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The alpha-effect in rotating convection: size matters

DAVID W. HUGHES[1] AND FAUSTO CATTANEO[2]

[1]Department of Applied Mathematics, University of Leeds, Leeds LS2 9JT, UK

[2]Department of Astronomy and Astrophysics, University of Chicago, Chicago, IL 60637, USA

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The results of numerical simulations of convection in a rotating layer are used to compute the α-effect of mean-field electrodynamics. The computations are carried out for different system sizes. It is found that the outcomes can depend critically on the system size, and that physically meaningful results can only be obtained if the system size is large compared with the typical eddy size.

1. Introduction

The average induction in a moving fluid plays a central role in our understanding of the generation of large-scale magnetic fields in astrophysical bodies. Its efficiency is measured by the α tensor of mean field electrodynamics, which gives the contribution to the mean electromotive force (emf) proportional to the mean magnetic field, and which is non-zero only for systems lacking reflectional symmetry. Since the physical nature of the α-effect is that of a regeneration term, its magnitude is directly related to the dynamo efficiency. Considerable effort has therefore been expended in the calculation of α for different classes of flows. One such class, which is particularly important astrophysically, is that of turbulent flows or, more generally, flows that are extremely spatially disorganized. The calculation of α for these flows poses a significant challenge. Broadly speaking, there are two ways of approaching the problem. One is to work with the full turbulent velocity field, either in terms of analytic theories of magnetohydrodynamic (MHD) turbulence or via large-scale numerical simulations. The other is to argue that most turbulent flows can be regarded as a collection of basic flow structures – for example, cells in turbulent convection, cyclonic eddies in rotating turbulence – to compute α for one such structure, and so regard its value as representative of the whole ensemble. In general, the second approach, though by no means straightforward, is more practicable, both analytically and numerically. The question naturally arises as to whether the two approaches give the same answer. One way to address this issue would be to proceed numerically, to calculate α for a spatially extended turbulent flow, and to compare this with the value obtained for an isolated flow structure. Until recently, such a comparison could not reasonably be undertaken, owing to the inherent difficulties of representing an extended turbulent system. However, with the constant increase in computational speed, such calculations have now become feasible.

Turbulent convection provides a particularly convenient and astrophysically relevant system in which to address these issues. In the presence of rotation the convection will typically have non-zero helicity, indicating a lack of reflectional symmetry, and it is therefore natural to expect the α tensor to be non-zero. Assuming that the α-effect is well defined for an extended system, it is natural to ask what is the smallest sub-system that correctly reproduces the same behaviour. The study of still smaller sub-systems is misleading, that of larger systems wasteful.

It is well known that turbulent convection appears as a pattern of convective cells; this is a robust feature, seen in experiments, numerical simulations and observations of naturally occurring convection. How many cells does one need to capture the behaviour of the full system? More specifically, is the behaviour of a single cell representative of the system as a whole? In answering these questions one can imagine two types of problem. The first is how one uses the value of α for a single representative cell, assuming that this can be calculated, to construct the average over a disordered distribution of such cells, i.e. what is the relationship between a typical value of α and its average value. The second concerns the very representation of a typical cell. As we shall see, there are cases where reduction in the system size leads to a change in the basic cell structure such that, in the end, they are no longer representative of the extended system.

It is extremely important, particularly in the light of the approach we shall adopt in this paper, to clarify the difference between dynamo action and mean induction, as measured, say, by the α-effect. Dynamo action describes the growth of magnetic fields. Traditionally a distinction has been made between small-scale and large-scale dynamo action. The former refers to the growth of magnetic fluctuations, without necessarily any accompanying growth of the mean field. It is believed to occur in any complicated three-dimensional flow provided the electrical conductivity (measured by the magnetic Reynolds number) is sufficiently high; it does not require helicity nor, indeed, any lack of reflectional symmetry (see, for example, Vainshtein & Kitchatinov 1986; Childress & Gilbert 1995). The latter is associated with the growth of the mean field. It requires mean induction, which proceeds only in flows lacking reflectional symmetry. It should be noted, however, that the presence of mean induction is not, of itself, sufficient to guarantee the growth of the large-scale component of the field. For instance, if the dissipation of the large-scale field overwhelms the mean induction then the large-scale field will decay. In the present paper we are interested in mean induction effects; they will be measured in terms of the mean current parallel to an imposed mean magnetic field, and *not* in terms of the growth of the mean field.

2. Formulation

Our present objective is to study how mean induction effects vary with system size. A simple system that leads to non-trivial behaviour consists of a plane layer of incompressible (Boussinesq) convecting fluid rotating about the vertical axis. The presence of rotation imparts a definite handedness to the system which, combined with the up–down symmetry of Boussinesq convection, leads to a helicity distribution that is antisymmetric about the midplane. The emf arising from the imposition of a uniform horizontal field will therefore likewise be antisymmetric about the midplane and thus can be measured by means of averages over the lower (or upper) half-volume. For this set-up the relevant parameter measuring the system size is the horizontal extent of the layer, which we vary from being comparable with to much larger than a typical convective cell size.

We consider a Cartesian layer of depth d, rotating with angular velocity Ω and with an imposed uniform horizontal magnetic field of strength B_0. We assume that the kinematic viscosity ν, thermal diffusivity κ and magnetic diffusivity η are constant.

Following standard practice, we adopt the layer depth d, the thermal relaxation time d^2/κ, and the temperature drop across the layer ΔT as the units of length, time, and temperature respectively. With these units, and in standard notation, the evolution equations read

$$(\partial_t - \sigma\nabla^2)\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \sigma Ta^{1/2}\boldsymbol{e}_z \times \boldsymbol{u} = -\nabla p + \boldsymbol{J}\times\boldsymbol{B} + \sigma Ra\,\theta\boldsymbol{e}_z, \tag{2.1}$$

$$(\partial_t - \sigma/\sigma_m\nabla^2)\boldsymbol{B} + \boldsymbol{u}\cdot\nabla\boldsymbol{B} = \boldsymbol{B}\cdot\nabla\boldsymbol{u}, \tag{2.2}$$

$$(\partial_t - \nabla^2)\,\theta + \boldsymbol{u}\cdot\nabla\theta = w, \tag{2.3}$$

$$\nabla\cdot\boldsymbol{B} = \nabla\cdot\boldsymbol{u} = 0, \tag{2.4}$$

where $\boldsymbol{J} = \nabla\times\boldsymbol{B}$ is the current density, w is the vertical velocity, and θ denotes the temperature fluctuations relative to a linear background profile (e.g. Chandrasekhar 1961). Four dimensionless numbers appear explicitly: the Rayleigh number $Ra = g\tilde{\alpha}\tilde{\beta}d^4/\kappa\nu$ (where g is the gravitational acceleration, $\tilde{\alpha}$ is the coefficient of thermal expansion and $\tilde{\beta}$ is the superadiabatic temperature gradient), which measures the strength of thermal buoyancy relative to dissipation; the Taylor number $Ta = 4\Omega^2 d^4/\nu^2$, and the kinetic and magnetic Prandtl numbers

$$\sigma = \frac{\nu}{\kappa} \quad \text{and} \quad \sigma_m = \frac{\nu}{\eta}\,. \tag{2.5}$$

It should be noted that there is no pre-factor for the Lorentz force term in (2.1), implying that we are measuring magnetic fields in terms of the equivalent Alfvén velocity and *not* in terms of the imposed field strength B_0. We do this for consistency with dynamo studies, for which $B_0 = 0$.

In the horizontal directions we assume that all fields are periodic with periodicity λ. In the vertical we consider standard illustrative boundary conditions on the temperature and velocity fields, namely that the boundaries are perfect thermal conductors, impermeable and stress-free. Formally these correspond to

$$\theta = w = \partial_z u = \partial_z v = 0 \quad \text{at} \quad z = 0, 1. \tag{2.6}$$

In terms of computational convenience there are two obvious possibilities for the choice of magnetic boundary condition, namely that the field is either purely horizontal or purely vertical on the upper and lower boundaries. The purpose of this work is to consider the evolution of mean quantities, which involves a definition of averages. The natural average in this system is one over horizontal planes, which involves averaging over many convective cells. From the point of view of generating large-scale fields with the simplest vertical structure, it is preferable to choose boundary conditions for which the field is purely horizontal, thereby admitting field configurations with only one node in the vertical. Thus we assume that the horizontal boundaries are perfectly electrically conducting, corresponding to the conditions

$$B_z = \partial_z B_x = \partial_z B_y = 0 \quad \text{at} \quad z = 0, 1. \tag{2.7}$$

In this system the importance of rotation is controlled by both the Taylor and Rayleigh numbers. We wish to explore regimes in which the rotation is significant, so that the resulting flows are helical. Our focus is on changes brought about by variations in system size, and we choose to explore these in different regimes such that the convection goes from being close to marginal to being quite vigorous. These requirements dictate that for vigorous convection Ra and Ta should be comparable and $\gtrsim 10^5$; substantially larger values become computationally difficult. We follow

Cattaneo & Hughes (2006) in picking these regimes by fixing $Ta = 5 \times 10^5$ and considering three values of Ra. In what follows we also fix $\sigma = 1$ and $\sigma_m = 5$.

We solve equations (2.1)–(2.4) numerically by standard pseudo-spectral methods optimized for machines with parallel architecture. Details concerning the numerical methods can be found in Cattaneo, Emonet & Weiss (2003).

The problem of dynamo action in a rotating convective layer has been considered by a number of authors. Childress & Soward (1972) computed the α-effect analytically for convection near marginal stability. Their work was later extended to the weakly nonlinear regime by Soward (1974). Subsequently there have been several numerical investigations of the problem. St Pierre (1993) considered subcritical dynamo action – the 'strong field branch' (for a discussion of subcritical effects in rotating convection see Eltayeb & Roberts 1970 and Fautrelle & Childress 1982). Supercritical dynamo action (the evolution from an initially weak field) in infinite Prandtl number convection and with electrically insulating boundary conditions was investigated by Jones & Roberts (2000) and extended to the case of higher Ta and an inclined rotation axis by Rotvig & Jones (2002). Precisely the same system as considered here has been studied by Stellmach & Hansen (2004) for aspect ratios close to unity, and by Cattaneo & Hughes (2006) for larger aspect ratios.

3. Results

In the absence of a magnetic field, and for Prandtl numbers greater than or equal to unity, convection sets in as a direct instability, the critical Rayleigh number being given by

$$Ra_c = Ra_0 + \frac{\pi^2 Ta}{k_h^2}, \tag{3.1}$$

where $k_h^2 = k_x^2 + k_y^2$ and $Ra_0 = (\pi^2 + k_h^2)^3/k_h^2$ is the critical value for convection in the absence of rotation (Chandrasekhar 1961). For our choice of Taylor number ($Ta = 5 \times 10^5$), Ra_c takes its minimum value of $Ra_c = 59\,008$ with $k_h = 11.4$. We could say therefore that, close to marginal stability, the typical convective cell has aspect ratio close to one half ($2\pi/11.4 \approx 0.55$). As the Rayleigh number increases, convection becomes more vigorous with a corresponding increase in the magnetic Reynolds number Rm. At $Ra \approx 170\,000$, Rm is sufficiently large that, in an extended system, the flow becomes unstable to dynamo action.

Our approach to calculating α is to measure the emf induced by the presence of an externally imposed uniform horizontal magnetic field $\boldsymbol{B}_0$. The α tensor is then defined by the relation

$$\alpha_{ij}(T) B_{0j} = \frac{1}{T} \int_0^T \mathscr{E}_i(t)\, \mathrm{d}t, \tag{3.2}$$

where $\mathscr{E} = \langle \boldsymbol{u} \times \boldsymbol{b} \rangle$ and where angle brackets denote a spatial average. In view of the antisymmetry of the helicity distribution about the midplane this could be an average over the lower half-volume, the upper half-volume, or, better still, one half of the difference of these two. In general we expect $\alpha(T)$ to converge to a well-defined value for sufficiently large T, i.e. values of T that are much larger than the correlation times for $\mathscr{E}$. However, since the latter is not known *a priori*, the rate of approach to the limiting value is also of interest (see Cattaneo & Hughes 2006).

The findings of this paper are based on roughly fifty individual numerical experiments. We have considered three values of the Rayleigh number: $Ra = 80\,000$, which is mildly supercritical; $Ra = 150\,000$, which leads to more vigorous convection, but is still below the threshold for small-scale dynamo action; and $Ra = 500\,000$, for which the convection is very vigorous and is well above the dynamo threshold. Because we keep Ta fixed, the Rossby number, the ratio of the rotational period to the turnover time, is different in these three cases, but is always smaller than one, so that the influence of rotation remains strong.

For each value of Ra we consider system sizes of $\lambda = 0.5$, comparable with a convective cell size, $\lambda = 1$, $\lambda = 2$ and $\lambda = 5$, for which the layer contains over a hundred cells. We consider purely hydrodynamic cases (no imposed field) and two values of the externally imposed field strength, namely $B_0 = 0.1$ and $B_0 = 10$. The smallest of these has no appreciable effect on the underlying flow and can be considered to be kinematic. The largest, although still small compared with the equipartition strength, is dynamically significant and leads to substantial changes in the basic flow. For most cases the convection is isotropic in the horizontal plane, and it suffices to consider a single orientation (along x, say) of the imposed field. For cases of small system size, however, the convection can become anisotropic; for these, the determination of the α tensor requires the imposition of fields in the x- and y-directions separately. The numerical resolution was varied according to system size and Rayleigh number; the smallest simulations ($\lambda = 0.5$, $Ra = 80\,000$) have a resolution of $32 \times 32 \times 65$, the largest ($\lambda = 5$, $Ra = 500\,000$) have a resolution of $256 \times 256 \times 97$.

3.1. *No imposed fields*

Figure 1 shows the density plots of the temperature fluctuations near the lower boundary for the three values of the Rayleigh number, with λ varying from 0.5 to 5, and with no magnetic field. For large aspect ratios, the convection exhibits an irregular time-dependent planform for all three values of Ra. For $Ra = 500\,000$ and $150\,000$ the convection is cellular, for $Ra = 80\,000$ the pattern is a mixture of cells and distorted rolls. At the smallest aspect ratio considered ($\lambda = 0.5$), the flow takes the form of a single steady roll for $Ra = 80\,000$, a regular time-periodic pattern consisting of alternating periodic rolls for $Ra = 150\,000$, and an irregular chaotic pattern for $Ra = 500\,000$. It should be noted that for any value of the Rayleigh number, the convection can be made steady, or indeed suppressed altogether, by adopting a small enough system size.

As expected, at fixed Taylor number the characteristic cell size increases with increasing Rayleigh number, as the influence of the rotation on the convection decreases. This is also manifest in the helicity distribution. At each depth we define the relative helicity by

$$h(z) = \frac{\langle \boldsymbol{u} \cdot \nabla \times \boldsymbol{u} \rangle_h}{\langle \boldsymbol{u}^2 \rangle_h^{1/2} \langle (\nabla \times \boldsymbol{u})^2 \rangle_h^{1/2}}, \tag{3.3}$$

where $\langle \cdot \rangle_h$ denotes an average over a horizontal plane. Figure 2 shows $h(z)$ for the three different values of Ra and for $\lambda = 5$. The decrease of helicity with increasing Ra is apparent. Nevertheless, it should be noted that even for the case of $Ra = 500\,000$, the flow is still substantially helical. Figure 2 also exhibits $h(z)$ for the case of $Ra = 80\,000$, $\lambda = 0.5$, thereby showing a fairly modest dependence on system size.

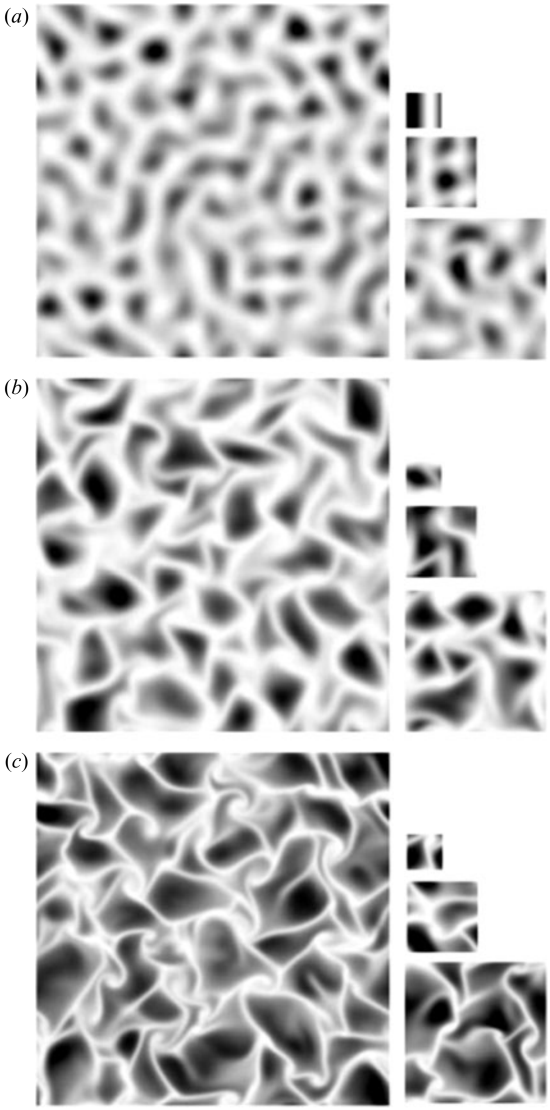

FIGURE 1. Density plots of temperature fluctuations near the lower boundary for (*a*) $Ra = 80\,000$, (*b*) 150 000 and (*c*) 500 000, and for $\lambda = 5$, 2, 1 and 0.5. Light (dark) tones correspond to hot (cold) fluid. As λ decreases, there is a definite tendency towards more ordered patterns of convection.

3.2. *Weak external fields*

Following the discussion above, our objective is to investigate the role of system size in determining average induction, as measured, say, by the α-effect. For large systems we expect α to be independent of system size. It is therefore natural to begin our investigation by considering the cases with $\lambda = 5$. Figure 3 shows the longitudinal component of the emf averaged over the lower half of the layer, for the three values of Ra and for $B_0 = 0.1$. For all three cases, even though at any instant the average involves many convective cells, the emf is a strongly fluctuating quantity, taking both positive and negative values. The time scale of the fluctuations decreases

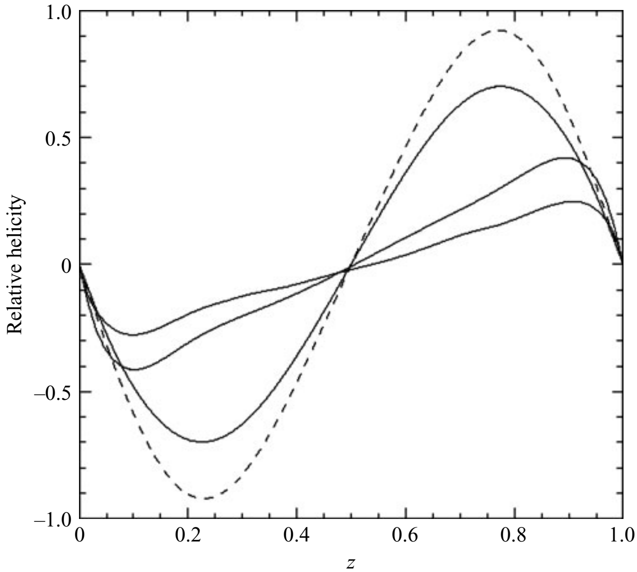

FIGURE 2. Snapshots of $h(z)$ (the horizontally averaged relative flow helicity). The solid curves correspond to cases with $\lambda = 5$ and $Ra = 80\,000$, $150\,000$ and $500\,000$; the helicity decreases with increasing Ra. The dashed curve corresponds to the case with $\lambda = 0.5$ and $Ra = 80\,000$.

with increasing Ra, which is to be expected since the characteristic turnover time becomes shorter as Ra increases. More interestingly, the characteristic amplitude of the fluctuations shows a dramatic increase with Ra. To understand this phenomenon we should note that the emf is a product of a fluctuating velocity and a fluctuating magnetic field. Below the dynamo threshold the velocity is determined solely by Ra; the fluctuating magnetic field is determined by some positive power of the magnetic Reynolds number Rm and by the magnitude of B_0. Increasing Ra in this regime leads to an increase in the r.m.s. velocity and, therefore, in Rm; this in turn leads to an increase in the amplitude of the magnetic fluctuations and, consequently, in the emf. Above the dynamo threshold, magnetic fluctuations will grow exponentially until they become nonlinearly saturated. Their final amplitude will depend on the equipartition strength and on Rm, but with a dependence that is completely different to that in the non-dynamo regime.

It is clear from inspection of figure 3 that even at this large system size, one that allows many convective cells, it is impossible to extract any useful information about α from any short time integration. The situation is improved by considering long-term time averaging, as defined by (3.2). Figure 4 shows the result of such averaging for the same cases as in figure 3. For $Ra = 80\,000$ and $150\,000$, the cumulative averages approach well-defined values of 0.1 and 0.05 respectively, with error bars of a few per cent. Here the error bars were computed by sub-sampling the original signal. For the third case, the error bars are substantially larger than the average and even the sign is indeterminate. It is clear that in order to be able to make any meaningful statement about the average value of the emf, a much longer time series is required, though it is important to note that the interval considered here contains over two hundred turnover times. We should remark on the magnitude of the average emf, and hence of the corresponding α. For $Ra = 80\,000$ and $150\,000$, α takes the values 1 and 0.5 respectively. In both cases this is much smaller than the r.m.s. velocities (18 and

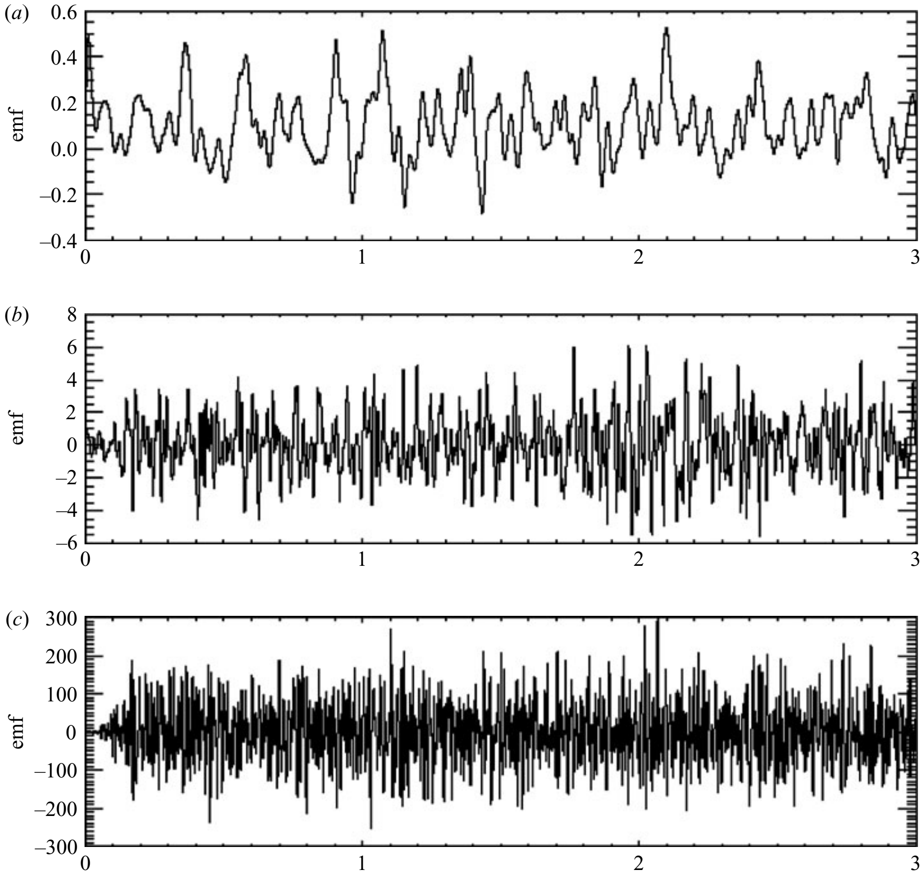

FIGURE 3. Time histories of the longitudinal component of the emf for $\lambda = 5$, $B_0 = 0.1$ and (*a*) $Ra = 80\,000$, (*b*) $Ra = 150\,000$, (*c*) $Ra = 500\,000$.

56), suggesting that the origin of the α-effect is collisional rather than turbulent. This is indeed confirmed to be the case by a more careful analysis involving varying the magnetic Prandtl number (Cattaneo & Hughes 2006).

We are now in a position to study the impact of reducing the system size. Figure 5 shows time histories of the volume-averaged longitudinal emfs for the three cases corresponding to figure 3, but with $\lambda = 0.5$. The cumulative average to time t has been overplotted as a thick line. The differences between the two cases are striking. For $Ra = 80\,000$, convection takes the form of a single steady roll, and the emf rapidly approaches a steady value. Here $\alpha \approx 8.5$, a value ten times larger than the corresponding value when $\lambda = 5$, and now comparable with the r.m.s. velocity, which again is approximately 18. For $Ra = 150\,000$, the convection and the emf are periodic, with a well-defined average value of $\alpha \approx 3$, six times the corresponding value when $\lambda = 5$. For these two cases the convection is strongly anisotropic, and therefore the emf depends on the orientation of the imposed field with respect to the convective pattern. For example, the values of α arising from a field orthogonal to those considered in figure 5 are zero. This is easily understood for the $Ra = 80\,000$ case; the convection consists of a roll along the y-direction and the imposition of a field in the y-direction therefore has no effect. The case of $Ra = 150\,000$ is more complicated.

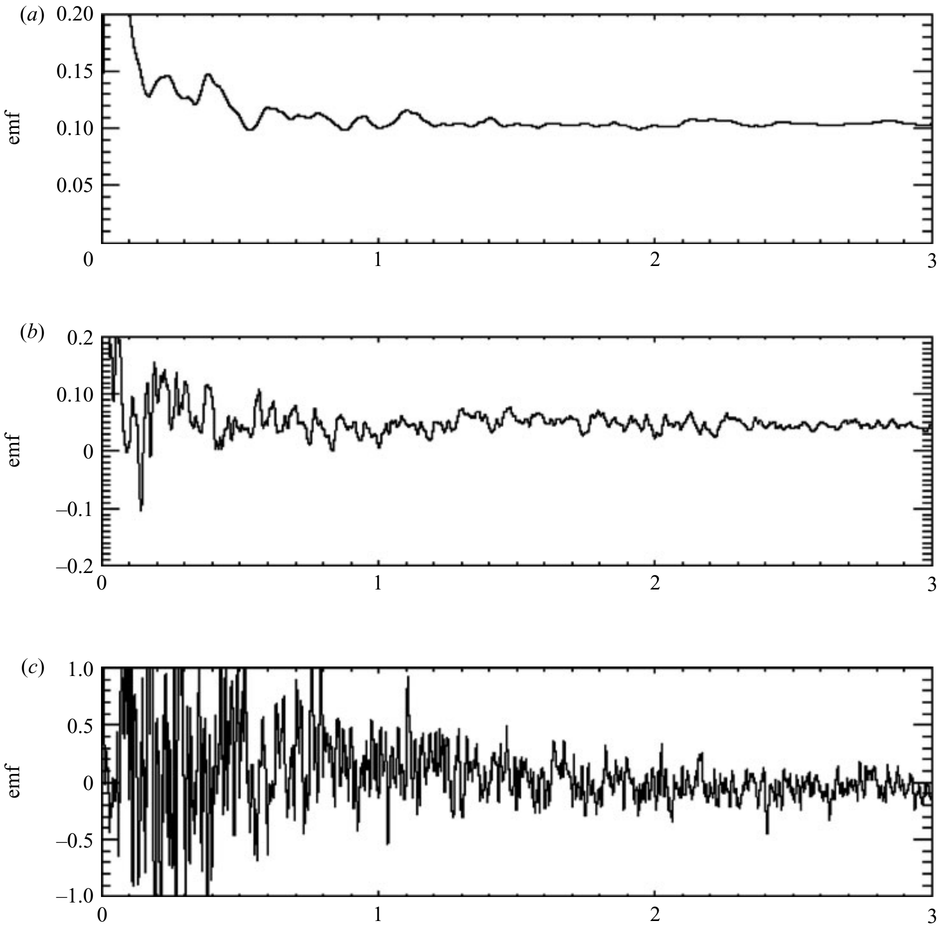

FIGURE 4. Cumulative averages of the longitudinal component of the emf as a function of averaging length, for the cases illustrated in figure 3.

The convective pattern here, though periodic, is not a simple roll; in this case, it is the emf integrated over a cycle that vanishes – rather remarkably since the emf itself is not an antisymmetric function of time.

The third case, $Ra = 500\,000$, looks just as messy as the corresponding case for $\lambda = 5$; time-averaging returns a similarly undetermined value. Superficially, therefore, one might be tempted to conclude that the cases with large and small system sizes are equivalent, albeit in a somewhat negative way. However this is not the case, there being at least two major differences between them. First, at $\lambda = 0.5$ the convection does not act as a dynamo, and all fluctuations in the emf are due entirely to the distortion of the imposed magnetic field. Second, the volume-averaged emf for the case of $\lambda = 0.5$ exhibits much greater intermittency than when $\lambda = 5$. This is illustrated by figure 6, which shows the probability density functions of the emfs for the two cases. The curve for $\lambda = 5$ is characteristic of a Gaussian process, whereas that for $\lambda = 0.5$ is closer to an exponential, at least near the core. This shows that the physical origin of the fluctuations is completely different in the two cases. For the case of large aspect ratio it is due to small-scale dynamo action; for the small aspect ratio it is more likely to be related to dramatic changes in the convective pattern.

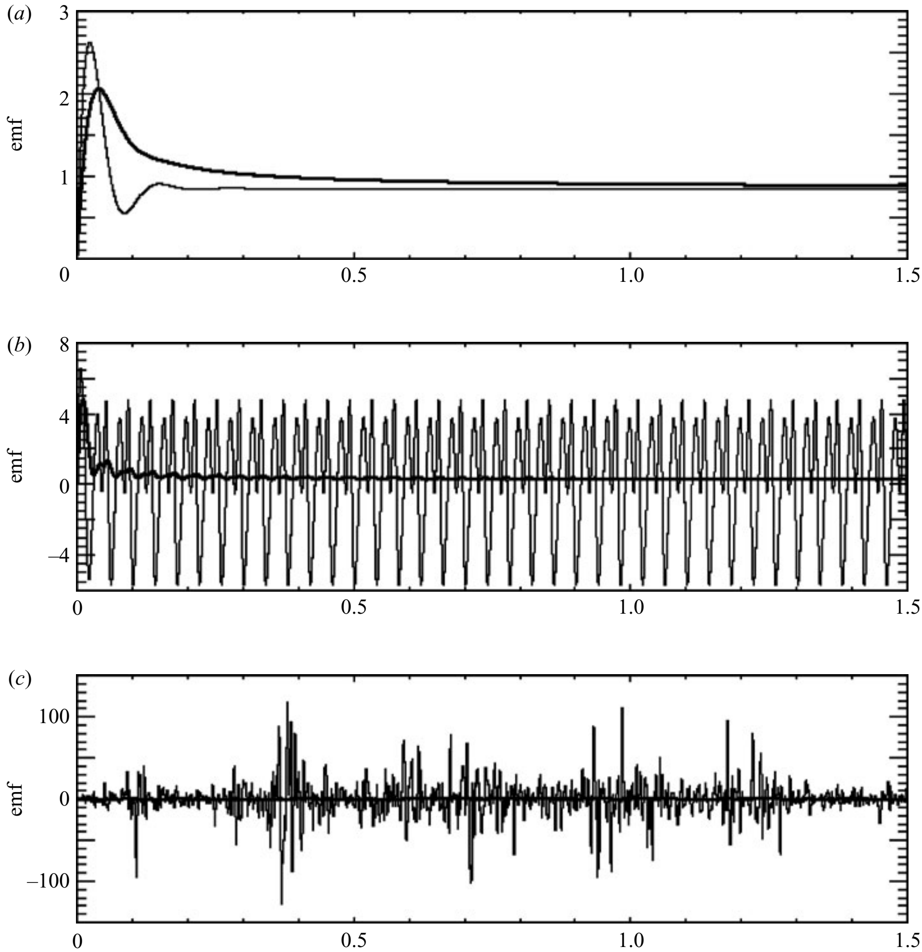

FIGURE 5. As for figures 3 and 4, but with $\lambda = 0.5$. Here we have plotted both the volume-averaged emf (thin line) and its cumulative average (thick line).

It is clear that the behaviour of the average induction for the cases with $\lambda = 0.5$ is not representative of that with $\lambda = 5$. It is therefore instructive to consider systems of intermediate size in order to determine at what point the departures from the large-scale behaviour become significant. To this end we have considered the cases of $\lambda = 1$ and $\lambda = 2$. The results for $Ra = 80\,000$ and $Ra = 150\,000$ are summarized in table 1. Inspection of the table shows that for system size $\lambda = 2$ the results are a reasonable representation of those obtained for a larger aspect ratio. For $\lambda = 1$ it could be argued that the values for α are not that dissimilar to those with larger aspect ratios. However, this is misleading since the convective pattern here is strongly anisotropic, and hence the values of the emf depend crucially on the direction of the imposed field. These results could have been anticipated to some extent by inspection of the convective patterns in figure 1, in which it can be seen that the $\lambda = 2$ cases can be viewed as subsets of the respective $\lambda = 5$ cases, whereas with $\lambda = 1$, the influence of the boundaries is overwhelming. This is emphasized by figure 7, in which the basic patterns for $\lambda = 1$ and $\lambda = 2$ have been replicated 16 and 4 times respectively for comparison purposes. The anisotropy of the $\lambda = 1$ cases is clearly apparent; the

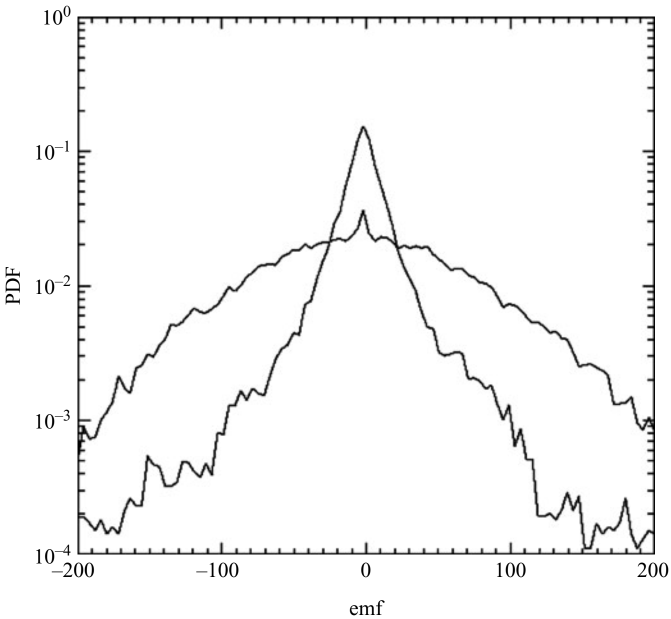

FIGURE 6. Probability density functions (PDFs) of the emfs for $Ra = 500\,000$, $B_0 = 0.1$ and $\lambda = 5$ and 0.5. For $\lambda = 5$ the PDF is Gaussian, for $\lambda = 0.5$ it is close to an exponential near the core.

λ	Orientation	$Ra = 80\,000$	$Ra = 150\,000$
0.5	x	8.47	-0.013 ± 0.002
0.5	y	0	2.85 ± 0.1
1.0	x	1.55 ± 0.22	0.27 ± 0.11
1.0	y	0.99 ± 0.11	0.62 ± 0.17
2.0	x	1.03 ± 0.05	0.57 ± 0.12
5.0	x	0.99 ± 0.02	0.47 ± 0.14

TABLE 1. Values of the longitudinal component of the α tensor for an imposed field of strength $B_0 = 0.1$. The orientation column indicates whether the imposed field is in the x- or y-direction.

patterns are time-dependent, but at any instance are strongly aligned with the x- or y-axes. We have not provided the results for $Ra = 500\,000$ since, for all values of λ, the error bars greatly exceed the average values, and so the results do not lend themselves to any meaningful comparison.

3.3. *Dynamic effects*

The results of the previous section were all obtained for cases where the imposed field had no effect on the convection. For dynamos, in general, the magnetic forces, even if negligible initially, will grow until they are of sufficient strength to alter the flow. It is therefore important to discuss how mean induction effects are modified by an increase in the imposed magnetic field. Here we shall be primarily concerned with a regime in which there is an initial departure from kinematic behaviour – i.e. fields that are strong enough to have a dynamic influence, but that are nonetheless weak compared to equipartition values.

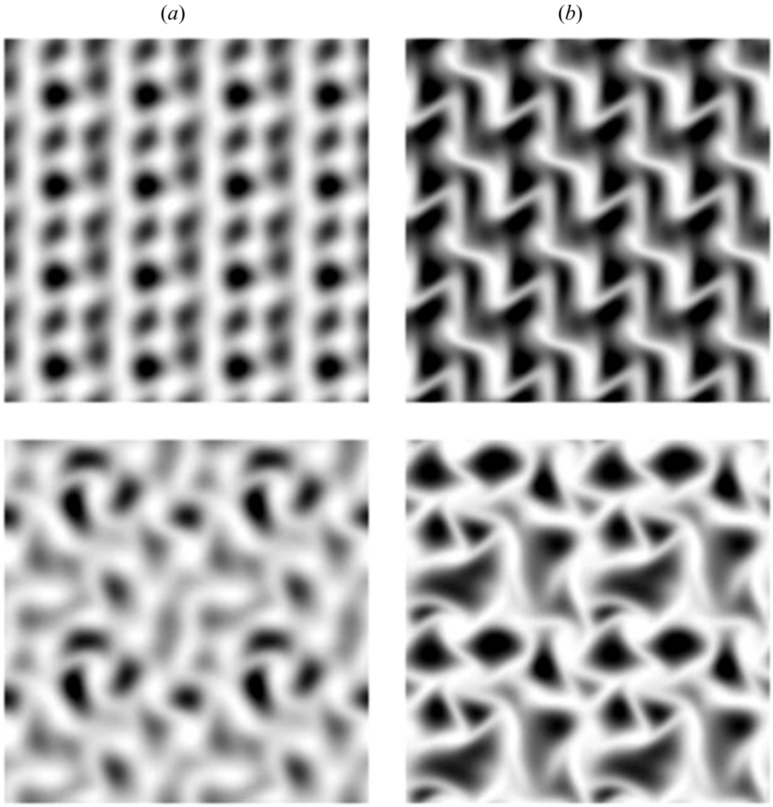

FIGURE 7. Convective patterns for the cases of (a) $Ra = 80\,000$ and (b) $Ra = 150\,000$ with $\lambda = 1$ (upper row), $\lambda = 2$ (lower row). For comparison purposes the cases with $\lambda = 1$ have been replicated 16 times, those with $\lambda = 2$, 4 times.

As in our discussion above of kinematic fields, it is natural first to consider large system sizes. Depending on the Rayleigh number, the presence of a mean magnetic field can lead to two different types of behaviour. At high values of Ra, at which convection dominates over rotational constraints, the magnetic field imparts a certain rigidity to the fluid, thereby reducing the vigour of the convective flow. By contrast, at smaller values of Ra, the effect of the magnetic field is to reduce the rotational constraint on the convection, thus leading to an increase in the vigour of the convection. This point is illustrated in figure 8, which shows the change in the kinetic energy following the imposition of a uniform field with $B_0 = 10$. Computation of the longitudinal α-effect for these three cases yields the values 0.75 for $Ra = 80\,000$, 0.33 for $Ra = 150\,000$, and 0.3 for $Ra = 500\,000$. For $Ra = 500\,000$, the presence of a dynamic magnetic field has modified the fluctuations in such a way that it is now possible – in contrast to the kinematic case – to assign a well-defined value to α. For $Ra = 150\,000$ and $Ra = 80\,000$, the values of α should be compared with the corresponding kinematic values contained in table 1. For $Ra = 150\,000$, both the α-effect and the r.m.s. velocity decrease, the former by 300%, the latter by only 7%. Even more bewildering is the case of $Ra = 80\,000$, for which the α-effect decreases by 25%, whereas the r.m.s. velocity actually increases by more than 30%. Thus it is clear that the dynamical influence of the magnetic field on the α-effect is subtle and complex.

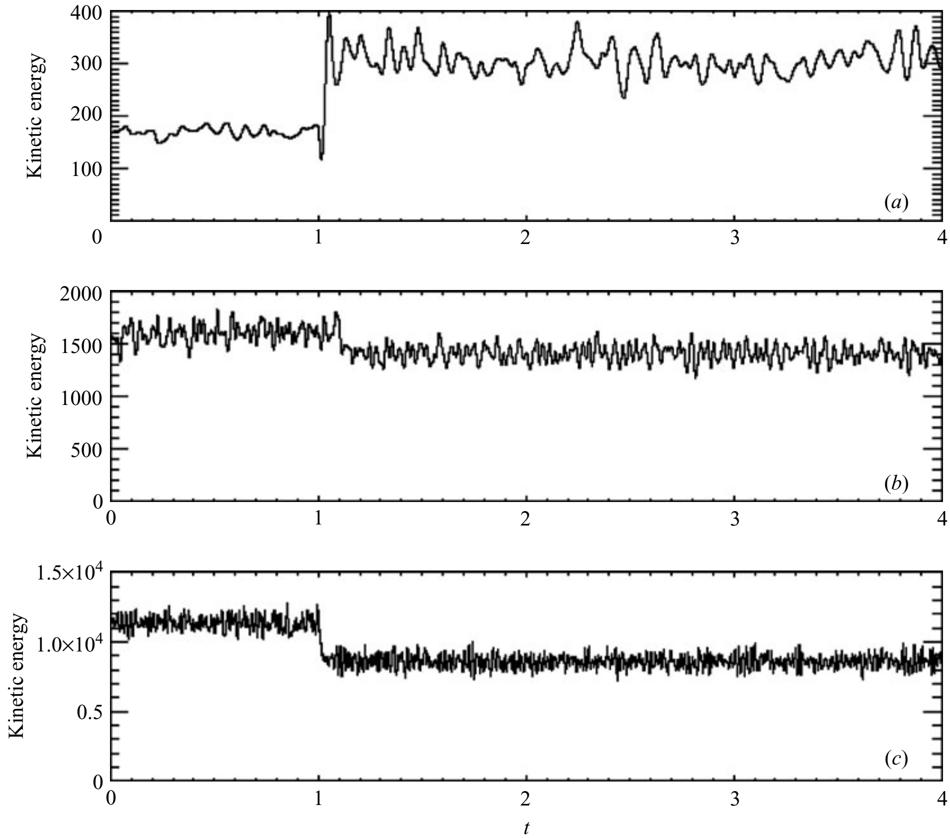

FIGURE 8. Kinetic energy as a function of time for the three cases of $\lambda = 5$ and (a) $Ra =$ 80 000, (b) 150 000 and (c) 500 000. At $t = 1$ a uniform field of strength $B_0 = 10$ is imposed.

λ	Orientation	$Ra = 80\,000$	$Ra = 150\,000$	$Ra = 500\,000$
0.5	x	0	0.0125	0.059 ± 0.069
0.5	y	0	0.0125	0.14 ± 0.05
1.0	x	1.10 ± 0.07	0.31 ± 0.04	0.16 ± 0.24
1.0	y	0.83 ± 0.022	0.35 ± 0.02	0.12 ± 0.34
2.0	x	0.77 ± 0.027	0.29 ± 0.03	0.23 ± 0.08
5.0	x	0.74 ± 0.015	0.32 ± 0.05	0.29 ± 0.016

TABLE 2. Values of the longitudinal component of the α tensor for an imposed field of strength $B_0 = 10$. The orientation column indicates whether the imposed field is in the x- or y-direction.

Although it is obvious that there are important and interesting effects associated with variations in Rayleigh number, the primary objective of this work is to study the impact of variations in system size. Table 2 summarizes the effects of such variations. If we examine the case of $Ra = 80\,000$, α increases slightly from 0.75 at $\lambda = 5$ to 1.1 at $\lambda = 1$, before plummeting to zero at $\lambda = 0.5$. Interestingly, measurements with the field in the orthogonal direction return the same values of α. It would be tempting therefore to interpret this as a manifestation of isotropy. However, although this is probably correct for $\lambda = 5$ and $\lambda = 2$, at smaller values of λ it is a different

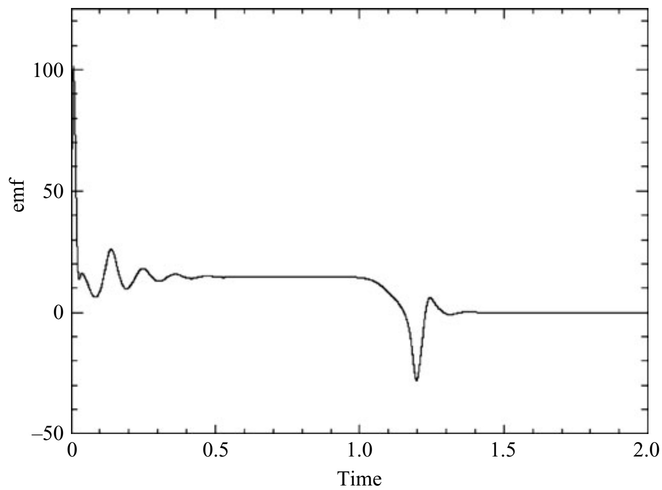

FIGURE 9. Time history of the longitudinal emf for $\lambda = 0.5$ and $Ra = 80\,000$. Initially the convective pattern (as defined by the roll axis) is transverse to the imposed field; at a later time the rolls become aligned with the imposed field, and the longitudinal emf vanishes.

phenomenon altogether, namely the dynamical alignment of the convective pattern with the direction of the mean field. This is most apparent for the case of $\lambda = 0.5$. Recall that in the kinematic case the α-effect is zero if the field is aligned with the basic roll pattern, but takes the value 8.47 for a transverse field. Here, even if initially the field is transverse to the basic roll, dynamical effects will cause the roll to realign and the α-effect eventually to vanish. This is illustrated in figure 9, which shows the longitudinal emf for a case in which the convective roll is initially transverse to the average field. Similar behaviour is exhibited for the case of $Ra = 150\,000$ except that for $\lambda = 0.5$ the basic convective pattern is not quite a roll, and so there is a small residual value for α.

For the case of $Ra = 500\,000$, $\lambda = 5$, the value for α, although now well-defined, is extremely small. The trend is for a slight reduction in this value with decreasing λ together with a marked increase in the errors, such that by $\lambda = 1$, the error and the mean value are comparable.

4. Discussion

One of the central themes of dynamo theory is to explain the origin of large-scale magnetic fields. One can consider two approaches: one in which the focus is on the growing field itself, the other in which attention is focused on the processes that lead to field growth, namely induction and diffusion. In some sense, the second approach, if both processes can be measured individually, is more general, for the following two reasons. One is that it clarifies the basic physical processes that can lead to field generation. The other is that it can be meaningfully carried out even in systems where no field growth occurs. To see this, recall that in the simplest possible formulation of a working mean field dynamo, the growth rate s of a magnetic field with characteristic scale k^{-1} is given by

$$s = \alpha k - \beta k^2, \tag{4.1}$$

where β is the turbulent diffusivity. Hence, if α is non-zero, magnetic fields of sufficiently large scale are guaranteed to grow. Thus, determination of α and β, by whatever means, leads to a full description of the dynamo process on any scale.† Expression (4.1) is valid for isotropic, homogeneous systems; in general one would need to consider the full α and β tensors. Typically, the α-effect can be determined for system sizes smaller than that necessary for field growth, thus affording considerable computational savings. This then leads to the crucial issue considered in this paper, namely that of determining the smallest system size that yields a sensible result for α.

Based on the results described above, the simple answer is 'not too small'. For the particular system and parameter values considered here, $\lambda = 2$ is in reasonable agreement with $\lambda = 5$, and we are confident that no new features will arise at yet larger values of λ. By contrast, the results with $\lambda = 1$, and worse still with $\lambda = 0.5$, give a misleading representation of the extended system. These values are of course specific to the convective model we have considered. However, the underlying causes that account for the discrepancies between large and small systems will have general applicability. We address this issue by considering carefully the process by which the value for the longitudinal α-effect is obtained. It is an average of the projection of a fluctuating vector quantity, the emf, along the mean field. Here the average is a two-step procedure, consisting of a volume average followed by a time average. If we assume that the turbulence has a finite correlation time then the above procedure should be equivalent to an average over an ensemble of volume averages sampled at different times. The result depends on the magnitude of the emf, on the angle between the emf and the imposed field, and on the statistical properties of the ensemble. Anything that can affect any of these three factors will in turn affect the determination of α.

For the values of λ that we have considered, the magnitude of the emf is comparable for large and small system sizes – though one can of course envisage cases for which λ is sufficiently small that the amplitude of the convection becomes extremely weak. By contrast, the distribution of angles between the emf and the imposed field can be markedly different owing, for instance, to the alignment of the convective pattern, either with the domain boundaries or, for stronger imposed fields, with the imposed field. Changes in the statistical properties of the ensemble are definitely present, and can either be blatant, such as the difference between ensembles generated by random as opposed to steady flows, or more subtle, such as the difference between Gaussian and exponential statistics exhibited in figure 6. The deterioration in the value of α has also been seen in studies of distorted cellular flows. Pétrélis & Fauve (2006) showed that phase demodulation of a steady cellular pattern always led to a reduction in α; a similar result for time-dependent flows has been obtained by Courvoisier, Hughes & Tobias (2008). The effect of the modulation is to cause the contributions to the α-effect from the individual cells to add incoherently rather than coherently.

Although the paper has been concerned primarily with the difficulties that may arise in adopting a system size that is too small, it is instructive to address the effects of going in the opposite direction, namely to very large system sizes. In view of the previous analogy between ensemble averaging and time averaging, one could ask how large a system needs to be in order to pin down α without the need for further time averaging. To fix ideas, consider the plots for $Ra = 150\,000$ and $Ra = 500\,000$ in figure 4. For $Ra = 150\,000$ the data set was sufficiently long that we were able to

† We should remark that for the convective system we have studied, the magnetic field can adopt arbitrarily large scales in the horizontal direction, but not in the vertical. Consequently k in equation (4.1) must be replaced by $(k_h^2 + \pi^2)^{1/2}$, and hence dynamo action cannot be guaranteed even for arbitrarily small k_h.

estimate the value of α to acceptable accuracy. No such measurement was possible for the case of $Ra = 500\,000$. It is nonetheless instructive to estimate the system size required to determine α with comparable accuracy over the same length of time. If we compare the two signals then it is clear that the one for higher Ra has larger characteristic amplitude and shorter correlation time. The necessary increase in system size, γ, is then given by

$$\gamma = \frac{\sigma_2}{\sigma_1}\sqrt{\frac{\tau_2}{\tau_1}}, \tag{4.2}$$

where σ_1 and σ_2 are the standard deviations of the signals corresponding to $Ra = 150\,000$ and $Ra = 500\,000$ respectively, and τ_1 and τ_2 are the corresponding correlation times. If we assume that the ratio of the correlation times is the same as the ratio of the turnover times then we obtain $\gamma \approx 25$. So we would need a domain of size 125 (in each horizontal direction), integrated over 3 time units. Alternatively, assuming that the correlation and turnover times are equal, we would need either an integration length of 3×625 for a domain of size 5, or a single snapshot of a whopping domain with size $125 \times \sqrt{3/\tau_2} \approx 2650$. This is clearly an impracticable size for any computation. The source of the difficulty can be understood from a simple dimensional argument. By definition, the magnitude of α is given by

$$\alpha \sim \frac{|\langle \boldsymbol{u} \times \boldsymbol{b} \rangle|}{B_0}. \tag{4.3}$$

Since α has the dimensions of velocity we may write $\alpha = Cu$, for some dimensionless C. If we define $\langle\cdot\rangle_n$ as the average over a set of n independent samples, then the key issue is to determine the smallest value of n, N say, for which

$$C \sim \frac{|\langle \boldsymbol{u} \times \boldsymbol{b} \rangle_n|}{uB_0} = \Gamma_N, \text{ say.} \tag{4.4}$$

Clearly, further progress requires some assumption about the magnitudes of C and b/B_0. A simple estimate for N gives

$$N \sim \left(\frac{b}{CB_0}\right)^2. \tag{4.5}$$

There are two possibilities for C: one is that the α-effect is turbulent, in which case $C = O(1)$, the other is that the α-effect is collisional, in which case $C = O(Rm^{-1})$. For the convective system considered here, α is collisional, as discussed in detail by Cattaneo & Hughes (2006). If we further assume that B_0 is the largest value for which we are still in the kinematic regime then, irrespective of whether we have a small-scale dynamo,

$$b \sim B_0 Rm^{\delta}, \tag{4.6}$$

where δ lies in the range $1/2 < \delta < 1$. This leads to the estimates of

$$N \sim Rm^{2\delta} \quad \text{for a turbulent } \alpha\text{-effect}, \tag{4.7}$$

and

$$N \sim Rm^{2\delta+2} \quad \text{for a collisional } \alpha\text{-effect}. \tag{4.8}$$

Even if one assumes the smallest value of δ ($\delta = 1/2$) these estimates give computational costs at high Rm that range from expensive, for turbulent α-effects, to prohibitive, for collisional α-effects.

We conclude by considering the consequences for the formulation of mean field electrodynamics of these enormous values of N, the number of independent samples needed in the averaging procedure. For most practical purposes the domain size is such that the number of uncorrelated contributions is always much less then N. Consequently averages have not converged, with the implication that the coefficients in the mean field equations – which are themselves expressed in terms of these averages – are rapidly fluctuating and should be interpreted as random variables. This is contrary to the philosophy of mean field theory, which is to produce a description in terms of smooth quantities.

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