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Topological constraints associated with fast dynamo action

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The conjecture of Vainshtein & Zel'dovich (1972) concerning the existence of a fast dynamo (i.e. one whose growth rate is independent of magnetic diffusivity η in the limit $\eta \to 0$) is discussed with particular reference to (i) the stretch–twist–fold cycle which can double the strength of a magnetic flux tube, and (ii) the space-periodic Beltrami flow of maximal helicity, which has been shown to be capable of space-periodic dynamo action with the same period as the velocity field, by Arnold & Korkina (1983) and by Galloway & Frisch (1984). The topological constraint associated with conservation of magnetic helicity is shown to preclude fast dynamo action unless the scale of the magnetic field is almost everywhere of order $\eta^{\frac{1}{2}}$ as $\eta \to 0$; in this case, the field structure is severely singular in the limit. A steady incompressible velocity field, quadratic in the space variables, is shown to mimic the action of the stretch–twist–fold cycle, and is proposed as a plausible candidate for fast dynamo action.

1. Introduction

The term 'dynamo action' in magnetohydrodynamics is generally used to describe the systematic and sustained generation of magnetic energy as a result of the stretching action of the velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ on the magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$. This action is described by the induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}) + \eta \nabla^2 \boldsymbol{B} \qquad (\nabla \cdot \boldsymbol{B} = 0), \tag{1.1}$$

where η is the magnetic diffusivity of the fluid (see e.g. Moffatt 1978).

In a purely kinematic approach to the dynamo problem, the velocity field $\boldsymbol{u}$ is regarded as *known*, and in particular the back-reaction of the magnetic field on $\boldsymbol{u}$ (via the Lorentz force distribution) is assumed negligible. This *known* velocity field *may* satisfy certain dynamic constraints (e.g. those imposed by the Euler equations or the Navier–Stokes equations, with or without Coriolis forces, buoyancy forces, etc.) but it is convenient to adopt a general approach in which $\boldsymbol{u}$ is *freed* from any such dynamic constraints, and we simply investigate the general behaviour of solutions of (1.1) for a wide class of velocity fields $\boldsymbol{u}$, which are supposed to satisfy only the mild kinematic constraint of incompressibility

$$\nabla \cdot \boldsymbol{u} = 0. \tag{1.2}$$

In the particular important situation in which $\boldsymbol{u}$ is *steady*, i.e. $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$, (1.1) admits solutions of the form

$$\boldsymbol{B}(\boldsymbol{x}, t) = \mathrm{Re}\{\hat{\boldsymbol{B}}(\boldsymbol{x})\, \mathrm{e}^{pt}\}, \tag{1.3}$$

where

$$p\hat{\boldsymbol{B}} = \nabla \wedge (\boldsymbol{u} \wedge \hat{\boldsymbol{B}}) + \eta \nabla^2 \hat{\boldsymbol{B}}. \tag{1.4}$$

Here p is the (possibly complex) growth rate associated with the field structure $\hat{\boldsymbol{B}}(\boldsymbol{x})$ (which satisfies $\nabla \cdot \hat{\boldsymbol{B}} = 0$). In conjunction with appropriate boundary conditions on $\hat{\boldsymbol{B}}(\boldsymbol{x})$, (1.4) constitutes an eigenvalue problem; and if any of the eigenvalues $p_1, p_2, p_3, \ldots$ have positive real part, then the corresponding field structures $\hat{\boldsymbol{B}}_1(\boldsymbol{x}), \hat{\boldsymbol{B}}_2(\boldsymbol{x}), \hat{\boldsymbol{B}}_3(\boldsymbol{x}), \ldots$, grow exponentially in time. The associated dynamo is oscillatory or non-oscillatory in type according as the imaginary part of p is non-zero or zero respectively.

The distinction between 'fast' and 'slow' dynamos has been introduced by Vainshtein & Zel'dovich (1972), and the distinction provides the basis for much of the discussion in the recently published monograph of Zel'dovich, Ruzmaikin & Sokoloff (1983). Suppose that the velocity field $\boldsymbol{u}(\boldsymbol{x})$ is characterized by a lengthscale l_0 and a velocity scale u_0, so that the timescale characteristic of the motion (the 'turnover time') is

$$t_0 = l_0/u_0. \tag{1.5}$$

The magnetic Reynolds number associated with the flow is

$$R_{\mathrm{m}} = u_0 l_0/\eta, \tag{1.6}$$

and we are particularly concerned in astrophysical contexts with the limiting behaviour when $R_{\mathrm{m}} \to \infty$. A dynamo with growth rate $p = p_{\mathrm{r}} + \mathrm{i}p_{\mathrm{i}}$† is said to be *slow* if

$$p_{\mathrm{r}} t_0 \to 0 \quad \text{as } R_{\mathrm{m}} \to \infty, \tag{1.7}$$

and it is said to be *fast* if

$$p_{\mathrm{r}} t_0 \to \text{const} > 0 \quad \text{as } R_{\mathrm{m}} \to \infty \tag{1.8}$$

(Zel'dovich & Ruzmaikin 1980). For a slow dynamo, the mechanism of field generation is diffusive in character (or at least involves magnetic diffusion in an essential way). All dynamos with laminar velocity fields $\boldsymbol{u}(\boldsymbol{x})$ for which detailed and rigorous calculations have been carried out are of the slow type; typically, for a slow dynamo,

$$p_{\mathrm{r}} t_0 = O(R_{\mathrm{m}}^{-q}) \quad \text{as } R_{\mathrm{m}} \to \infty, \quad \text{where } 0 < q < 1. \tag{1.9}$$

The fast dynamo, if it exists, becomes (in some sense) insensitive to the value of η as $\eta \to 0$. The first thing to do is therefore to examine the properties of (1.4) when we simply put $\eta = 0$, i.e.

$$p\hat{\boldsymbol{B}} = \nabla \wedge (\boldsymbol{u} \wedge \hat{\boldsymbol{B}}). \tag{1.10}$$

There are certainly solutions of the equation for which $p = 0$, viz those for which

$$\hat{\boldsymbol{B}} = \mu(\boldsymbol{x})\, \boldsymbol{u}(\boldsymbol{x}), \tag{1.11}$$

where $\mu(\boldsymbol{x})$ is any scalar function of position satisfying

$$\boldsymbol{u} \cdot \nabla \mu = 0. \tag{1.12}$$

This is not, however, a fast dynamo, since $p = 0$. For certain obvious choices of $\boldsymbol{u}$, there are also solutions of (1.10) for which p is pure imaginary. For example, if $\boldsymbol{u}$ is a rigid-body rotation with angular velocity $\boldsymbol{\Omega}$, and $\hat{\boldsymbol{B}}(\boldsymbol{x})$ is a sinusoidal function $\sim \mathrm{e}^{\pm \mathrm{i}m\phi}$ of the azimuth angle ϕ about the axis of rotation, then $p = \pm \mathrm{i}m\Omega$; but again $p_{\mathrm{r}} = 0$, and this is not a fast dynamo.

In §2, we shall in fact show that there are no localized solutions of (1.10) for which $p_{\mathrm{r}} \neq 0$. This means that any fast dynamo must involve diffusive effects in a crucial

† Suffixes r and i will throughout refer to real and imaginary parts.

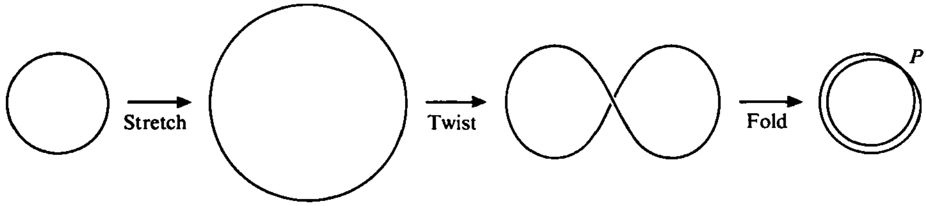

FIGURE 1. The stretch–twist–fold cycle; is this a fast dynamo?

way, and it can do this only if the field $\hat{B}$ varies on a scale $O(\eta^{\frac{1}{2}})$ so that $\eta \nabla^2 \hat{B} = O(1)$ as $\eta \to 0$. We shall show in §3 that, if the relative helicity of a fast-dynamo magnetic field is $O(1)$, then this scale-refinement effect must occur throughout a fraction of the available volume that remains $O(1)$ as $\eta \to 0$ (and not, for example, only in the neighbourhood of a set of singular surfaces).

These results do not prove the existence of a fast dynamo – they merely describe what it must look like if it does exist. It is desirable to have a much more detailed picture, and for this purpose there are two candidates for fast-dynamo action which deserve detailed study.

(*a*) *The stretch, twist and fold dynamo.* This is the prototype fast dynamo proposed by Vainshtein & Zel'dovich (1972). An initially circular flux tube of small cross-section is subjected to a stretch, twist and fold sequence as indicated in figure 1, like the doubling of an elastic band. To get back exactly to the initial configuration, with a doubling of the field strength, a little diffusion is evidently needed to eliminate the crossing of field lines in the neighbourhood of the point P; but if it is accepted that this can be achieved, then the doubling time should be of order $l_0/u_0 = t_0$, the timescale for the stretch–twist–fold cycle, independent of η. We shall study this process closely in §4, and show that the effect of diffusion is crucial in determining the field structure that may develop under many iterations of the cycle. We shall also construct an Eulerian velocity field which incorporates the stretch, twist and fold ingredients, and which is proposed as a candidate for a localized fast dynamo.

(*b*) *The space-periodic Beltrami dynamo.* A second veliocity field that has attracted recent attention in the fast-dynamo context (Arnold & Korkina 1983; Galloway & Frisch 1984) is the space-periodic field

$$\boldsymbol{u} = (U_3 \sin kz + U_2 \cos ky,\ U_1 \sin kx + U_3 \cos kz,\ U_2 \sin ky + U_1 \cos kx), \quad (1.13)$$

which satisfies the Beltrami condition

$$\boldsymbol{\omega} = \nabla \wedge \boldsymbol{u} = k\boldsymbol{u}, \quad (1.14)$$

and which is therefore a field of maximal mean helicity $\langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle = k \langle \boldsymbol{u}^2 \rangle$ (angular brackets indicating an average over a cube of side $2\pi/k$). The field (1.13) is of intrinsic interest because of the chaotic character of the streamlines when $U_1 U_2 U_3 \neq 0$ (Arnold 1965; Hénon 1966), a property that may be conducive to fast-dynamo action. Mean-field and first-order-smoothing techniques (Roberts 1970; Childress 1970; see also Moffatt 1978, chap. 7) may be used to show that the velocity field (1.13) will act as a dynamo when R_m is *small*, the field $\boldsymbol{B}$ then growing on a scale L *large* compared with $l_0 \sim k^{-1}$. As R_m increases, the scale L decreases, and ultimately the techniques of mean-field electrodynamics are inapplicable. The approach of Arnold & Korkina (1983) and of Galloway & Frisch (1984) is to restrict attention to fields $\boldsymbol{B}(\boldsymbol{x}, t)$ that are space-periodic with the same period $2\pi/k$ as $\boldsymbol{u}$ (and with zero mean over the basic cube), and to compute the field evolution. Results obtained for R_m up to 200

(Galloway & Frisch 1984) are suggestive of fast-dynamo action, although a more detailed analysis of field structure than is yet available will be required to confirm this behaviour. We return to this problem in §5, where some general aspects of space-periodic dynamos are discussed.

2. Topological constraints on a non-diffusive fast dynamo

Consider first the perfectly conducting situation in which $\eta = 0$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ satisfies the frozen field equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \wedge (\boldsymbol{u} \wedge \boldsymbol{B}). \tag{2.1}$$

Suppose that $\boldsymbol{B}$ is localized in the sense that

$$e^{kr} |\boldsymbol{B}| \rightarrow 0 \quad \text{as } r \equiv |\boldsymbol{x}| \rightarrow \infty \tag{2.2}$$

for some $k > 0$, and let $\boldsymbol{A}(\boldsymbol{x}, t)$ be a vector potential for $\boldsymbol{B}$. Then it is well known (Woltjer 1958) that

$$\mathscr{H} = \int \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V = \text{const.} \tag{2.3}$$

This invariant, the helicity of the field $\boldsymbol{B}$, is essentially topological in character (Moffatt 1969), and is in fact a generalization of the Hopf invariant, described as the *asymptotic Hopf invariant* by Arnold (1974).

A magnetic field with non-zero helicity is one for which there is a net linkage of lines of force. The fact that lines of force are frozen in the fluid implies that this net linkage cannot change, and this is reflected mathematically in the conservation of $\mathscr{H}$. It is therefore obvious that a field that has non-zero helicity cannot be amplified by dynamo action, since this would imply a corresponding exponential increase in $\mathscr{H}$.

This argument does not exclude the possibility that a field for which $\mathscr{H} = 0$ (i.e. for which the net linkage is zero) may be amplified by dynamo action, with at most time-periodic change of structure, when $\eta = 0$. This possibility may however be eliminated by the following argument.

We are concerned with the existence of solutions of (1.10) with $p \neq 0$ for given $\boldsymbol{u}(\boldsymbol{x})$. We may include the possibility of compressible flow by introducing a density field $\rho(\boldsymbol{x})$, and a (steady) mass conservation equation

$$\nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{2.4}$$

We shall consider two cases:

case A. $\boldsymbol{u}(\boldsymbol{x})$ is localized in the sense that there exists a finite closed surface S on which $\boldsymbol{u} \cdot \boldsymbol{n} = 0$;

case B. $\boldsymbol{u}$ and $\boldsymbol{B}$ are space-periodic with the same basic cuboid of periodicity, whose surface we again denote by S.

In either case, let the volume interior to S be denoted by V.

Now if $\hat{\boldsymbol{A}}$ is a vector potential for $\hat{\boldsymbol{B}}$, we may 'uncurl' (1.10) to obtain

$$p\hat{\boldsymbol{A}} = \boldsymbol{u} \wedge \hat{\boldsymbol{B}} - \nabla \hat{\varphi} \tag{2.5}$$

for some scalar field $\hat{\varphi}$. If $p \neq 0$, we may introduce the change of gauge $\hat{\boldsymbol{A}}_1 = \hat{\boldsymbol{A}} + p^{-1} \nabla \hat{\varphi}$, so that

$$p\hat{\boldsymbol{A}}_1 = \boldsymbol{u} \wedge \hat{\boldsymbol{B}}, \quad \nabla \wedge \hat{\boldsymbol{A}}_1 = \hat{\boldsymbol{B}}, \tag{2.6}$$

from which it immediately follows that

$$\hat{A}_1 \cdot (\nabla \wedge \hat{A}_1) = 0. \tag{2.7}$$

If $\hat{A}_1(x)$ were a *real* vector field, then this would be recognized as the condition for the existence of a family of surfaces $g(x) = \text{const.}$, everywhere orthogonal to $\hat{A}_1$, i.e. for the existence of scalar functions $f(x)$, $g(x)$ such that

$$\hat{A}_1 = f\nabla g, \quad \hat{B} = \nabla f \wedge \nabla g. \tag{2.8}$$

In fact, the result (2.8) still holds† when $\hat{A}_1$ is complex, but now $f(x)$ and $g(x)$ are *complex* scalar fields. Substituting (2.8) back in (2.6) gives

$$pf\,\nabla g = u \wedge (\nabla f \wedge \nabla g) = \nabla f\, u\cdot\nabla g - \nabla g\, u\cdot\nabla f, \tag{2.9}$$

and crossing this with ∇f gives

$$(pf + u\cdot\nabla f)\,\hat{B} = 0. \tag{2.10}$$

Hence at every point of space, either $\hat{B} = 0$ or

$$pf + u\cdot\nabla f = 0. \tag{2.11}$$

Suppose first that $\hat{B}$ is non-zero throughout V, so that (2.11) holds thoughout V. We easily deduce that

$$(p+p^*)\int_V \rho\,|f|^2\,\mathrm{d}V = -\int_S \rho(u\cdot n)\,|f|^2\,\mathrm{d}S. \tag{2.12}$$

In case A the surface integral vanishes because $u\cdot n = 0$ on S; in case B it vanishes by periodicity. So in either case it follows that $p_r = \frac{1}{2}(p+p^*) = 0$, and so we do not have a fast dynamo.

Now suppose that there exists a surface S_1 inside S on which $\hat{B} = 0$, but such that $\hat{B} \neq 0$ in the volume V_1 interior to S_1. Then $B(x,t) = 0$ (all t) on S_1 and $B(x,t) \neq 0$ in V_1. By Alfvén's theorem, it follows that $u\cdot n = 0$ on S_1 (since otherwise these conditions could not persist). We may then apply the above argument to the volume V_1, and again we conclude that $p_r = 0$.

We may conclude therefore that, in all cases considered, a non-diffusive fast dynamo is impossible.

Note that, if $p = \mathrm{i}p_i \neq 0$, then from (2.9), at all points where $\hat{B} \neq 0$, we must have

$$\left.\begin{aligned} pf + u\cdot\nabla f &= 0,\\ u\cdot\nabla g &= 0\end{aligned}\right\} \quad \nabla f \wedge \nabla g \neq 0. \tag{2.13}$$

This can happen only if the streamlines of the flow are the intersections of the surfaces $g_r = \text{const}, g_i = \text{const}$, i.e. only for a very special (non-generic) class of velocity fields.

3. Topological constraints on a diffusive fast dynamo

In the presence of weak molecular diffusivity, helicity is no longer conserved. In fact, from (1.1) and the 'uncurled' equation

$$\frac{\partial A}{\partial t} = u \wedge B - \nabla\varphi - \eta\,\nabla \wedge (\nabla \wedge A) \tag{3.1}$$

† This is a consequence of Frobenius' theorem in $\mathbb{C}^3$ (see e.g. Boothby 1975, p. 159). The result may be proved by elementary methods (following Ince 1925, p. 52) by regarding $\hat{A}_1$ as an analytic function of 3 complex variables (x, y, z) and ultimately restricting to the real axes.

we may readily obtain the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}V = -2\eta\int \boldsymbol{B}\cdot\nabla\wedge(\nabla\wedge\boldsymbol{A})\,\mathrm{d}V, \tag{3.2}$$

where the integrals are throughout all space. Hence, even if η is very small, the helicity can change significantly when the field gradient becomes large.

We may obtain an upper bound on the growth rate p_{r} of a dynamo as follows. First we average (3.2) over a time $2\pi/p_{\mathrm{i}}$; denoting this average by an overbar, (3.2) then gives

$$p_{\mathrm{r}}\int \overline{\boldsymbol{A}\cdot\boldsymbol{B}}\,\mathrm{d}V = -\eta\int \overline{\boldsymbol{B}\cdot\nabla\wedge(\nabla\wedge\boldsymbol{A})}\,\mathrm{d}V. \tag{3.3}$$

By the Schwarz inequality,

$$\int \overline{\boldsymbol{B}\cdot\nabla\wedge(\nabla\wedge\boldsymbol{A})}\,\mathrm{d}V \leqslant \left(\int \overline{\boldsymbol{B}^2}\,\mathrm{d}V\right)^{\frac{1}{2}}\left(\int (\nabla\wedge(\nabla\wedge\boldsymbol{A}))^2\,\mathrm{d}V\right)^{\frac{1}{2}}. \tag{3.4}$$

Defining the lengthscale l_B characteristic of the field $\boldsymbol{B}$ by

$$\int \overline{(\nabla\wedge(\nabla\wedge\boldsymbol{A}))^2}\,\mathrm{d}V = l_B^{-4}\int \overline{\boldsymbol{A}^2}\,\mathrm{d}V \tag{3.5}$$

and the relative helicity $\mathscr{H}_{\mathrm{R}}$ (satisfying $|\mathscr{H}_{\mathrm{R}}| \leqslant 1$ by

$$\mathscr{H}_{\mathrm{R}} = \frac{\int \overline{\boldsymbol{A}\cdot\boldsymbol{B}}\,\mathrm{d}V}{\left(\int \overline{\boldsymbol{A}^2}\,\mathrm{d}V\int \overline{\boldsymbol{B}^2}\,\mathrm{d}V\right)^{\frac{1}{2}}}, \tag{3.6}$$

we easily obtain from (3.3)–(3.6)

$$|p_{\mathrm{r}}| \leqslant \eta/l_B^2\,|\mathscr{H}_{\mathrm{R}}| \tag{3.7}$$

or equivalently

$$|\mathscr{H}_{\mathrm{R}}|^{\frac{1}{2}}\,l_B \leqslant \eta^{\frac{1}{2}}/|p_{\mathrm{r}}|. \tag{3.8}$$

Now in general a velocity field that has non-zero helicity will tend to generate a magnetic field with non-zero helicity, so that in general there is no reason to expect that $\mathscr{H}_{\mathrm{R}}$ should be small. If $|\mathscr{H}_{\mathrm{R}}| = O(1)$ then (3.8) implies that l_B/l_0 is at most $O(R_{\mathrm{m}}^{-\frac{1}{2}})$; from the definition (3.5) it then seems likely that the scale of $\boldsymbol{B}$ must be $O(R_{\mathrm{m}}^{-\frac{1}{2}})$ or less over an $O(1)$ fraction of the flow domain.

The diffusive fast dynamo (if it exists!) therefore generates a magnetic field whose gradient is typically $O(R_{\mathrm{m}}^{\frac{1}{2}})$, and which evidently becomes non-differentiable over a substantial part of the flow domain in the limit $R_{\mathrm{m}}\to\infty$ ($\eta\to 0$). The Lorentz force distribution in such a dynamo is

$$\boldsymbol{F}(\boldsymbol{x},t) = \boldsymbol{j}\wedge\boldsymbol{B} = \mu_0^{-1}(\nabla\wedge\boldsymbol{B})\wedge\boldsymbol{B} \sim B^2/\mu_0 l_B, \tag{3.9}$$

and this also will vary on the scale l_B. This force will ultimately generate an additional velocity field $\boldsymbol{u}_1(\boldsymbol{x},t)$ on this same lengthscale, a process which must ultimately be responsible for equilibration of the growing field.

Such a dynamo is totally different from the slow dynamos that emerge from, for example, the two-scale analysis of mean-field electrodynamics. In these dynamos, the growing mean magnetic field has a scale L *large* compared with l_0, and there is also a fluctuating ingredient on the scale l_0 driven directly by the velocity field $\boldsymbol{u}$. The Lorentz force acts either to suppress the turbulence (Moffatt 1972) or to drive a large-scale mean velocity (Malkus & Proctor 1975). In the fast dynamo considered

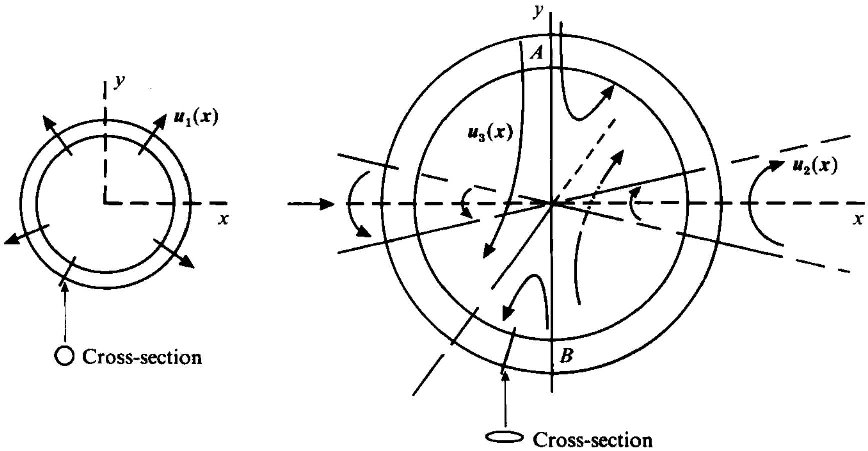

FIGURE 2. The stretch $\boldsymbol{u}_1(\boldsymbol{x})$ leads to flattening of the flux-tube cross-section to elliptic form. This is followed by the twist $\boldsymbol{u}_2(\boldsymbol{x})$ and the plane strain $\boldsymbol{u}_3(\boldsymbol{x})$ to bring the points A and B towards each other.

here, the scale of the magnetic field is *small* compared with the scale of the velocity field $\boldsymbol{u}(\boldsymbol{x})$, and the Lorentz force acts to *generate* small-scale velocity fluctuations $\boldsymbol{u}_1(\boldsymbol{x}, t)$. This is indeed a novel situation in the dynamo context.

In §4 we examine in detail the stretch, twist and fold dynamo described in §1, with a view to understanding just how it is that large field gradients can develop.

4. Stretch, twist and fold dynamo

Let us consider again the distortion process depicted in figure 1, but now taking account of the finite cross-section of the flux tube. Suppose that the centreline of the flux tube is initially the circle $x^2+y^2 = a_0^2$ in the plane $z = 0$, and that its cross-section is initially a circle of radius $c \ll a_0$. We shall suppose first that $\eta = 0$, i.e. that diffusion is totally negligible.

The initial process of stretching (figure 2) may be achieved by the uniform incompressible straining field

$$\boldsymbol{u}_1(\boldsymbol{x}) = (\alpha x, \alpha y, -2\alpha z), \tag{4.1}$$

with $\alpha > 0$. Under the action of this field, the radius of the flux tube increases exponentially:

$$a(t) = a_0 \,\mathrm{e}^{\alpha t}, \tag{4.2}$$

being doubled after a time $t_1 = \alpha^{-1} \ln 2$. At the same time the cross-section of the tube is flattened by the strain into an ellipse

$$\tfrac{1}{4}(x-2a_0)^2+16z^2 = c^2, \tag{4.3}$$

with semiaxes in the ratio 8:1. Note that the volume of the flux tube $V \approx 2\pi^2c^2a_0$ remains constant.

Consider now the twist stage. A twist about the x-axis is well-represented by the velocity field

$$\boldsymbol{u}_2(\boldsymbol{x}) = (0, -\omega(x)\, z, \omega(x)\, y), \tag{4.4}$$

where $\omega(x)$ is antisymmetric about $x = 0$, the simplest possibility (uniform twist) being $\omega(x) = -fx$, where f is constant ($f > 0$ for a right-handed twist), so that

$$\boldsymbol{u}_2(\boldsymbol{x}) = (0, fxz, -fxy). \tag{4.5}$$

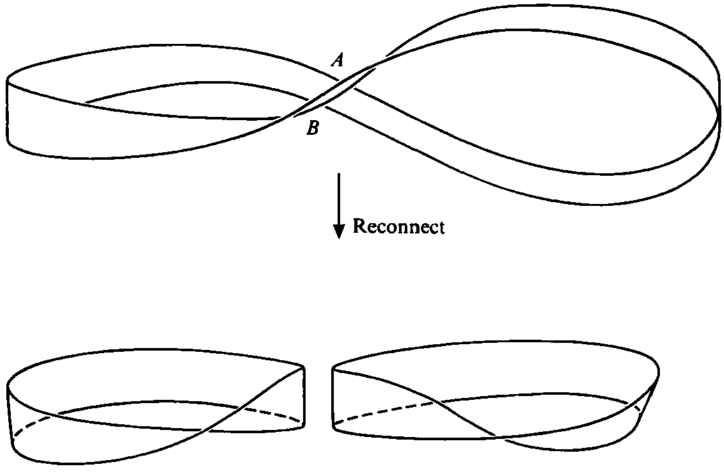

FIGURE 3. Twisting and reconnection of paper tape, or flux tube of elliptic cross-section.

This is a twist of the kind that is applied to an elastic band in the doubling process, but it fails to mimic this process in that the points $(0, \pm 2a_0, 0)$ (A and B in figure 2) remain fixed under the velocity field (4.5) and do not approach each other as suggested in figure 1. This approach in an 'experiment' with elastic band or paper tape is a natural consequence of the resistance of these materials to stretching, a feature that does not arise for the magnetic flux tube if the Lorentz force is negligible. (It is interesting to note here that a *strong* magnetic field subjected to twist *would* presumably respond in a nearly inextensible manner, so that, in a dynamic regime in which Lorentz forces *are* important, the elastic-band analogy may be more relevant.)

In order to achieve the approach of the points A and B, we have to supplement the twist field (4.5) by a further strain field which compresses along the y-axis, but leaves the scale of the loop along the x-axis undisturbed. The two-dimensional strain field

$$\boldsymbol{u}_3(\boldsymbol{x}) = (0, -\beta y, \beta z) \tag{4.6}$$

with $\beta > 0$ will do for this purpose. If the fields $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$ act simultaneously for a time $t_2 = \pi/4a_0 f$, then the distance between A and B will be reduced to

$$\delta = 2a_0\,\mathrm{e}^{-\beta t_2} = 2a_0\,\mathrm{e}^{-\pi\beta/4a_0 f}. \tag{4.7}$$

Now, however, we are twisting not simply a closed curve, but a flux tube with initially elliptic cross-section. The twisting of a paper tape provides a better analogy. As a simple experiment will demonstrate, a right-handed twist applied to a paper tape induces a left-handed twist of the tape about its own centreline (figure 3).† If the tape is broken and reconnected at the points A and B (simulating the diffusion process), then the two loops thus created have the form of Möbius strips, each one having a net left-handed twist of π.

Similarly the flux tube will develop what may be described as 'intrinsic twist' as a result of the action of the velocity field $\boldsymbol{u}_2(\boldsymbol{x}) + \boldsymbol{u}_3(\boldsymbol{x})$. When reconnection takes place, this intrinsic twist manifests itself as helicity of the magnetic field. A 'poloidal' magnetic field has been generated, superposed on the original toroidal field round each

† Certain features of this process have been recently considered by Berger & Field (1984).

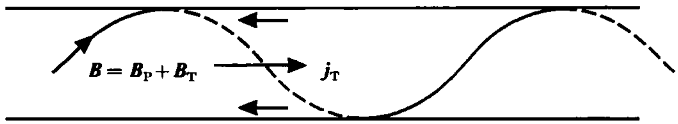

FIGURE 4. Helical field $\boldsymbol{B}$ in a twisted tube of force; the poloidal part of the field $\boldsymbol{B}_P$ is associated with a toroidal current j_T which has zero integral over the tube cross-section.

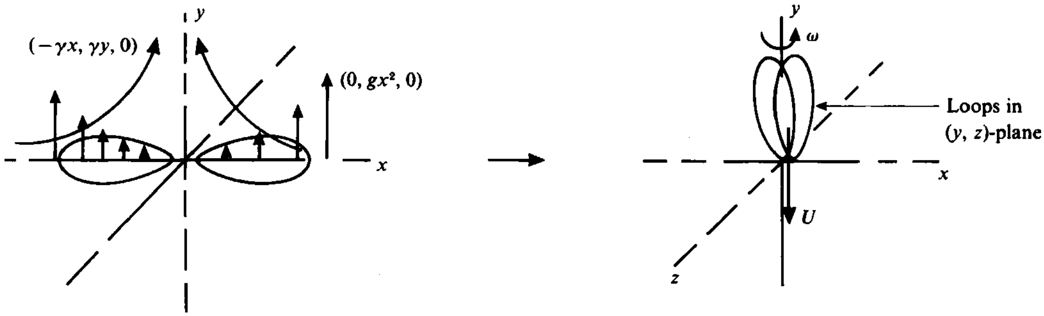

FIGURE 5. Folding of two loops into near-coincidence, followed by translation U and rotation ω to return to configuration of figure 2.

loop. Thus the process is not unlike the process by which poloidal loops of field are generated by cyclonic eddies (Parker 1955), the process which underlies the α-effect of Steenbeck, Krause & Rädler (1966). In the present context, the poloidal field is associated with a toroidal current flowing round each flux tube; the net flux of current along the tube is however zero, since the poloidal field is confined to the neighbourhood of the flux tube (figure 4).

To complete the stretch–twist–fold cycle, we require a velocity field $\boldsymbol{u}_4(\boldsymbol{x})$ that represents the action of folding the two loops of figure 5 into near-coincidence. A field that will achieve this is

$$\boldsymbol{u}_4(\boldsymbol{x}) = (-\gamma x, \gamma y + gx^2, 0), \tag{4.8}$$

with $\gamma > 0$, $g > 0$. The gx^2 ingredient deforms the loops out of the (x, z)-plane, and the remaining (plane-strain) part of (4.8) compresses both loops towards each other on the (y, z)-plane. A small value of g ($ga_0 \ll \gamma$) will suffice to achieve the necessary effect, in a time t_4 satisfying $\gamma t_4 \gtrsim 1$.

We now have a double loop in the (y, z)-plane. To complete the process and to return to the initial configuration, we require a translation (of order a_0) and a rotation $\frac{1}{2}\pi$ about the y-axis; the velocity field

$$\boldsymbol{u}_5(\boldsymbol{x}) = (0, \omega, 0) \wedge \boldsymbol{x} - (0, U, 0) \tag{4.9}$$

with $U \approx 2a_0 \omega/\pi$ will achieve this effect in a time $t_5 = 2\omega/\pi$.

The velocity fields $\boldsymbol{u}_1(\boldsymbol{x}), \boldsymbol{u}_2(\boldsymbol{x}), \ldots, \boldsymbol{u}_5(\boldsymbol{x})$ should thus, in succession, and applied for suitable time intervals, achieve an approximate doubling of the initial magnetic field, but at the cost of generating a net twist in both of the new flux tubes. If these velocity fields act simultaneously, so that we have the steady velocity field

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_1(\boldsymbol{x}) + \boldsymbol{u}_2(\boldsymbol{x}) + \ldots + \boldsymbol{u}_5(\boldsymbol{x})$$

$$= (\alpha' x + \omega z, \beta' y + gx^2 + fxz - U, \gamma' z - \omega x - fxy), \tag{4.10}$$

where

$$\alpha' = \alpha - \gamma, \quad \beta' = \alpha + \gamma - \beta, \quad \gamma' = -2\alpha + \beta, \tag{4.11}$$

then we may expect that the initial loop will be continuously deformed in a similar way, and indeed that the doubling process should be repeated again and again if the velocity field (4.10) is sustained. The particle trajectories associated with (4.10) are given by the dynamical system

$$\left.\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= \alpha' x + \omega z,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \beta' y + g x^2 + f x z - U,\\ \frac{\mathrm{d}z}{\mathrm{d}t} &= \gamma' z - \omega x - f x y,\end{aligned}\right\} \tag{4.12}$$

where $\alpha'+\beta'+\gamma' = 0$, and where approaching of the two loops in the twist and fold process is achieved if $\alpha' < 0$ and $\beta' < 0$. This volume-preserving ($\nabla\cdot\boldsymbol{u} = 0$) system clearly deserves close study, for various values of α', β', ω, g, f, U; it seems highly likely that, in general, the trajectories are chaotic.

The vorticity associated with the velocity field (4.10) is

$$\boldsymbol{\omega}(\boldsymbol{x}) = (-2fx,\ 2\omega+fy,\ 2gx+fz), \tag{4.13}$$

and the helicity, integrated over any sphere $|\boldsymbol{x}| < R$, is

$$\mathscr{H}(R) = \int_{|\boldsymbol{x}|<R} \boldsymbol{u}\cdot\boldsymbol{\omega}\,\mathrm{d}V = -\tfrac{4}{5}\pi R^5 f\alpha'. \tag{4.14}$$

The motion therefore has a net right-handed or left-handed sense according as $f\alpha' < 0$ or > 0. It is the helicity (4.14) which, in conjunction with weak diffusion, is responsible for generating helicity (of opposite sign!) in the magnetic field.

The motion (4.10) is of course unbounded at infinity, and there is no guarantee that the trajectories of fluid particles will remain within a sphere $r < R$, no matter how large R may be. It is easy, however, to modify the velocity field (4.10) so that nearly all of the trajectories *do* all return to the neighbourhood of the origin. To do this, let $\boldsymbol{A}(\boldsymbol{x})$ be the vector potential of $\boldsymbol{u}(\boldsymbol{x})$, a cubic function of the coordinates: in fact

$$\boldsymbol{A}(\boldsymbol{x}) = [gx^2z - Uz + \tfrac{1}{2}fx(z^2+y^2),\ \gamma zx - \tfrac{1}{2}\omega(x^2+z^2),\ -\beta xy]. \tag{4.15}$$

Now define the modified vector potential

$$\boldsymbol{A}^{\mathrm{M}}(\boldsymbol{x}) = \boldsymbol{A}(\boldsymbol{x})\,\mathrm{e}^{-r/R} \tag{4.16}$$

and the modified (solenoidal) velocity field

$$\boldsymbol{u}^{\mathrm{M}}(\boldsymbol{x}) = \nabla \wedge \boldsymbol{A}^{\mathrm{M}}(\boldsymbol{x}). \tag{4.17}$$

Then clearly $\boldsymbol{u}^{\mathrm{M}}$ coincides with $\boldsymbol{u}$ for $r \ll R$, and yet is exponentially small for $r \gtrsim R$, so that nearly all of the streamlines are forced to return to the interior of the sphere $r = R$. Any magnetic field that is initially confined to the sphere $r \lesssim R$ will then probably remain so confined for all time (under the frozen field assumption).

Let us now consider what happens when the stretch–twist–fold cycle is repeated. The stretch is now applied to two adjacent flux tubes each of 8:1 elliptical cross-section twisted in the form of a Möbius strip. The initial stretch in the (x, y)-plane again flattens the cross-section: where the long axis of the ellipse is initially parallel to the plane, the ellipse is further stretched till its axes are in the ratio 64:1, and where the long axis is initially perpendicular to the plane, the cross-section returns to the original circular form (but with $\frac{1}{4}$ the original radius). The twist about the x-axis

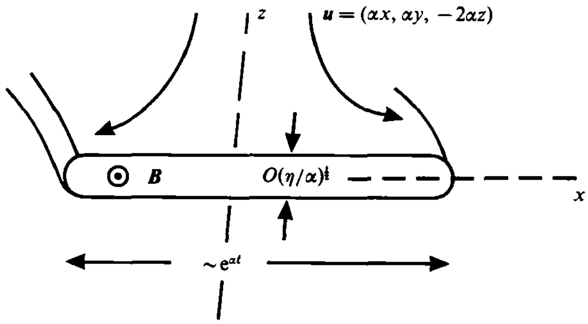

FIGURE 6. Section of flux tube when diffusion limits compression.

again induces additional intrinsic twist in the flux tube – the twisting and reconnection of a Möbius strip leads to a Möbius strip (twist π) and a strip with twist 2π; in the case of flux tubes, if symmetry is maintained between the two daughter tubes, then each will have a twist of $\frac{3}{2}\pi$. The folding stage again leads to the superposition of these two tubes, which will moreover be linked with the neighbouring 'cousin' tubes.

It is clear that, even after only two stretch, twist and fold cycles, we have generated a field of considerable complexity. To be sure, the toroidal flux has increased fourfold; but poloidal field varying on a scale $\frac{1}{64}c$ has been generated – and after n cycles the scale of variation would be $c/2^{3n}$. This is in effect an exponential decrease of scale $l_B \sim c\,\mathrm{e}^{-t/t_0}$, where t_0 is the timescale of the stretch–twist–fold cycle. Clearly, molecular diffusion, neglected in the discussion so far, must intervene to eliminate these field variations as well as to achieve the reconnection of flux tubes. It seems clear that we are dealing with a fast dynamo of the diffusive rather than the non-diffusive type (see §3). Let us now consider the effects of diffusion in more detail.

For the sake of argument, suppose that the initial stretch (4.1) is maintained for a long time until the smaller dimension of the cross-section of the tube is reduced to $O(\eta/\alpha)^{\frac{1}{2}}$, at which diffusion becomes important. This lengthscale does not then reduce further (figure 6). However, the larger dimension of the cross-section continues to increase like $\mathrm{e}^{\alpha t}$, and, since the total toroidal flux in the tube is constant during the stretch process, the field intensity must *decrease* like $\mathrm{e}^{-\alpha t}$. In fact, if we move with the flux tube, the relevant local solution of (1.1) with the velocity field (4.1) is $\boldsymbol{B} = (0, B(z,t), 0)$, with

$$B(z,t) = B_0\,\mathrm{e}^{-\alpha t}\,\mathrm{e}^{-\alpha z^2/\eta}. \tag{4.18}$$

This type of behaviour in which the decrease of scale in one direction is limited by molecular diffusion was discussed in the context of scalar-field diffusion by Batchelor (1959); its importance in the dynamo context has recently been emphasized by Zel'dovich *et al.* (1984).

When we are on this small lengthscale, it is evident that, in the stretch–twist–fold cycle, the toroidal field is *not* doubled – it is halved! The toroidal *flux* is nevertheless doubled, because there are now *two* adjacent flux tubes each with *double* the original cross-section. Repetition of the cycle leads to continued increase of the net cross-section, and the structure of the field that finally emerges from many repetitions will undoubtedly be very different from the initial simple circular flux tube of small cross-section.

5. The space-periodic Beltrami dynamo

A Beltrami flow is one for which $\nabla \wedge \boldsymbol{u} = k_0 \boldsymbol{u}$, where k_0 is constant. Such a flow has maximal helicity, and for this reason is of particular interest in the dynamo context. The helical wave

$$\boldsymbol{u}_1(\boldsymbol{x},t) = (0,\ U_1 \sin(k_0 x - \omega_1 t),\ U_1 \cos(k_0 x - \omega_1 t)) \tag{5.1}$$

satisfies the Beltrami condition, and has helicity

$$\mathscr{H}_1 = \langle \boldsymbol{u}_1 \cdot \nabla \wedge \boldsymbol{u}_1 \rangle = k_0 U_1^2. \tag{5.2}$$

Similarly, defining

$$\boldsymbol{u}_2(\boldsymbol{x},t) = (U_2 \cos(k_0 y - \omega_2 t), 0, U_2 \sin(k_0 y - \omega_2 t), \tag{5.3}$$

$$\boldsymbol{u}_3(\boldsymbol{x},t) = (U_3 \sin(k_0 z - \omega_3 t), U_3 \cos(k_0 z - \omega_3 t), 0), \tag{5.4}$$

the velocity field

$$\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}_1(\boldsymbol{x},t) + \boldsymbol{u}_2(\boldsymbol{x},t) + \boldsymbol{u}_3(\boldsymbol{x},t) \tag{5.5}$$

satisfies $\nabla \wedge \boldsymbol{u} = k_0 \boldsymbol{u}$ and has helicity

$$\mathscr{H} = \langle \boldsymbol{u} \cdot \nabla \wedge \boldsymbol{u} \rangle = k_0 (U_1^2 + U_2^2 + U_3^2). \tag{5.6}$$

The flow (5.5) is an exact solution of the Euler equation in a rotating fluid; in a frame rotating with angular velocity $\boldsymbol{\Omega}$ this equation may be written

$$\frac{\partial \boldsymbol{u}}{\partial t} + 2\boldsymbol{\Omega} \wedge \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \tfrac{1}{2}\boldsymbol{u}^2\right) + \boldsymbol{u} \wedge \boldsymbol{\omega}, \tag{5.7}$$

and it is easily verified that this is satisfied by (5.5) provided

$$\boldsymbol{\Omega} = -\tfrac{1}{2}(\omega_1, \omega_2, \omega_3), \tag{5.8}$$

and provided p is suitably chosen. When $\omega_1 = \omega_2 = \omega_3 = 0$, we have the steady flow (1.14) discussed briefly in §1.

It is well known that flows of this type are capable of dynamo action on lengthscales L large compared with k_0^{-1} (Childress 1970; Roberts 1970). Indeed, there is an α-effect associated with the motion (5.5), which, on first-order-smoothing theory, is given by the tensor

$$(\alpha_{ij}) = \begin{pmatrix} \alpha^{(1)} & 0 & 0 \\ 0 & \alpha^{(2)} & 0 \\ 0 & 0 & \alpha^{(3)} \end{pmatrix}, \tag{5.9}$$

where

$$\alpha^{(n)} = -\frac{\eta U_n^2 k_0^2}{\omega_n^2 + \eta^2 k_0^4} \quad (n = 1,2,3) \tag{5.10}$$

(Moffatt 1978, §7.7). When the frequencies ω_n are non-zero, $\alpha^{(n)} \to 0$ as $\eta \to 0$, a property that persists at all higher orders of perturbation theory (Dillon 1974).

If, for simplicity, we consider the isotropic situation

$$\left.\begin{aligned} U_1 = U_2 = U_3 = U, \\ \omega_1 = \omega_2 = \omega_3 = \omega, \end{aligned}\right\} \tag{5.11}$$

then $\alpha_{ij} = \alpha \delta_{ij}$, where

$$\alpha = -\frac{\eta k_0 \mathscr{H}}{3(\omega^2 + \eta^2 k_0^4)}. \tag{5.12}$$

Similarly, there is an augmentation of the molecular diffusivity given by

$$\beta = \frac{2\eta k_0 E}{3(\omega^2+\eta^2 k_0^4)}, \tag{5.13}$$

where $E = \frac{3}{2}U^2$ is the mean kinetic-energy density of the motion ($\mathscr{H} = 2k_0 E$). The important point here is that

$$\frac{|\alpha|}{\beta} = O\left(\frac{|\mathscr{H}|}{E}\right) = O(k_0), \tag{5.14}$$

and, although this is a result of first-order-smoothing theory, it may be expected to hold at higher orders of perturbation theory also.

The mean-field equation, describing evolution of a field $\boldsymbol{B}_0$ on a scale large compared with k_0^{-1}, is

$$\frac{\partial \boldsymbol{B}_0}{\partial t} \approx \alpha \boldsymbol{\nabla} \wedge \boldsymbol{B}_0 + (\eta+\beta)\nabla^2 \boldsymbol{B}_0. \tag{5.15}$$

This has non-oscillatory dynamo solutions of force-free structure ($\boldsymbol{\nabla} \wedge \boldsymbol{B}_0 = K\boldsymbol{B}_0$) whose growth rate p is given by

$$p = \alpha K - (\eta+\beta) K^2. \tag{5.16}$$

The maximum growth rate occurs for

$$K = K_{\mathrm{m}} = \frac{|\alpha|}{2(\eta+\beta)}, \tag{5.17}$$

and, if we adopt the expressions (5.12) and (5.13), then

$$\frac{K_{\mathrm{m}}}{k_0} = \frac{U^2 k_0^2}{2[U^2 k_0^2 + (\omega^2+\eta^2 k_0^4)]}, \tag{5.18}$$

so that $K_{\mathrm{m}} \ll k_0$ (as required for self-consistency of the approximation (5.15)) provided

$$\omega^2 + \eta^2 k_0^4 \gg U^2 k_0^2. \tag{5.19}$$

As U increases (for fixed ω and η), the preferred scale of growth of the field $\boldsymbol{B}_0$ decreases towards the scale k_0^{-1} of the velocity field, and the methods of mean-field theory become progressively less reliable.

The alternative approach (Arnold & Korkina 1983; Galloway & Frisch 1984) is then to restrict attention to a field $\boldsymbol{B}(\boldsymbol{x}, t)$ with the same periodicity as the field $\boldsymbol{u}$ and with zero mean over a basic cube of side $2\pi/k_0$. The results (2.5) and (2.6) apply equally if V is taken to be this cube – so, if $\eta = 0$, the magnetic helicity in the cube is constant. The arguments of §§2 and 3 then imply that, if we have a fast dynamo ($p_{\mathrm{r}} = O(1)$) with non-zero magnetic helicity, then the scale of variation of the magnetic field must be of the order $R_{\mathrm{m}}^{-\frac{1}{2}} k_0^{-1}$ nearly everywhere. In spectral terms, the spectrum of $\boldsymbol{B}$ may peak at wavenumbers of order k_0, but the spectrum of $\boldsymbol{\nabla} \wedge \boldsymbol{B}$ must have strong contributions at wavenumbers of order $R_{\mathrm{m}}^{\frac{1}{2}} k_0$. The situation is consistent with the statement of Galloway & Frisch (1984)† that "spectra of the growing or decaying modes show that the value of the wavenumber k at which the energy peaks is surprisingly insensitive to R, though the length of the tail appears to increase roughly as the square root of R". In the range of wavenumbers $k_0 \ll k \ll R_{\mathrm{m}}^{\frac{1}{2}} k_0$, the magnetic spectrum function $M(k)$ presumably has a power-law dependence of the form

$$M(k) \sim k^{-q}, \quad 0 < q \leqslant 3. \tag{5.20}$$

† Galloway & Frisch restrict attention to the steady case ($\omega_1 = \omega_2 = \omega_3 = 0$) and present results for $R(= R_{\mathrm{m}}) < 200$.

If the process of field distortion is qualitatively similar to that of the stretch–twist–fold dynamo of §4 then it would appear that the fluctuation of $\nabla \wedge \boldsymbol{B}$ that is generated at any scale k^{-1} is related to the twist effective at that scale, and for the motion (5.5) considered, this twist is independent of k. This suggests that $(k^2M(k))\,k$ should be independent of k, i.e. that $q = 3$ in (5.20). The integral of $k^2M(k)$ is then logarithmically divergent as $\eta \rightarrow 0$, corresponding to the non-analytical character of the magnetic field in this limit. The results available at present (Galloway, private communication) suggest that $M(k)$ in fact falls off more slowly than k^{-3} in the range $k_0 \ll k \ll R_{\mathrm{m}}^{\frac{1}{2}}\,k_0$; the reasons for this are not as yet clear.

6. Conclusions

The fast dynamo is a dynamo whose growth rate p_{r} is independent of η in the limit $\eta \rightarrow 0$. Its existence, as a phenomenon distinct from the more familiar slow dynamo, has not yet been rigorously established for a steady flow. This paper has been devoted to a discussion of its structural properties, if it does exist. We have shown firstly that a 'non-diffusive' fast dynamo does not exist, and secondly that, for a 'diffusive' fast dynamo, the scale of variation of the magnetic field of a growing helical mode must typically be $O(R_{\mathrm{m}}^{-\frac{1}{2}})$ and diffusion must play the key role in resolving the conflict between magnetic-helicity invariance and exponential field growth. The stretch–twist–fold dynamo has been examined in some detail, and the mechanism by which fine structure appears in the magnetic field has been revealed. A similar fine structure must appear in the space-periodic dynamo of Arnold & Korkina (1983) if, as suggested by the computations of Galloway & Frisch (1984), this is indeed a fast dynamo in the limit $R_{\mathrm{m}} \rightarrow \infty$.

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