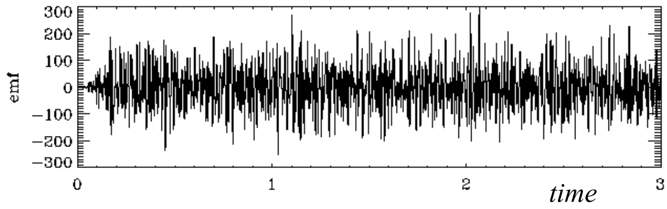
Ra = 150,000

$\langle u^2 \rangle \approx 3000$



Ra = 500,000

$\langle u^2 \rangle \approx 22000$



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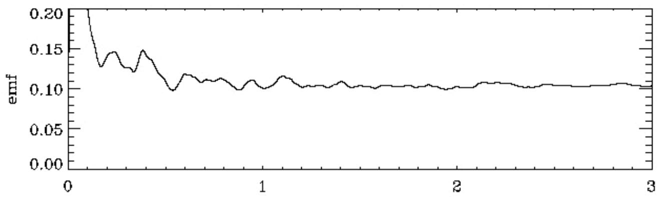
Cumulative time average of longitudinal emf

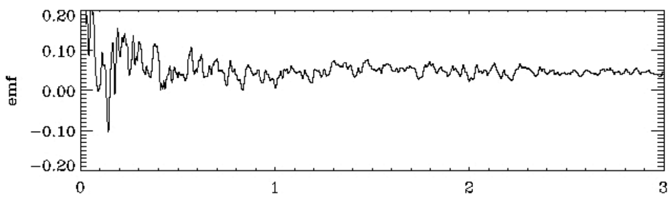
Aspect ratio = 5

$B_0 = 0.1$

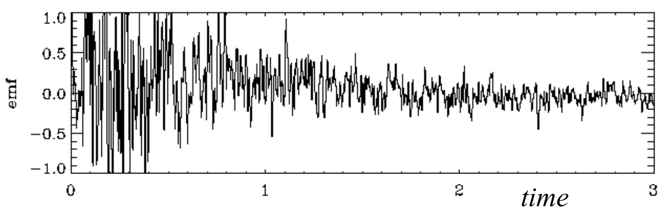
Ra = 80,000



Ra = 150,000



Ra = 500,000



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Longitudinal emf versus time

Aspect ratio = 0.5

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Ra = 80,000

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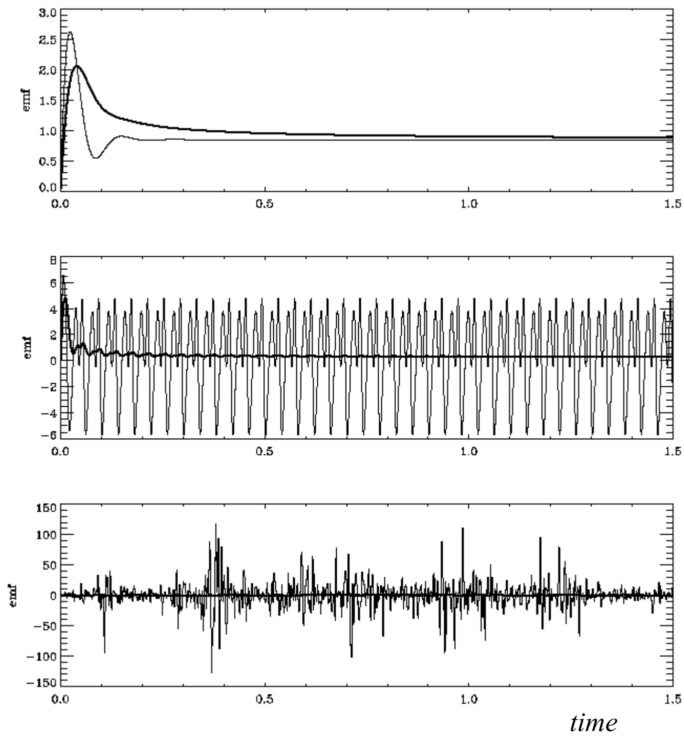
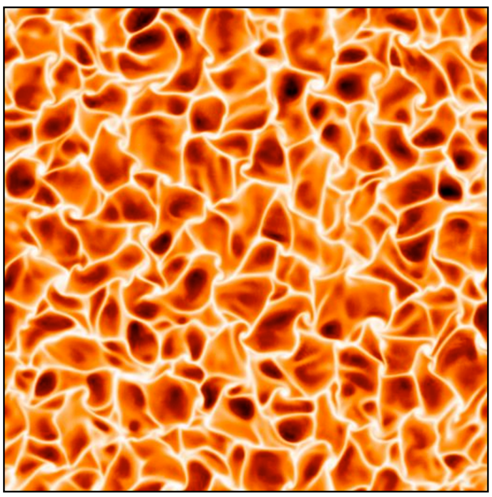
Ra = 500,000



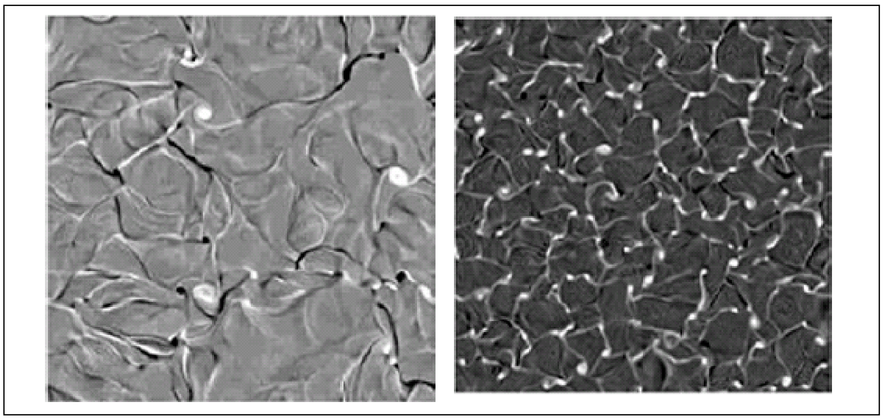
A Potentially Large-Scale Dynamo



$Ra = 10^6$, $Ta = 5 \times 10^5$
Box size: 10 x 10 x 1,
Resolution: 512 x 512 x 97
Snapshot of temperature.
No imposed mean magnetic field.

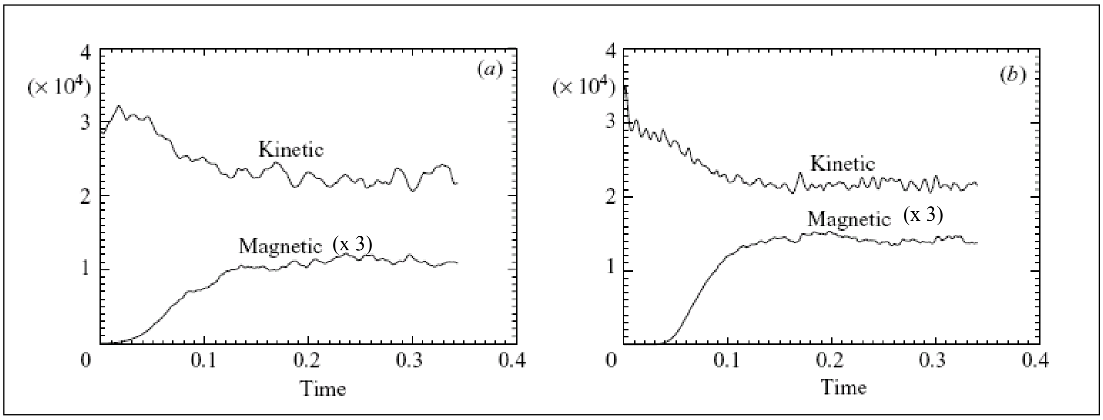
For comparison we consider a case with no rotation and with $Ra = 5 \times 10^5$

Comparison of vorticity



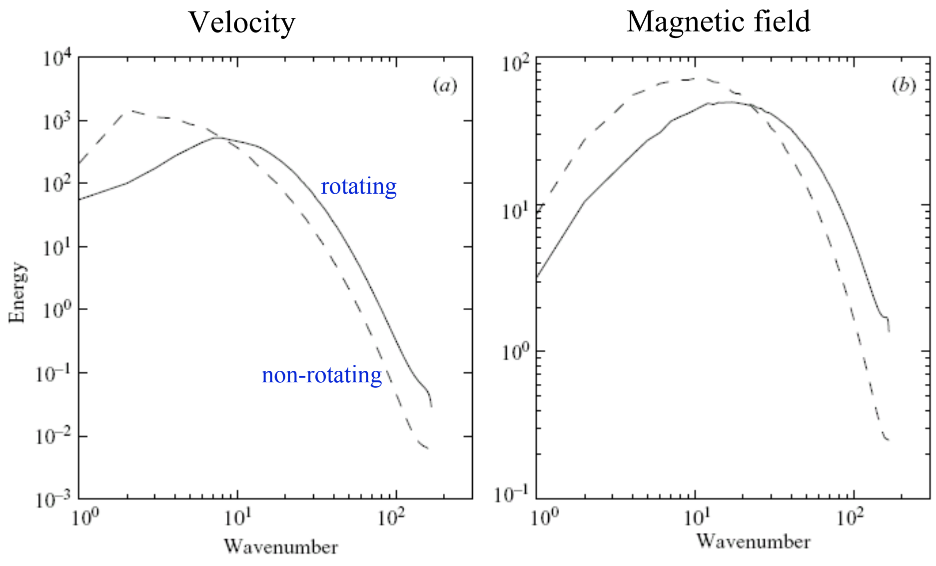
Ta = 0, *Ra* = 5 x 10^5 *Ta* = 5 x 10^5, *Ra* = 10^6

Dynamo action



Ta = 0, *Ra* = 5 x 10^5 *Ta* = 5 x 10^5, *Ra* = 10^6

Horizontal power spectra



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Conclusions

- Plenty of local **u** x **b**. However, no coherence – spatially or temporally.
- However the overall magnetic energy grows on a fast timescale.
- Efficient dynamo action, but no evidence of large-scale dynamo action – despite significant kinetic helicity. What is seen at high *Rm* is essentially just a modification to the small-scale dynamo that is present in the absence of rotation.
- Small box sizes lead to a very unrepresentative representation of the extended system.