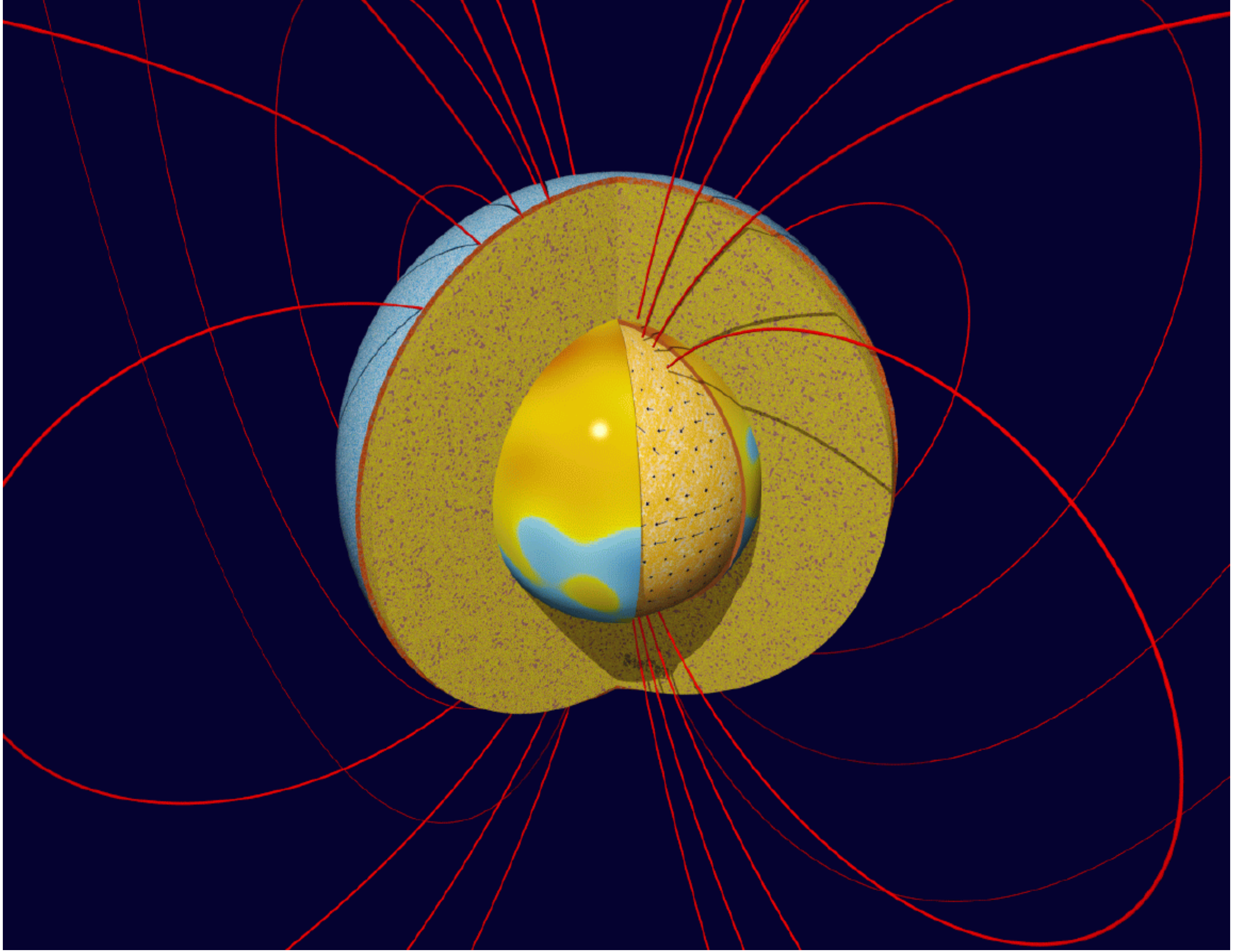
Fusing core dynamics and observations

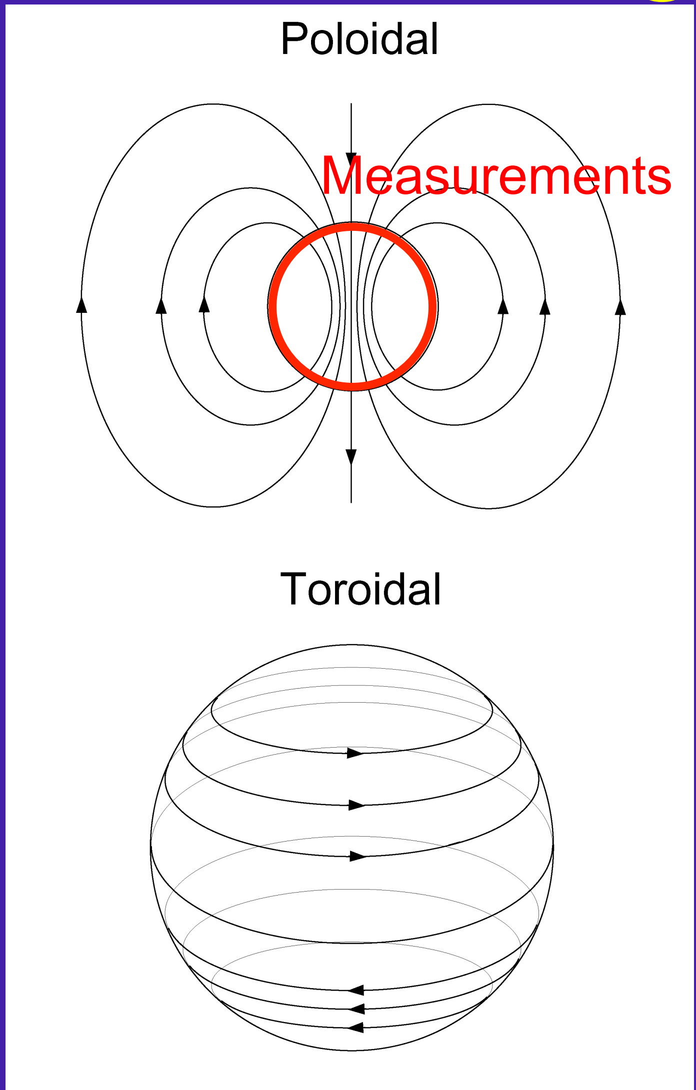
Andy Jackson, Kuan Li
ETH Zurich
&
Phil Livermore
University of Leeds

Outline

- Rationale for 4DVar
- Proof of concept: Hall effect dynamo
- 4DVar applied to a model of core dynamics



Visible and hidden parts of the magnetic field



Based only on poloidal magnetic field data on the boundary of the core, can we infer interior properties (including those of the toroidal field)?

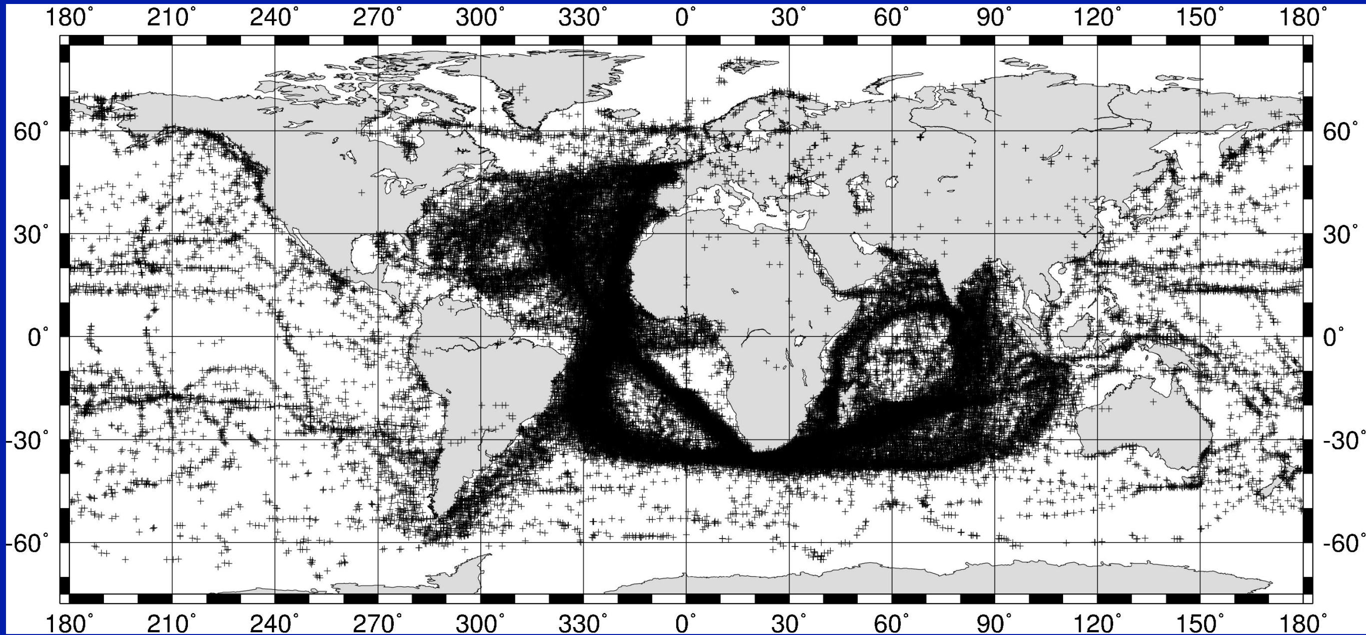
Technology transfer from meteorology: 4DVar variational data assimilation

2-D observations in time of a time-evolving system can give information about the third (hidden) dimension

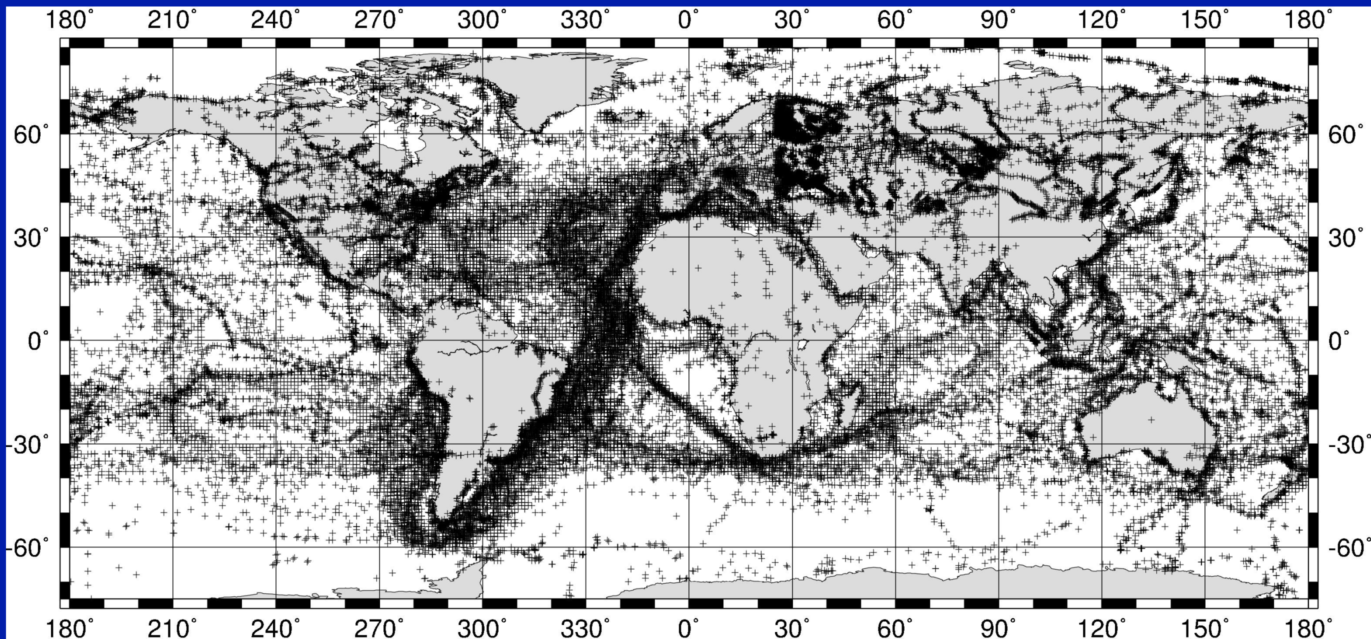
$$\frac{\partial X}{\partial t} = f(X)$$

Applications:
Seismology (X=velocity),
Mantle convection (X=temperature),
Core convection (X=magnetic field)

Declination
1700-1799

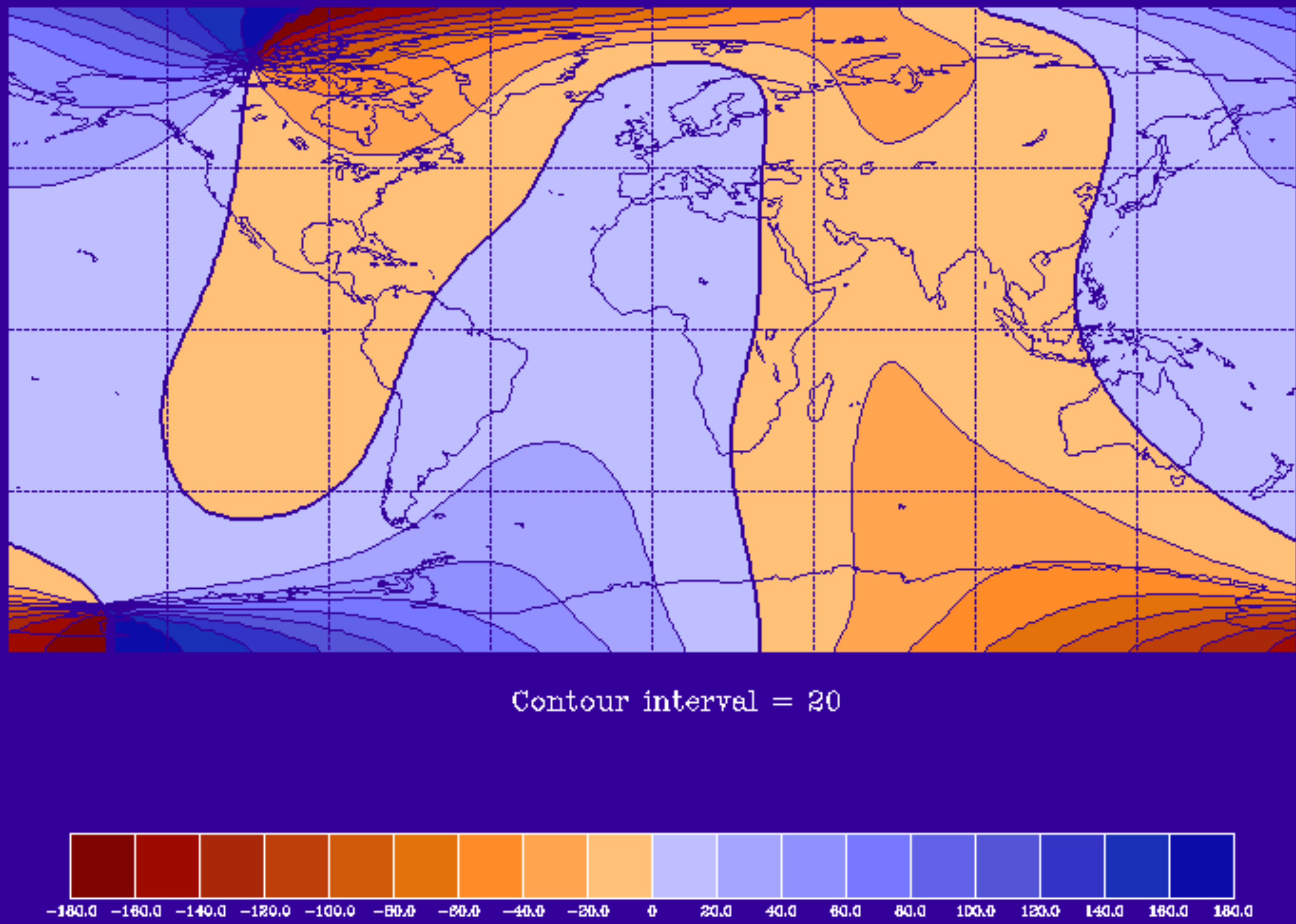


Declination
1800-1930

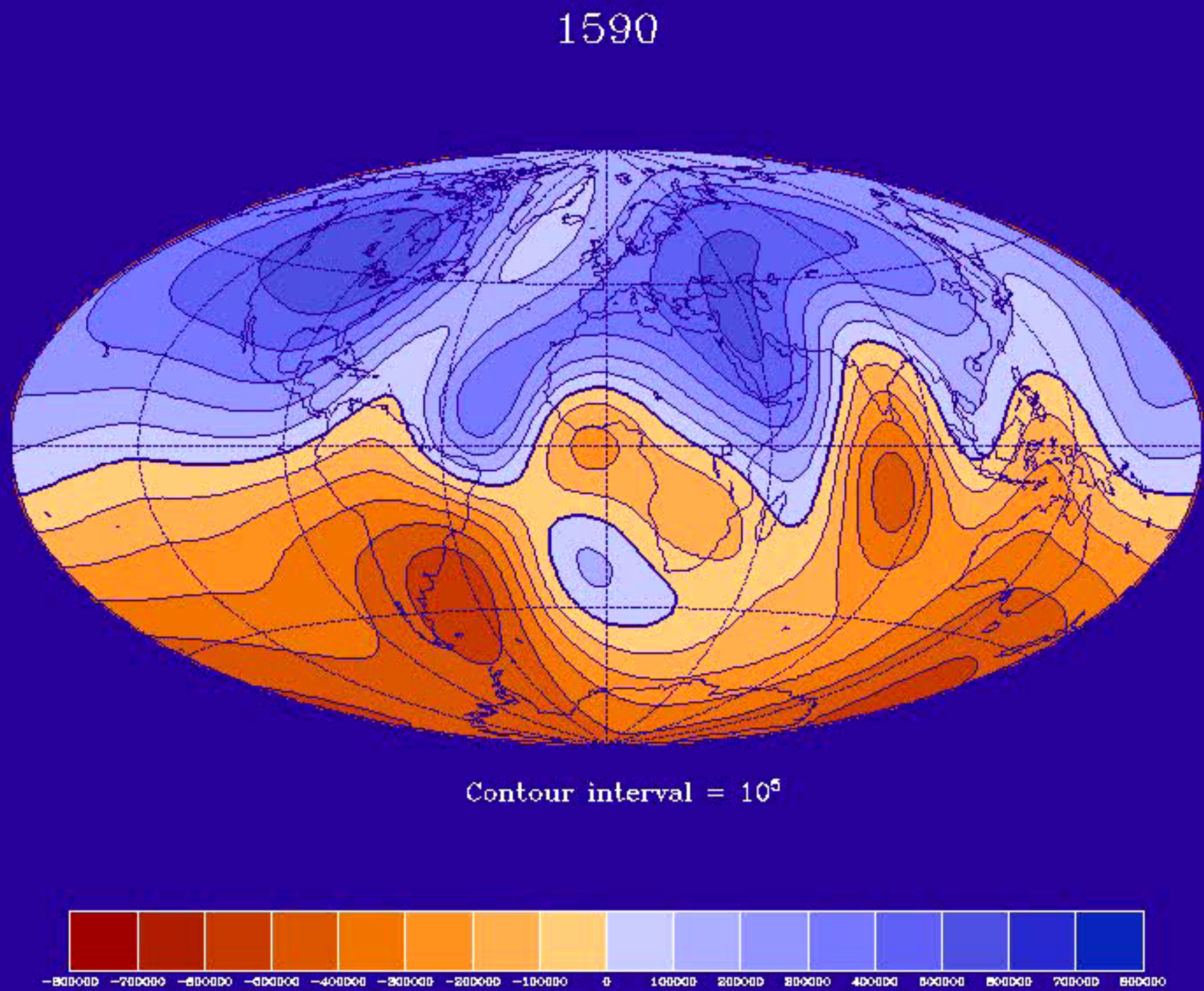


Declination (Earth's Surface)

1590



Radial Magnetic Field (Core Surface)

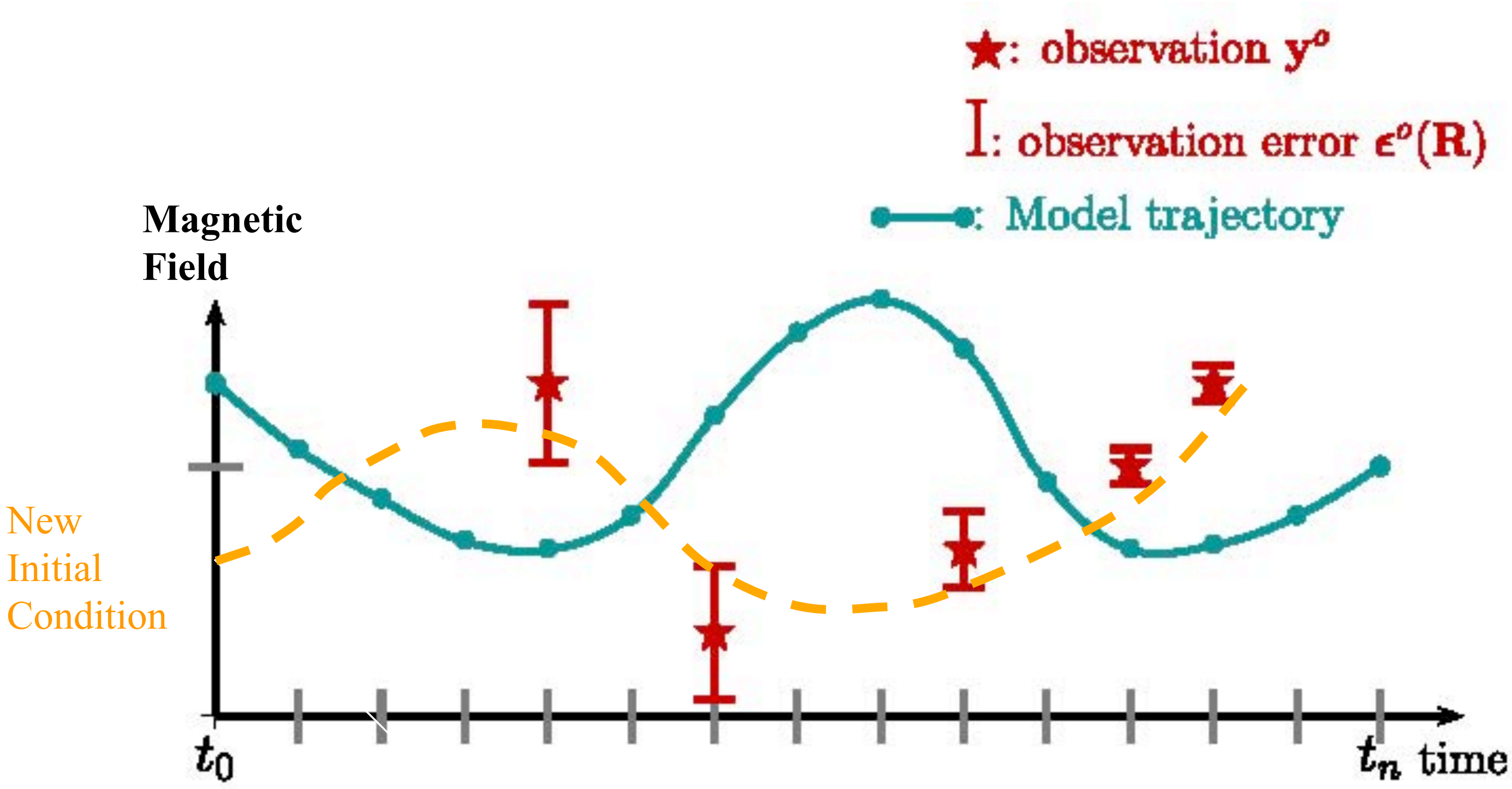


Why data assimilation? (PDE constrained optimisation)

- Data
- Dynamical model
    - Magnetic part (induction equation)
    - Momentum part (Navier Stokes)
    - Energy part (temperature equation)

- Unknown is the initial state of the model

System Evolution



Fournier et al 2010

A dynamical model for the core

$$\cancel{\frac{\partial \mathbf{u}}{\partial t}} + \cancel{(\mathbf{u}\cdot\nabla)\mathbf{u}} + 2\mathbf{\Omega}\times\mathbf{u} + \frac{1}{\rho}\nabla p - \frac{1}{\rho}\mathbf{J}\times\mathbf{B} - \cancel{\nu\nabla^2\mathbf{u}} - \alpha T\hat{\mathbf{r}} = 0$$

$$\boxed{\frac{\partial \mathrm{B}}{\partial t} - \nabla\times(\mathbf{u}\times\mathrm{B}) - \eta\nabla^2\mathrm{B} = 0}$$

Begin by solving this part

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T - \kappa\nabla^2 T - h = 0$$

B = Magnetic field
u = Fluid velocity
T = Temperature
J = Current density
Ω = Rotation vector

$Ro_m = \eta / L^2\Omega \sim 10^{-9}$
Magnetic Rossby Number

Misfit of poloidal field measurements at core surface

Misfit = χ^2 = *(Observed-Predicted)*^{2}

Kinematic Induction Equation (u given and constant) and its adjoint

$$\frac{\partial B}{\partial t} = \nabla \wedge (v \wedge B) + \eta \nabla^2 B$$

B = Magnetic field
v = Fluid velocity
η = Magnetic Diffusivity

$B^\dagger$= Adjoint Magnetic field

$\nabla . B^\dagger = 0$

Same boundary conditions as B

$$-\frac{\partial B^\dagger}{\partial t} = (\nabla \wedge B^\dagger) \wedge v + \eta \nabla^2 B^\dagger - \nabla p^\dagger + f(\text{Misfit})$$

Equation operates in *reverse* time

Li et al PRE 2011

Dynamo Equations

$$
\begin{aligned}
\mathrm{Eq}_1 &:= \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} + \frac{1}{\rho}\nabla p - \frac{1}{\rho}\mathbf{J}\times\mathbf{B} - \nu\nabla^2\mathbf{u} - \alpha T\hat{\mathbf{r}} = 0 \\
\mathrm{Eq}_2 &:= \frac{\partial \mathrm{B}}{\partial t} - \nabla\times(\mathbf{u}\times\mathrm{B}) - \eta\nabla^2\mathrm{B} = 0 \\
\mathrm{Eq}_3 &:= \frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T - \kappa\nabla^2 T - h = 0
\end{aligned}
$$

Misfit derivatives with respect to initial condition

$$
\begin{aligned}
\nabla_{\mathbf{u}_0}\chi^2 &= -\mathbf{u}_0^\dagger \\
\nabla_{\mathrm{B}_0}\chi^2 &= -\mathrm{B}_0^\dagger \\
\nabla_{T_0}\chi^2 &= -T_0^\dagger
\end{aligned}
$$

Adjoint differential equation to integrate in reverse time

$$
\begin{aligned}
0 &= -\frac{\partial \mathbf{u}^\dagger}{\partial t} + \nabla\times(\mathbf{u}\times\mathbf{u}^\dagger) + \mathbf{u}^\dagger\times(\nabla\times\mathbf{u}) + 2\mathbf{u}^\dagger\times\boldsymbol{\Omega} + \nabla p_1^\dagger - \mathrm{B}\times(\nabla\times\mathrm{B}^\dagger) - \nu\nabla^2\mathbf{u}^\dagger + T\nabla T^\dagger, \\
0 &= -\frac{\partial \mathrm{B}^\dagger}{\partial t} - (\nabla\times\mathrm{B}^\dagger)\times\mathbf{u} + \nabla p_2^\dagger - \frac{1}{\rho\mu_0}\Big[\nabla\times(\mathrm{B}\times\mathbf{u}^\dagger) + \mathbf{u}^\dagger\times(\nabla\times\mathrm{B})\Big] - \eta\nabla^2\mathrm{B}^\dagger + O^\dagger[O\mathrm{B} - y], \\
0 &= -\frac{\partial T^\dagger}{\partial t} - \mathbf{u}\cdot\nabla T^\dagger - \alpha\mathbf{u}^\dagger\cdot\hat{\mathbf{r}} - \kappa\nabla^2 T^\dagger,
\end{aligned}
$$

Toy problem I – Hall Effect

A neutron star toy problem

A toy problem to illustrate the physics

- Evolution of the field is given by

$$\frac{\partial B}{\partial t} = R_m \nabla \wedge (\nabla \wedge B \wedge B) + \nabla^2 B$$

Induction through Hall effect

Ohmic diffusion

- The Hall effect is thought to be responsible for the field regeneration

- The initial condition B(t=0) determines the subsequent evolution

- Can we determine the initial condition, and thus the 3-D field at all times?

A closed-loop proof of concept

- Observations of B_r are taken every 100 years for 7k years (~1/4 magnetic decay time)
- Note – no constraints at all on toroidal field
- In our simulation R_m=20 [advection is weak]

The adjoint method

- Forward problem based on current estimate of B(0)
- Calculate residuals
- Backward propagation (reverse time) of adjoint equation

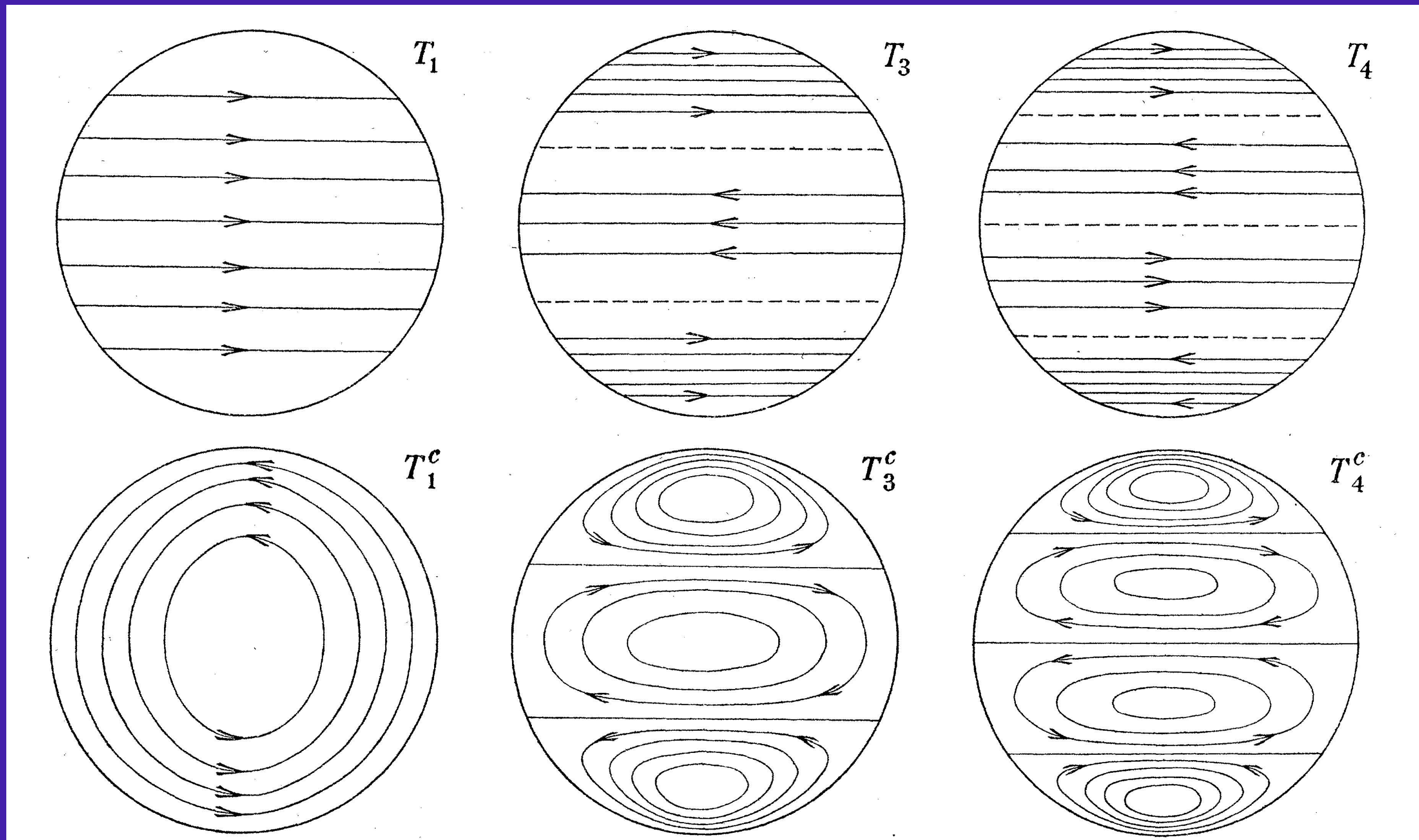
$$
\begin{aligned}
\nabla_{\mathbf{B}_0}\chi^2 &= -\mathbf{B}^\dagger(t=0) \\
-\frac{\partial \mathbf{B}^\dagger}{\partial t} &= -\nabla p^\dagger + R_B\left\{\left(\nabla\times\mathbf{B}^\dagger\right)\times(\nabla\times\mathbf{B}) + \nabla\times\left[\mathbf{B}\times(\nabla\times\mathbf{B}^\dagger)\right]\right\} + \nabla^2\mathbf{B}^\dagger \\
&\quad -\text{forcing by data residuals}
\end{aligned}
$$

- Use gradient vector to update estimate of B(0)
- Go again

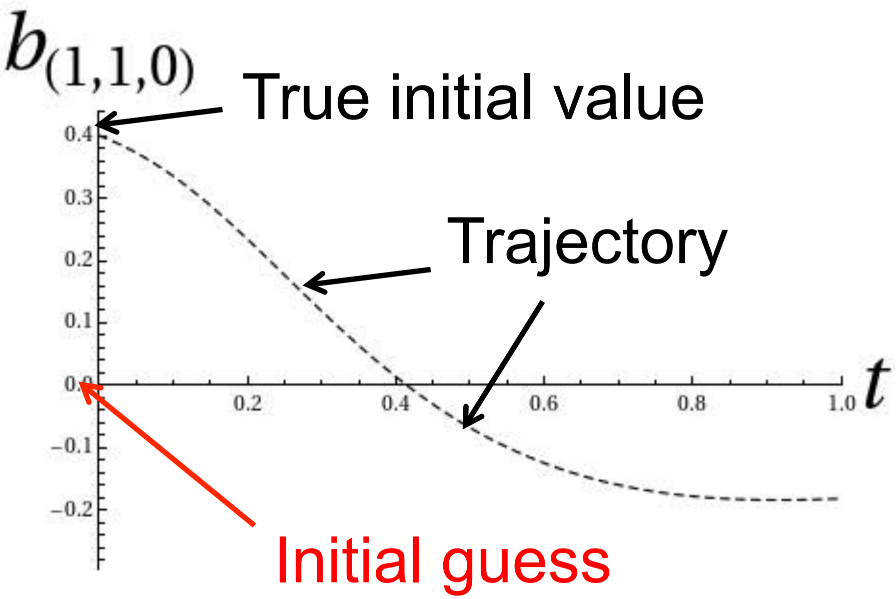
Mie (toroidal-poloidal) representation

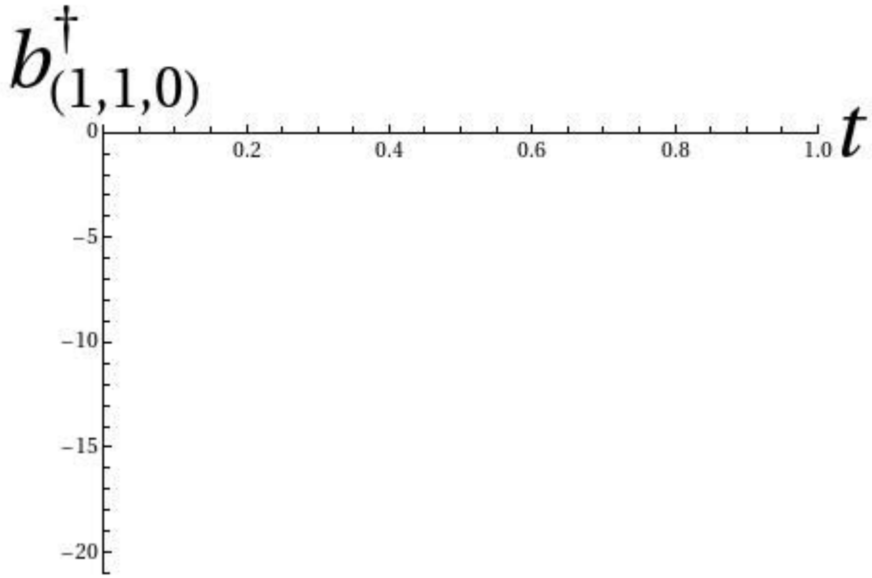
$\nabla.B=0 \quad \Rightarrow \quad B = \nabla\wedge T\mathbf{r} + \nabla\wedge\nabla\wedge P\mathbf{r}$

$T=T_l^m$ etc based on spherical harmonics



Iterative reconstruction of l=1 toroidal coefficient

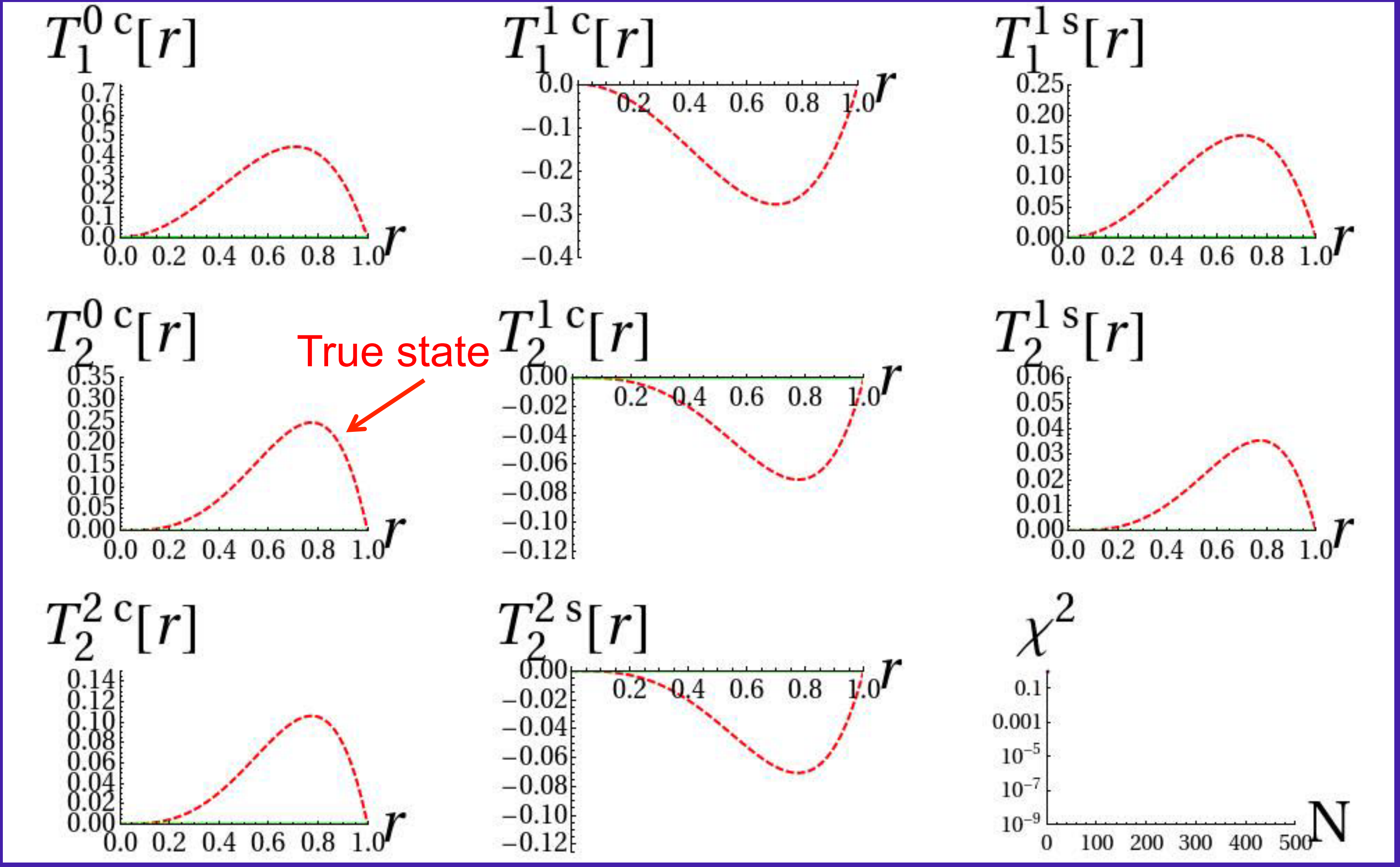




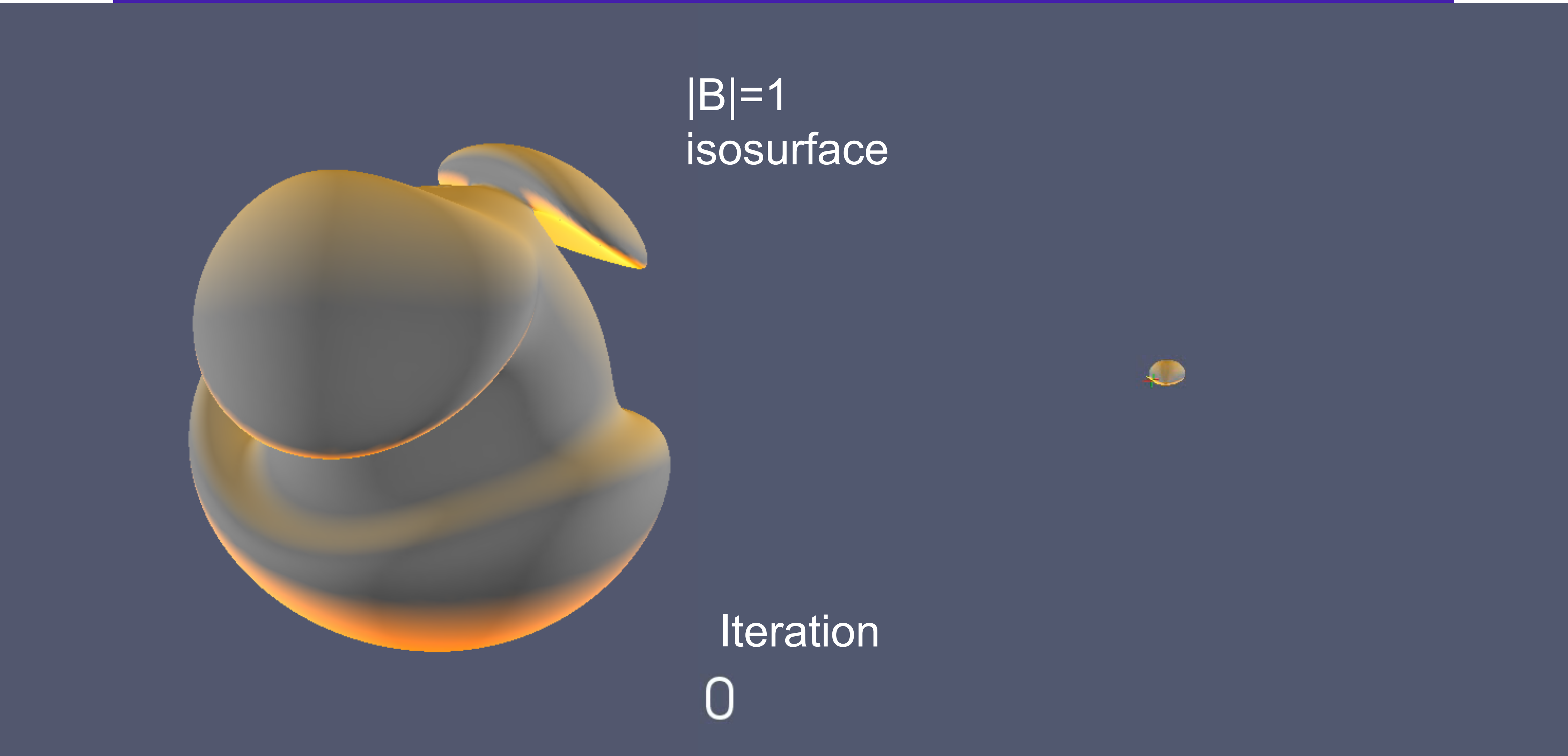
Forward model

Adjoint model

Reconstructed toroidal field (2-D surface poloidal observations, R_m=20, 7k years)



R_m=5 30,000 years surface poloidal observations



True Initial State Reconstructed Initial State

Kuan Li & Andrey Sheyko

Convergence

- There is no proof of uniqueness
- For the case of perfect data, we have never been trapped in local minima
- In the case of real, noisy data this will need testing

Example II
Coupled Navier-Stokes/Induction

Dynamo Equations

$$\text{[illegible]} := \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} + \frac{1}{\rho}\nabla p - \frac{1}{\rho}\mathbf{J}\times\mathbf{B} - \nu\nabla^2\mathbf{u} - \alpha T\hat{\mathbf{r}} = 0$$

$$\text{[illegible]} \frac{\partial \mathbf{B}}{\partial t} - \nabla\times(\mathbf{u}\times\mathbf{B}) - \eta\nabla^2\mathbf{B} = 0$$

$$\mathrm{Eq}_3 \; : \text{[illegible]} \; \mathbf{u}\cdot\nabla T - \kappa\nabla^2 T - h = 0$$

Misfit deriv[illegible]s with respect to initial condition

$$\text{[illegible]}^2 = -\mathbf{u}_0^\dagger$$

$$\nabla_{\mathbf{B}_0}\text{[illegible]} -\mathbf{B}_0^\dagger$$

$$\nabla_{T_0}\chi^2 \; \text{[illegible]}$$

Adjoint differential equation to int[illegible]e in reverse time

$$0 = -\frac{\partial \mathbf{u}^\dagger}{\partial t} + \nabla\times(\mathbf{u}\times\mathbf{u}^\dagger) + \mathbf{u}^\dagger\times(\nabla\times\mathbf{u}) + 2\mathbf{u}^\dagger\times\boldsymbol{\Omega} + \nabla p_1^\dagger - \mathbf{B}\times \text{[illegible]}^\dagger) - \nu\nabla^2\mathbf{u}^\dagger + T\nabla T^\dagger,$$

$$0 = -\frac{\partial \mathbf{B}^\dagger}{\partial t} - (\nabla\times\mathbf{B}^\dagger)\times\mathbf{u} + \nabla p_2^\dagger - \frac{1}{\rho\mu_0}\left[\nabla\times(\mathbf{B}\times\mathbf{u}^\dagger) + \mathbf{u}^\dagger\times(\nabla\times\mathbf{B})\right] \text{[illegible]}^\dagger + O^\dagger[O\mathbf{B} - y],$$

$$0 = -\frac{\partial T^\dagger}{\partial t} - \mathbf{u}\cdot\nabla T^\dagger - \alpha\mathbf{u}^\dagger\cdot\hat{\mathbf{r}} - \kappa\nabla^2 T^\dagger,$$

$$2\rho\Omega \wedge \mathbf{u} = -\nabla p + \mathbf{J} \wedge \mathbf{B} \qquad + \nu\nabla^2\mathbf{u} \tag{3}$$

The system evolves according to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla(\mathbf{u} \wedge \mathbf{B}) + \eta\nabla^2\mathbf{B} \tag{4}$$

Such 'slow' relative motion is governed by the equations

$$2\rho(\mathbf{\Omega} \times \mathbf{v}) = (\mathbf{j} \times \mathbf{B}) - \nabla p' + \rho' \nabla \phi, \tag{1·7}$$

$$\operatorname{div} \mathbf{v} = 0, \tag{1·8}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v} \times \mathbf{B}) + \frac{\eta}{4\pi} \nabla^2 \mathbf{B}, \tag{1·9}$$

$$\frac{\partial \rho'}{\partial t} = -(\mathbf{v} \,.\, \nabla)\,\rho' + S + \kappa \nabla^2 \rho'. \tag{1·10}$$

G.A. Glatzmaier, P.H. Roberts / Physics of the Earth and Planetary Interiors 91 (1995) 63–75

$$0 = -\boldsymbol{\nabla} p + \mathrm{Ra}\, g' T \hat{\boldsymbol{r}} + \boldsymbol{v} \times \hat{\boldsymbol{z}} + \mathrm{E}\nabla^2 \boldsymbol{v} + P(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}) + \boldsymbol{P}\nabla^2 \boldsymbol{B}$$

$$\frac{\partial T}{\partial t} = -\boldsymbol{v} \cdot \boldsymbol{\nabla} T + \nabla^2 T$$

Additional workload – adjoint u

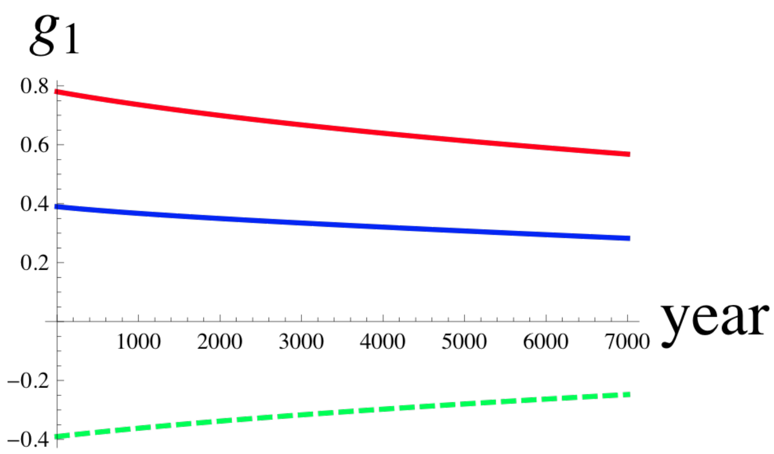
The adjoint N-S equations The adjoint flow is $\mathbf{u}^\dagger$

$$\begin{aligned} \hat{\mathbf{z}} \times \mathbf{u} &= -\nabla\pi + E\nabla^2\mathbf{u} + (\nabla \times \mathbf{B}) \times \mathbf{B} \text{ Forward} && (7) \\ \mathbf{u}^\dagger \times \hat{\mathbf{z}} &= -\nabla p_1^\dagger + E\nabla^2\mathbf{u}^\dagger + \mathbf{B} \times (\nabla \times \mathbf{B}^\dagger) \text{ Adjoint} && (8) \end{aligned}$$

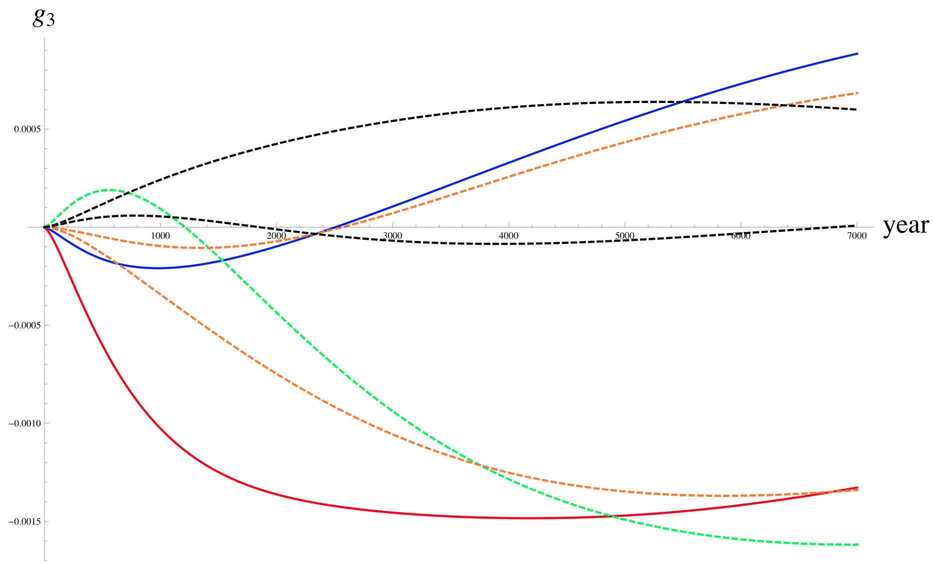
The flow $\mathbf{u}$ is a slave of $\mathbf{B}$

Proof of concept (core dynamics)

- Observations of B_r are taken every 25years for 7k years (~1/4 magnetic decay time)
- Note – no constraints at all on toroidal field
- In our simulation R_m=varies between 1-10 [advection is weak]
- Ekman number 10^{-2} (development phase!)

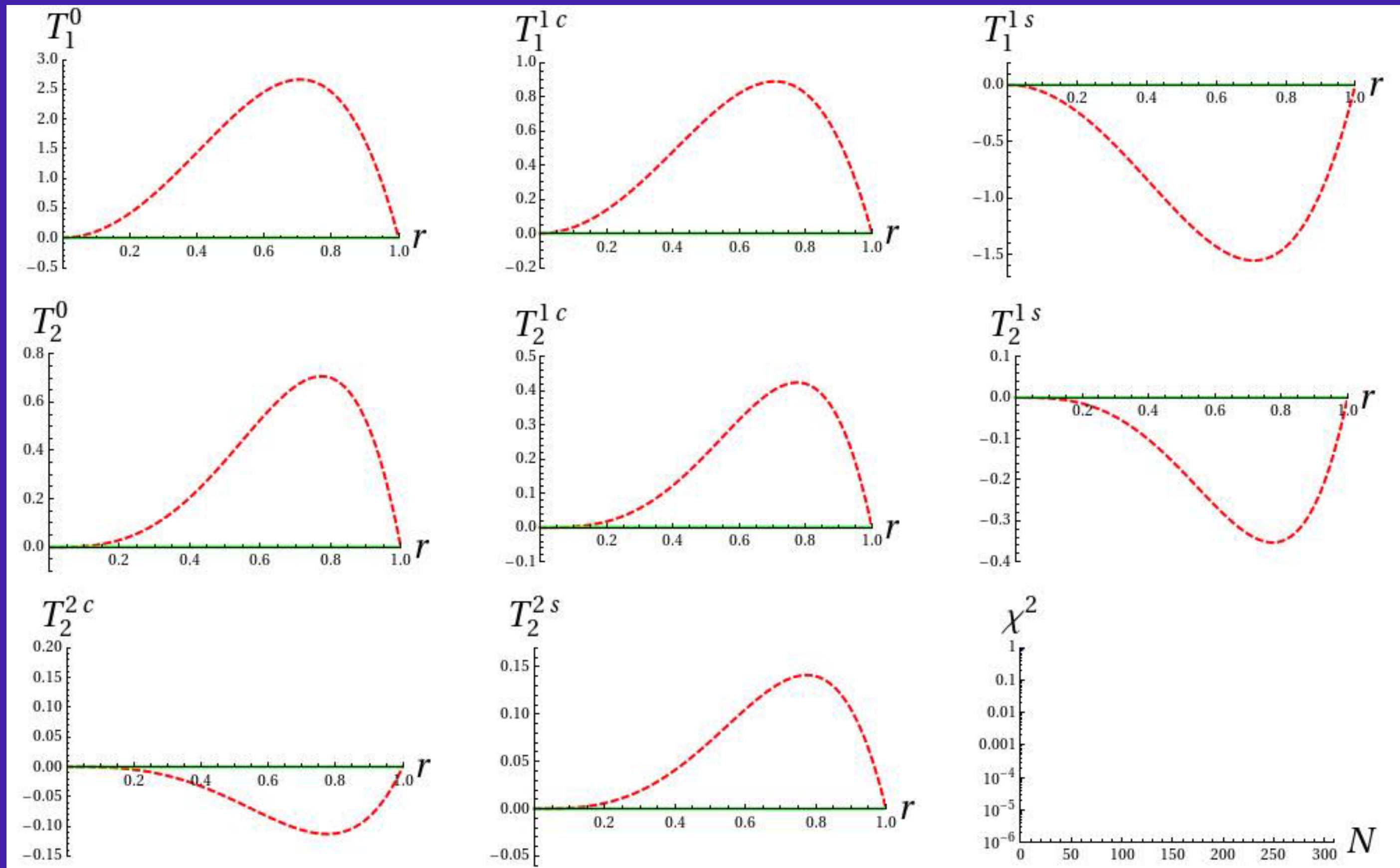


g_1
0.8
0.6
0.4
0.2
−0.2
−0.4
1000
2000
3000
4000
5000
6000
7000
year



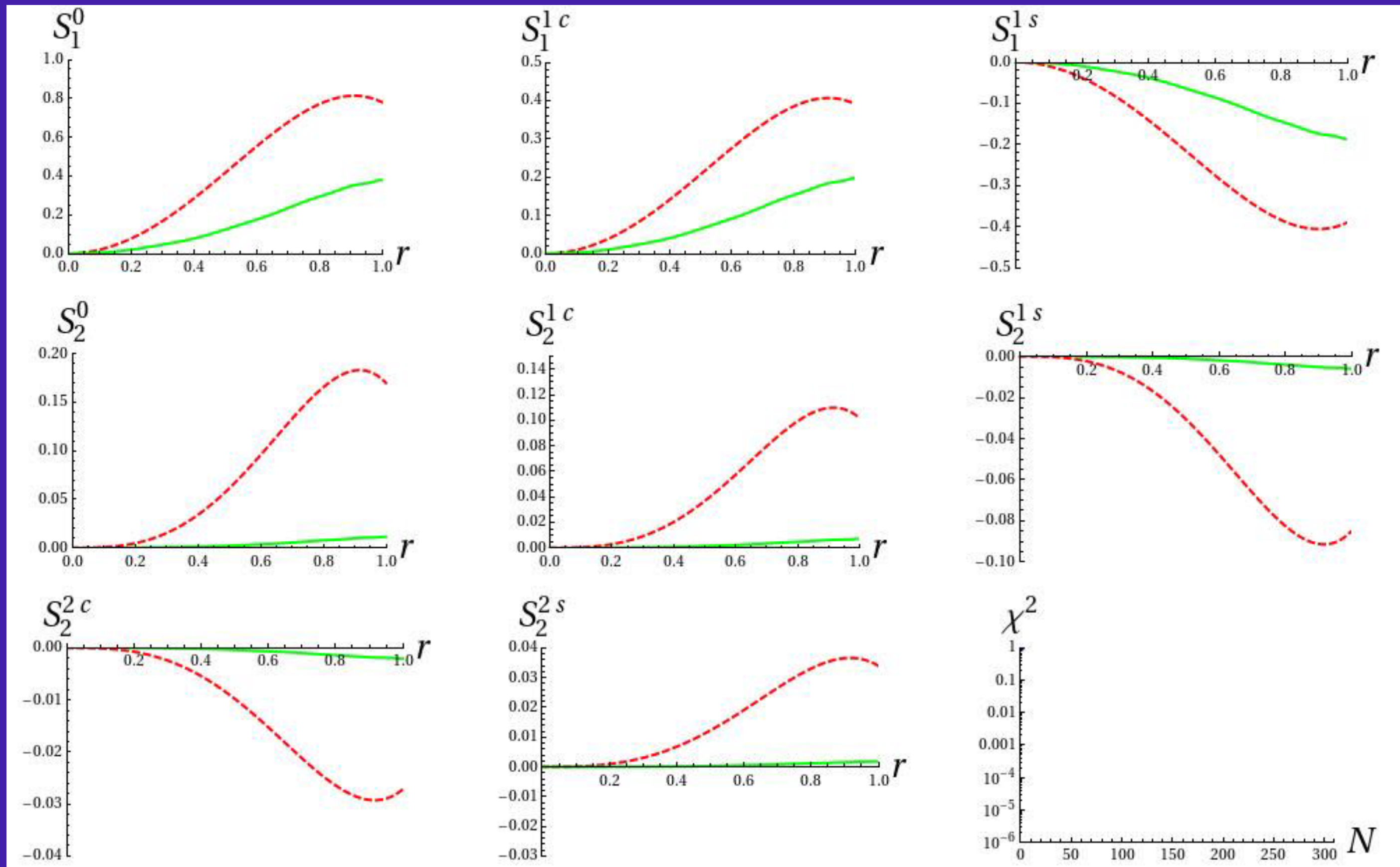
g_3
0.0005
−0.0005
−0.0010
−0.0015
year

Only B on boundary – toroidal reconstruction



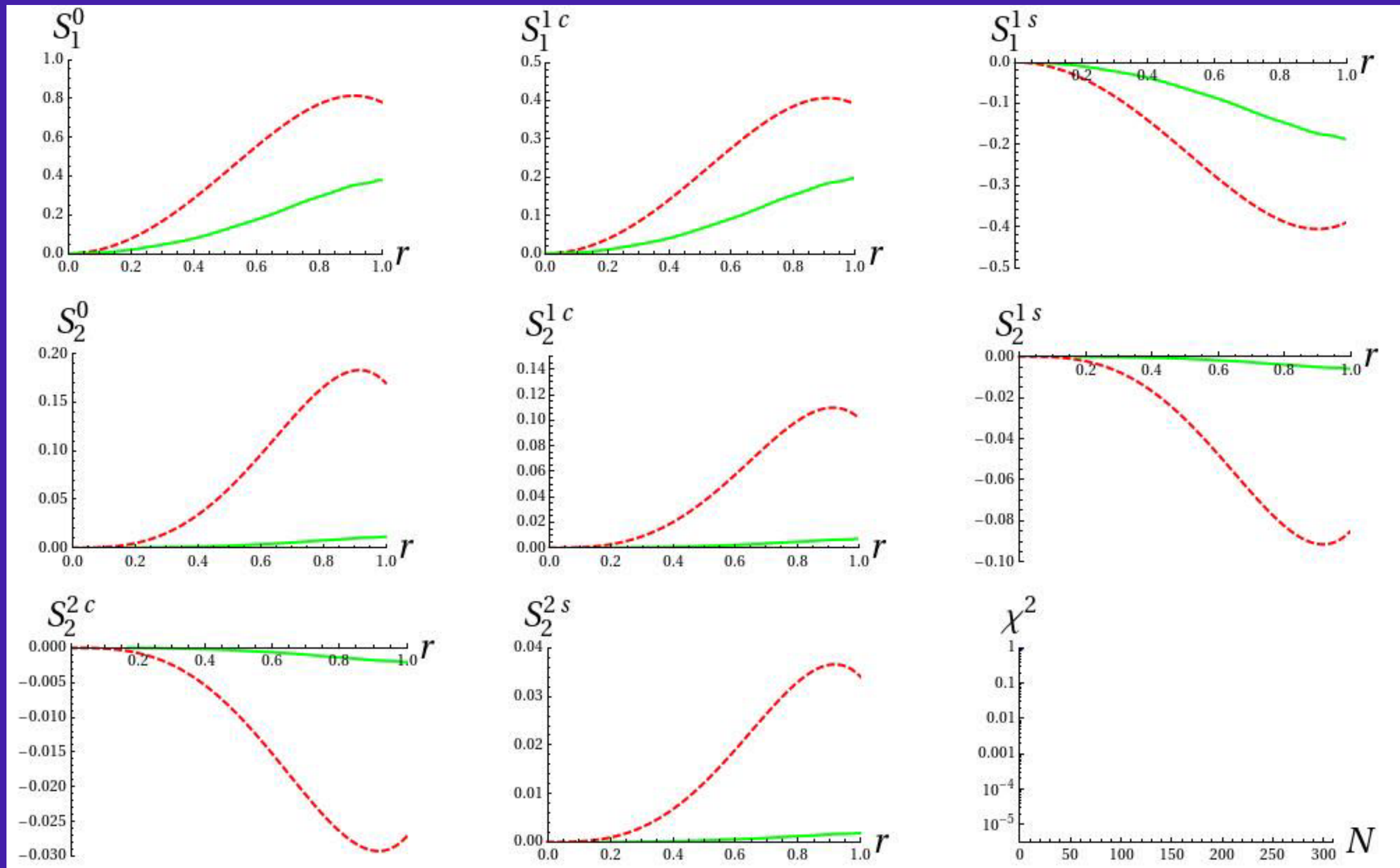
Naïve implementation

- Observations of B on the boundary as before – fails to reconstruct properly
- We now pretend to observe the flow as well as B on the boundary
- This removes the non-uniqueness

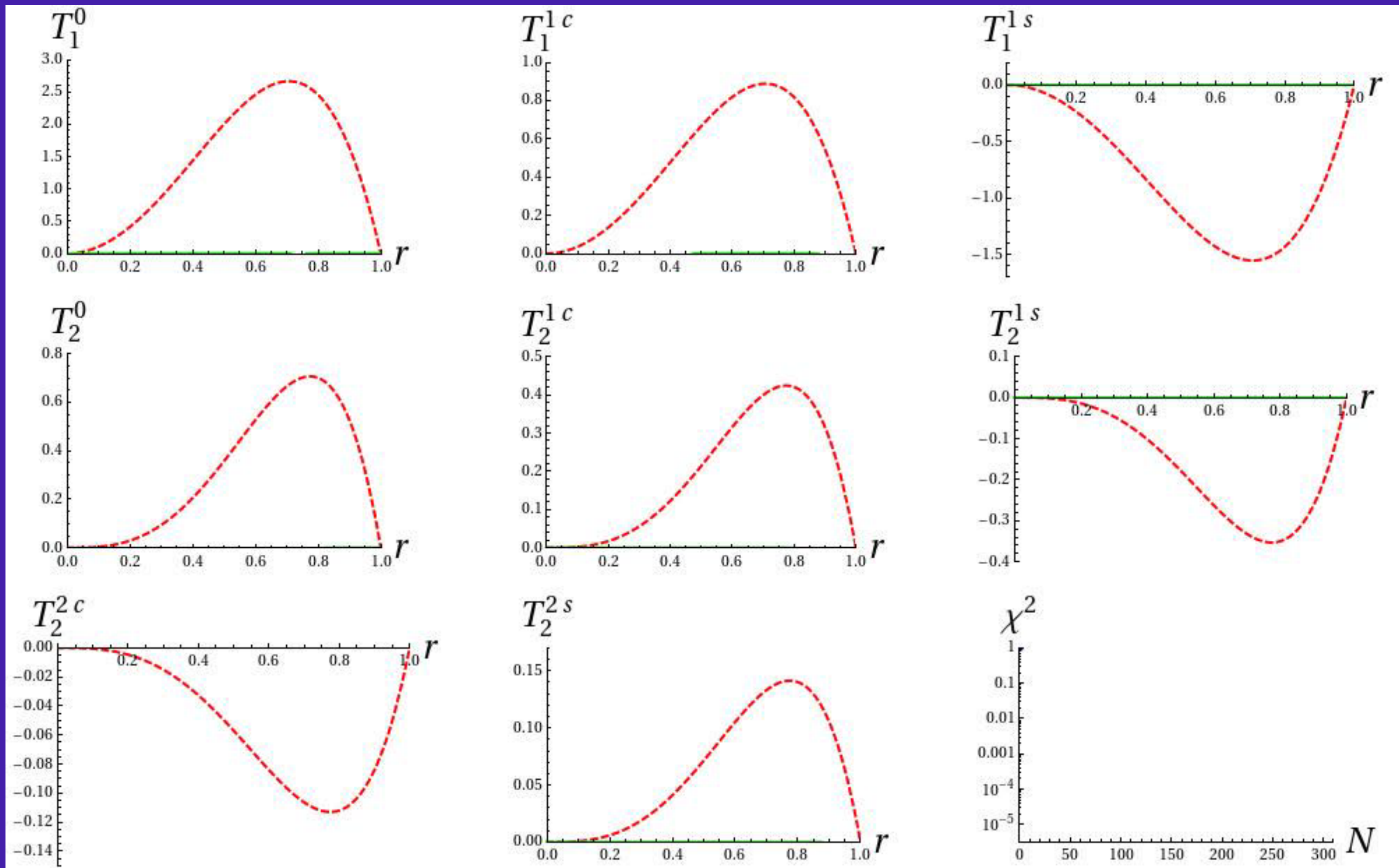


S_1^0
S_1^{1c}
S_1^{1s}
S_2^0
S_2^{1c}
S_2^{1s}
S_2^{2c}
S_2^{2s}
χ^2
r
N

B and u on boundary – poloidal reconstruction



B and u on boundary – toroidal reconstruction



Outlook

- Have working scheme for 4DVar
- Idea is to apply to last 400 years of Earth's B field observations
- Non-uniqueness needs to be handled:
  - Add constraints such as e.g. monotonicity
  - Make $|B|$ increase with depth?
- Time span of data:
  - $\Delta T \approx 1/R_m$

Summary

- Variational data assimilation presents itself as a useful technique for interrogating the core
- We have a differential form for the adjoint of all the dynamo equations, which can be efficiently used with a pseudospectral method
- We have demonstrated convergence on a very nonlinear toy problem
- Core's dynamical evolution
  - Ingredients are in place
  - 300 years of data seems to be almost sufficient

http://tinyurl.com/mhd-test

Numerical benchmarks for spherical convection, dynamos and forced flow

Click on the links to retrieve the definitions.

1) Spherical shell dynamo with pseudo vacuum magnetic boundary conditions

This dynamo is a solution documented first by Harder & Hansen (GJI, 2005).
Since the boundary conditions are local, this benchmark is useful for non-spectral codes (e.g. finite volume, finite element, spectral element etc).
Computations have been carried out by Andrey Sheyko.

2) Whole sphere dynamo

Two benchmarks are available in a whole sphere with internal heating:
a) Non-magnetic convection
b) A dynamo that is purely oscillatory (boundary conditions on B are the usual insulating type)
Computations have been carried out by Philippe Marti.

3) Whole sphere boundary driven flow *UPDATED 12/9/12*

This is a purely hydrodynamical test.
Computations have been carried out by Nathanael Schaeffer

Please supply requested values to A Jackson: email ajackson AT ethz.ch

Two separate papers will be written with the results.
Order of authorship will be determined by the order in which the results are received at ETH Zurich by the coordinators.
Please supply any pertinent grant acknowledgements at the time of submission of results.

Deadline for receipt of results is 10 October 2012. It is envisaged that the papers will be distributed to all authors by 20 October - revisions are required by 30 October. The papers are planned to be included in the SEDI special issue.

Coordinators:
Andy Jackson
Andrey Sheyko
Philippe Marti
Nathanael Schaeffer