Introduction to self-excited dynamo action

M.R.E.Proctor

DAMTP, University of Cambridge

Mathematical Aspects of Natural Dynamos, Caramulo, September 2003

- Faraday’s Law and the Induction Equation

  Maxwell’s equations. Faraday’s Law of induction. Ohm’s Law and reduction to induction equation. The magnetic Reynolds number. Reduction for two-dimensional and axisymmetric cases. Boundary conditions.

- Simple dynamos

  Mechanical generators and Faraday disc dynamo. Importance of chirality and geometric complexity. Difficulty of *homogeneous* dynamos.Dynamos as a high Rm phenomenon. Field line stretching: the Stretch-Twist-Fold mechanism. Singular character of perfectly conducting case. Fast and slow dynamos.

- Anti-dynamo theorems

  Definitions of dynamo action in finite or periodic conducting domains. Decay of fields in stationary media. Non-normality of induction equation. Bounds on flow velocity: Childress, Backus, Busse. Dissipation bound. Geometrical results: Cowling’s theorem, toroidal theorem, Zel’dovich’s theorem.

- Steady and time-dependent velocities

  Influence of time dependence in enhancing stretching. Two-dimensional unsteady flows acting as fast dynamos. ‘Pulsed’ flows: dynamos with almost-always two-dimensional flows.

- Two-scale dynamos

  Parker’s ‘cyclonic event’ model. Mean-field electrodynamics and the α-effect. Importance of broken mirror-symmetry. Approximation methods for α; difficulties at large values of small-scale Rm. α^2- and $\alpha\omega$-dynamos. Applications to solar and planetary dynamos.

0. Faraday's Law and the Induction Equation

- Basis of dynamo action is Faraday's Law for *e.m.f.* in a circuit due to flux change. Neglecting displacement current we have

$$\frac{\partial \mathbf{B}}{\partial t} = -\boldsymbol{\nabla} \wedge \mathbf{E}; \quad \boldsymbol{\nabla} \cdot \mathbf{B} = 0; \quad \boldsymbol{\nabla} \wedge \mathbf{B} = \mu_0 \mathbf{j}$$

  - To close system have Ohm's Law relating $\mathbf{E}'$ (comoving electric field) and $\mathbf{j}$:

$\mathbf{j} = \sigma \mathbf{E}' = \sigma(\mathbf{E} + \mathbf{u} \wedge \mathbf{B})$. Simplifying, obtain the Induction Equation

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \wedge (\mathbf{u} \wedge \mathbf{B})[\text{Advection}] \quad - \boldsymbol{\nabla} \wedge (\eta \boldsymbol{\nabla} \wedge \mathbf{B})[\text{Diffusion}]; \quad \eta = (\mu_0 \sigma)^{-1}$$

- Note that here η (magnetic diffusivity) assumed isotropic (and often uniform too). OK for fluids – unlike wires! In some cases Ohm's Law may be too simple (Hall effect etc.) Note also formal similarity to vorticity equation.

- Balance between advection and diffusion provided by Magnetic Reynolds number $\mathrm{Rm} = \mathcal{U}\mathcal{L}/\eta$ where $\mathcal{U}$, $\mathcal{L}$ are velocity and length scales (cf. Reynolds number)
- Nondimensionalizing $\mathbf{u}$ with $\mathcal{U}$, t with $\mathcal{L}/\mathcal{U}$ get dimensionless I.E.:

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \wedge (\mathbf{u} \wedge \mathbf{B}) - \mathrm{Rm}^{-1} \boldsymbol{\nabla} \wedge (\boldsymbol{\nabla} \wedge \mathbf{B})$$

- If $\mathbf{u}$, $\mathbf{B}$ are axisymmetric can write in polars (s, ϕ, z); $\mathbf{B} = B\mathbf{e}_\phi + \boldsymbol{\nabla} \wedge (A\mathbf{e}_\phi) = B\mathbf{e}_\phi + \mathbf{B}_p$, $\mathbf{u} = \mathbf{u}_p + U\mathbf{e}_\phi$; then get

$$\frac{\partial A}{\partial t} + s^{-1}\mathbf{u}_p \cdot \boldsymbol{\nabla}(sA) = \mathrm{Rm}^{-1}(\Delta - s^{-2})A$$

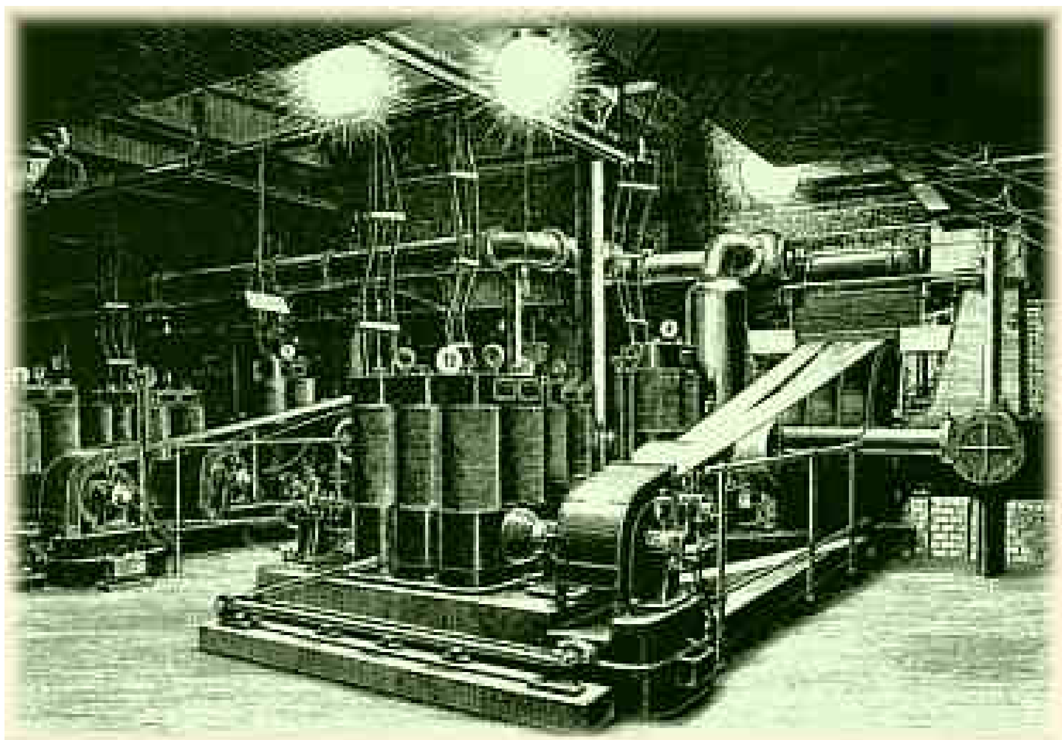
$$\frac{\partial B}{\partial t} + s\mathbf{u}_p \cdot \boldsymbol{\nabla}(B/s) = s\mathbf{B}_p \cdot \boldsymbol{\nabla}(U/s) + \mathrm{Rm}^{-1}(\Delta - s^{-2})B$$

- For two-dimensional case write $\mathbf{B} = B\mathbf{e}_z + \boldsymbol{\nabla} \wedge (A\mathbf{e}_z)$; get

$$(\partial_t + \mathbf{u}_p \cdot \boldsymbol{\nabla})A = \mathrm{Rm}^{-1}\Delta A; \quad (\partial_t + \mathbf{u}_p \cdot \boldsymbol{\nabla})B = \mathbf{B}_p \cdot \boldsymbol{\nabla}(\mathbf{u}_z) + \mathrm{Rm}^{-1}\Delta B$$

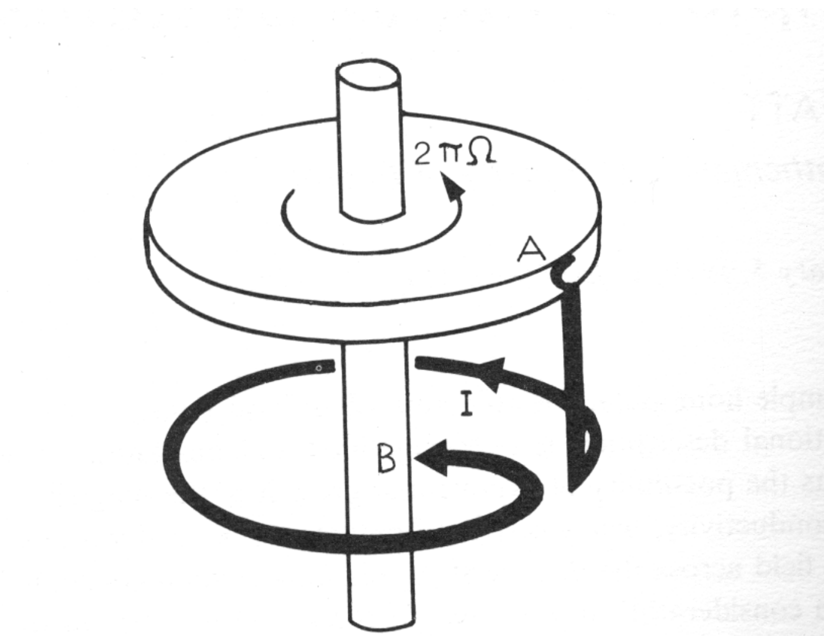
1. Simple Dynamos

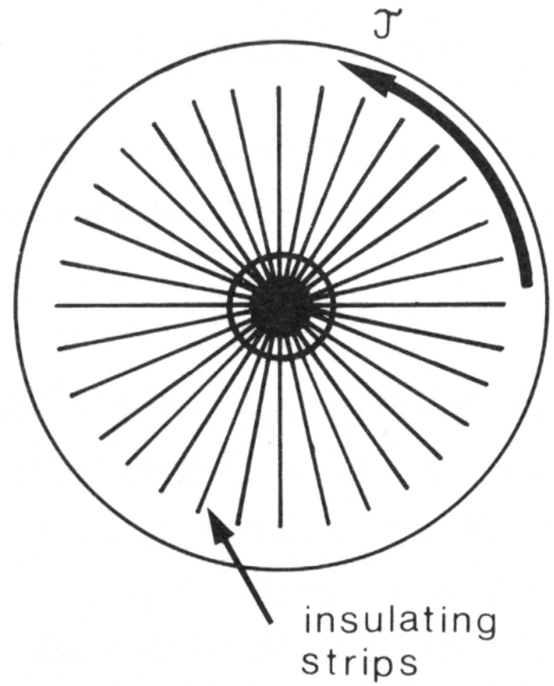
- What is a dynamo? Essentially a mechanism for turning mechanical energy into magnetic energy.



  - Will only consider Kinematic dynamo: neglect Lorentz force; $\mathbf{u}$ is prescribed.

- Mechanical example: Faraday (segmented) dynamo (Moffatt 1979).





  - Simple equations relate current in the wire I, current round disc J, angular velocity Ω and fluxes Φ_I, Φ_J. Get

$$\Phi_I = LI + MJ, \quad \Phi_J = MI + L'J, \quad RI = \Omega\Phi_J - \frac{d\Phi_I}{dt}; \quad R'J = -\frac{d\Phi_J}{dt}$$

- Seek solutions $\propto e^{pt}$. Growth if $\omega M > R$. Growthrate is

$$p_+ = \left(\sqrt{(RL' + R'L)^2 + 4R'(\Omega M - R)(LL' - M^2)} - (RL' + R'L)\right)/2(LL' - M^2).$$

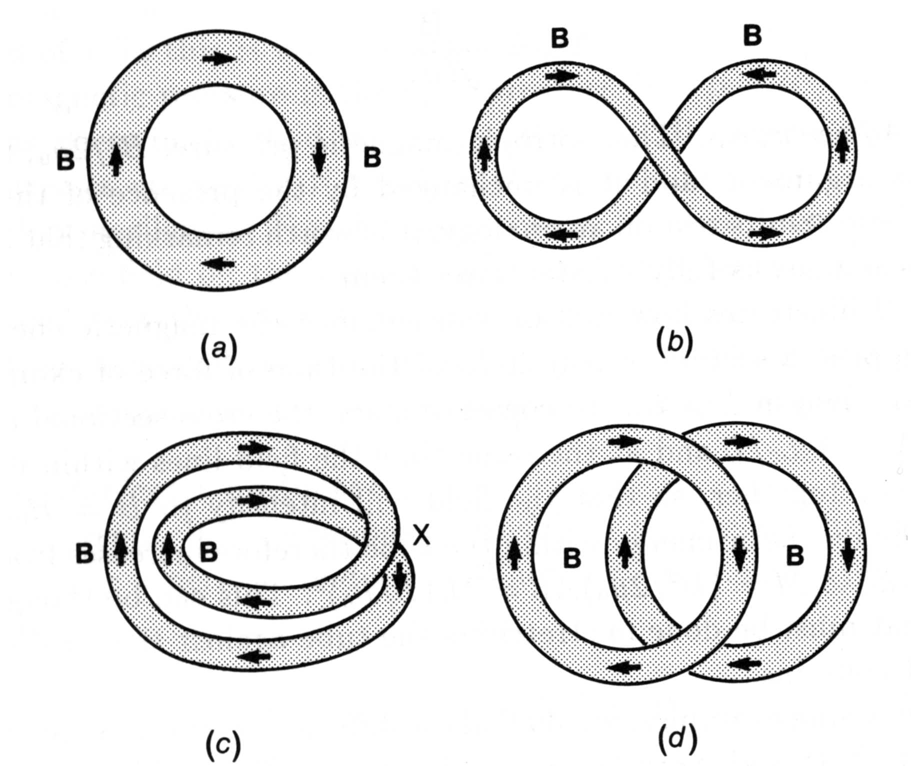
$p_+ > 0$ for all $\Omega > R/M$ but $p \sim \sqrt{\Omega R'}$ as $\Omega \to \infty$. Thus growthrate is controlled by diffusion and not exclusively by advection. This is an example of a slow dynamo (see later).

- The Faraday dynamo is **untypical** of fluid dynamos!

  - Current travels in wires: anisotropic, inhomogeneous conductivity!

  - System lacks mirror-symmetry: cf. e.g. Earth, with symmetry under reflection and exchange of poles.

- How can a homogeneous fluid act as a dynamo? Best understood in case of large Rm, when field lines almost frozen into fluid (Alfvén's Theorem, cf. Kelvin's Theorem for vorticity). Magnetic energy then enhanced by stretching (advection) repeated stretching and folding can lead to exponential growth of energy, e.g. Vainshtein-Zel'dovich dynamo (Stretch-Twist-Fold):



- STF and other mechanisms suggest possibility of growth of magnetic energy at a rate independent of diffusion - a fast dynamo. Role of small diffusion is complex; depends on amount of stretching (good) to folding (bad). In a bounded domain, some folding must occur; in $\mathbb{R}^2$ there is always too much folding, so all fields ultimately decay.

- Consider simple 2D example: flow field $\mathbf{u} = (-x, 0, z)$, $\mathbf{B} = (0, 0, B(x,t))$. B obeys $\dot{B} - xB_x = B + \mathrm{Rm}^{-1} B_{xx}$. If $B(x,0) = \mathrm{Re}(\beta_0 e^{\mathrm{i}k_0 x})$ then

$$B(x,t) = \mathrm{Re}(\beta_0 e^{t - k_0^2(e^{2t}-1)/2\mathrm{Rm}} e^{\mathrm{i}k_0 e^t x})$$

so $|B|$ eventually decays superexponentially due to exponentially increasing gradients. But transient growth of energy can occur for long times $\sim \ln(\mathrm{Rm}/k_0^2)$. As $\mathrm{Rm} \to \infty$ energy can increase indefinitely.

- Fast and slow dynamos. In astrophysical applications need to understand behaviour of rate of growth of field at large Rm. Can energy/flux/dipole moment grow at a rate indept. of η?

  - Slow dynamo. Growthrates (on advective timescale) $\to 0$ as $\mathrm{Rm} \to \infty$.

  - Fast dynamo. Growthrates (or at least $\lim_{sup}$ if many modes) do not tend to zero at large Rm. In this case field appears on all scales as $\mathrm{Rm} \to \infty$. Diffusion can never be neglected. This is necessary to get round flux conservation as diffusion becomes negligible.

2. “Anti-Dynamo Theorems”

- Statement of the dynamo problem. (i) Suppose $\mathbf{B}$ is defined in a finite volume $\mathcal{D}$, surrounded (in ${}^c\mathcal{D}$) by an insulator (this could be generalised so that ${}^c\mathcal{D}$ is a stationary conductor). In ${}^c\mathcal{D}$ we have $\boldsymbol{\nabla}\wedge\mathbf{B}=0$, with all components of $\mathbf{B}$ continuous at $\partial\mathcal{D}$; $|\mathbf{B}|\sim\mathcal{O}(|\mathbf{x}|^{-3})$ as $|\mathbf{x}|\to\infty$. (ii) $\mathbf{B}$ is defined in a periodic domain $\mathcal{D}\in\mathbb{R}^3$, with $\int_{\mathcal{D}}\mathbf{B}\,d\mathbf{x}=0$. In each case $\mathbf{u}$ satisfies $\boldsymbol{\nabla}\cdot\mathbf{u}=0$, has time-bounded norm (e.g. $\mathsf{U}\equiv\max_{\mathcal{D}}(|\mathbf{u}|)$, $\mathsf{S}\equiv\max_{\mathcal{D},i}(|\partial_j u_i|)$, $\mathsf{E}^{\frac{1}{2}}\equiv\left(\int_{\mathcal{D}}|\boldsymbol{\nabla}\mathbf{u}|^2 d\mathbf{x}\right)^{\frac{1}{2}}$, ..., etc.) In case (i) suppose that $\mathbf{u}=0$ on $\partial\mathcal{D}$. Then define magnetic energy $\mathcal{M}=\frac{1}{2}\int|\mathbf{B}|^2\,d\mathbf{x}$; integral over $\mathbb{R}^3$ in case (i), over $\mathcal{D}$ in case (ii). We say that we have dynamo action if $\mathcal{M}$ does not tend to zero as $t\to\infty$. Then in case (i)

$$\frac{d\mathcal{M}}{dt}=\mathcal{P}-\eta\mathcal{J}$$

$$=\int_{\mathcal{D}}\mathbf{B}\cdot(\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u})d\mathbf{x}-\eta\int_{\mathcal{D}}|\boldsymbol{\nabla}\wedge\mathbf{B}|^2 d\mathbf{x}=\int_{\mathcal{D}}(\mathbf{u}\wedge\mathbf{B})\cdot(\boldsymbol{\nabla}\wedge\mathbf{B})d\mathbf{x}-\eta\int_{\mathbb{R}^3}|\boldsymbol{\nabla}\mathbf{B}|^2 d\mathbf{x}$$

(for case (ii), all integrals are over $\mathcal{D}$).

- Necessary conditions for dynamo action. Need the induction term to be larger than dissipation term. Use Poincaré Inequality $\mathcal{F} \equiv \mathcal{J}/(2\mathcal{M}) \geq c^{-2}$; $c \propto (\int_{\mathcal{D}} d\mathbf{x})^{\frac{1}{3}}$. (For sphere, radius a, $c = a/\pi$; periodic cube, side a, $c = a/2\pi$). Have bounds on $\mathcal{P}$:

$$\text{(a)} \quad \mathcal{P} \leq \mathsf{U} \int_{\mathcal{D}} |\mathbf{B} \cdot \boldsymbol{\nabla} \mathbf{B}| d\mathbf{x} \leq \mathsf{U} (2\mathcal{M})^{\frac{1}{2}} \mathcal{J}^{\frac{1}{2}} \quad \text{[Childress]}$$

$$\text{(b)} \quad \mathcal{P} \leq \mathsf{S} \cdot 2\mathcal{M} \quad \text{[Backus]}$$

$$\text{(c)} \quad \mathcal{P} \leq \mathsf{E}^{\frac{1}{2}} \left(\int_{\mathcal{D}} |\mathbf{B}|^4 d\mathbf{x} \right)^{\frac{1}{2}} \quad \text{[Proctor]}$$

Thus (using for (c) the inequality $\int_{\mathcal{D}} |\mathbf{B}|^4 d\mathbf{x} \leq c_1^2 \cdot (2\mathcal{M})^{\frac{1}{2}} \cdot \mathcal{J}^{\frac{3}{2}}$ with $c_1 \in \mathbb{R}$), we get three bounds on the growthrate:

$$(a)\quad (2\mathcal{M})^{-1}\frac{\partial \mathcal{M}}{\partial t} \le \mathsf{U}\mathcal{F}^{\frac{1}{2}} - \eta\mathcal{F} \le \mathcal{F}^{\frac{1}{2}}(\mathsf{U} - \eta c^{-1})$$

$$(b)\quad (2\mathcal{M})^{-1}\frac{\partial \mathcal{M}}{\partial t} \le \mathsf{S} - \eta\mathcal{F} \le \mathsf{S} - \eta c^{-2}$$

$$(c)\quad (2\mathcal{M})^{-1}\frac{\partial \mathcal{M}}{\partial t} \le c_1\mathsf{E}^{\frac{1}{2}}\mathcal{F}^{\frac{3}{4}} - \eta\mathcal{F} \le \mathcal{F}^{\frac{3}{4}}(c_1\mathsf{E}^{\frac{1}{2}} - \eta c^{-\frac{1}{2}})$$

So if $\mathcal{M} \not\to 0$ must have $\mathsf{U} > \eta/c$, $\mathsf{S} > \eta/c^2$, $\mathsf{E} > \eta^2/cc_1^2$. (For (a) only need $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\mathcal{D}$). Can also get bounds on growthrate $s \equiv 2\mathcal{M}^{-1}d\mathcal{M}/dt$.

(a) $s \le \max[(\mathsf{U}/c - \eta/c^2), \mathsf{U}^2/4\eta]$; (c) $s \le \max[(c_1\mathsf{E}^{\frac{1}{2}}c^{-\frac{3}{2}} - \eta c^{-2}), 27c_1^4\mathsf{E}^2/256\eta^3]$

∘ N.B. These are only necessary conditions! Dynamo action not guaranteed even if inequalities violated. Even optimal condition $\eta < \max \int_{\mathcal{D}} \mathbf{B}\cdot(\mathbf{B}\cdot\nabla\mathbf{u})d\mathbf{x}/\mathcal{J}$ gives Euler-Lagrange equation different from I.E. This is because I.E. is non-normal; eigenvectors not orthogonal. When inequalities violated, can get transient growth of $\mathcal{M}$ even though there are no unstable eigenfunctions.

- A non-necessary condition! There is no lower bound on the energy $\int_{\mathcal{D}} |\mathbf{u}|^2 d\mathbf{x}$ for a working dynamo. Consider velocity $\mathbf{u}$ in sphere radius R surrounded by stationary conductor. For steady dynamo I.E. invariant under $\mathbf{x} \to \mathbf{x}/R$, $\mathbf{u} \to R\mathbf{u}$, energy $\sim |\mathbf{u}|^2 \times R^3 \sim R$. Argument can be extended to external insulator.
- Geometrical constraints. These are of two kinds: (i) constraints on the flow; (ii) constraints on the field.

  - (i) Flow constraints. Consider case(a) above. Define $P = \mathbf{B} \cdot \mathbf{r}$, $Q = \mathbf{u} \cdot \mathbf{r}$.

Then

$$\frac{\partial P}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} P = \mathbf{B} \cdot \boldsymbol{\nabla} Q + \eta \Delta P \ \text{ in } \mathcal{D}$$

with $\Delta P = 0$ in ${}^c\mathcal{D}$, and $P, \partial P/\partial r$ continuous on $\partial\mathcal{D}$. If $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ we can write $\mathbf{u} = \mathbf{u}_{tor} + \mathbf{u}_{pol} \equiv \boldsymbol{\nabla} \wedge (T\mathbf{r}) + \boldsymbol{\nabla} \wedge \boldsymbol{\nabla} \wedge (S\mathbf{r})$. It follows that $Q = -\mathcal{L}^2 S$, where $-\mathcal{L}^2$ is the angular momentum operator.

Thus if $\mathbf{u}$ is toroidal ($\mathbf{u}_{pol} = 0$) then $Q = 0$ and vice versa. In this case we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{D}} P^2 d\mathbf{x} = -\eta \int_{\mathbb{R}^3} |\boldsymbol{\nabla} P|^2 d\mathbf{x} < -\eta c^{-2} \int_{\mathcal{D}} P^2 d\mathbf{x} \quad \Rightarrow |P| \to 0.$$

Neglecting P, now have both $\mathbf{u}$, $\mathbf{B}$ toroidal, and $\boldsymbol{\nabla}\wedge(\mathbf{u}\wedge\mathbf{B}_{tor}) = \boldsymbol{\nabla}\wedge(-\mathbf{r}(\mathbf{u}\cdot\boldsymbol{\nabla}T))$. After integrating get $\partial T/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}T = \eta\Delta T$, with $T = 0$ on $\partial\mathcal{D}$; then can show $\int_{\mathcal{D}} T^2 d\mathbf{x} \to 0$ also. This is the toroidal theorem, that a toroidal velocity field cannot act as a dynamo. A similar result holds in cartesian coords when $\mathbf{u}\cdot\mathbf{z} = 0$: then $\partial B_z/\partial t + \mathbf{u}\cdot\boldsymbol{\nabla}B_z = \mathbf{B}\cdot\boldsymbol{\nabla}u_z + \eta\Delta B_z$, etc. [Zel'dovich].

- When flow has poloidal parts dynamo action cannot be excluded generally. Can bound poloidal field/total field ratio [Busse]. Poloidal energy equation

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathcal{D}} P^2 d\mathbf{x} = -\int_{\mathcal{D}} Q\mathbf{B}\cdot\boldsymbol{\nabla}P d\mathbf{x} - \eta\int_{\mathbb{R}^3} |\boldsymbol{\nabla}P|^2 d\mathbf{x}$$

$$\leq \max_{\mathcal{D}} Q \left(2\mathcal{M}\cdot 2\int_{\mathbb{R}^3} |\mathbf{B}_{pol}|^2\, d\mathbf{x}\right)^{\frac{1}{2}} - 2\eta\int_{\mathbb{R}^3} |\mathbf{B}_{pol}|^2\, d\mathbf{x}$$

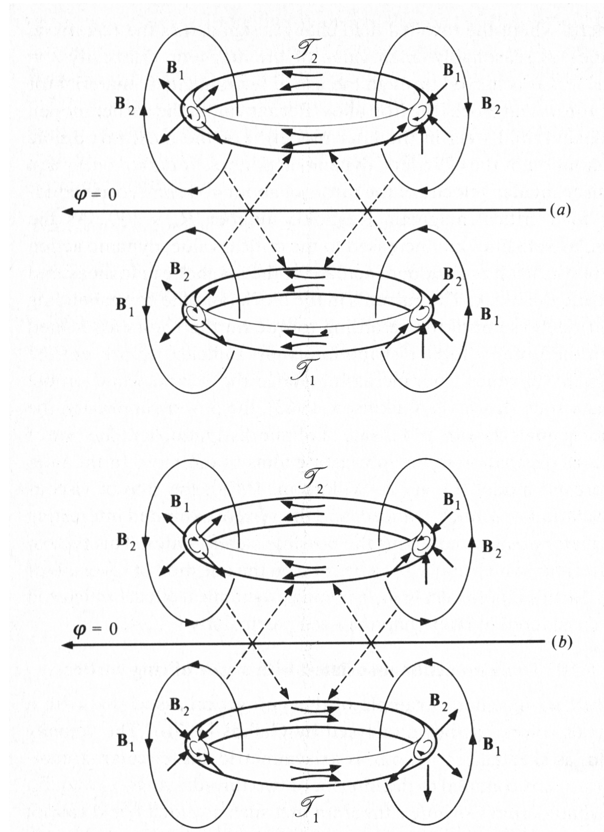
Thus we have

$$\max_{\mathcal{D}} Q \geq \eta \left(\frac{\int_{\mathbb{R}^3} |\mathbf{B}_{pol}|^2 \, d\mathbf{x}}{\mathcal{M}} \right)^{\frac{1}{2}}$$

- Can use continuity arguments for eigenspectrum to show that in fact for any given $\mathbf{u}_{tor}$, no dynamo action unless $\max_{\mathcal{D}} Q/\eta$ sufficiently large.
- Can also show [Proctor] that

$$\int_{\mathcal{D}} |\boldsymbol{\nabla} \wedge \mathbf{u}_{pol}|^2 d\mathbf{x} \geq \frac{\eta^2}{c_2} \left(\frac{\int_{\mathbb{R}^3} |\mathbf{B}_{pol}|^2 \, d\mathbf{x}}{\int_{\mathcal{D}} |\boldsymbol{\nabla} \wedge \mathbf{B}|^2 d\mathbf{x}} \right), \quad c_2 \propto \int_{\mathcal{D}} d\mathbf{x}$$

- Dynamos can be found when $\mathbf{u}$ is purely poloidal (e.g. Gailitis ring dynamo).



- Constraints on the field. The main theorem is Cowling's Theorem: An axisymmetric magnetic field cannot be a dynamo. Note that if **B** is axisymmetric then so is **u** but NOT vice versa. There are several proofs of this in various cases.

- Braginskii's proof: $\boldsymbol{\nabla}\cdot\mathbf{u}=0$. Write $\mathbf{B}=\boldsymbol{\nabla}\wedge(\chi\mathbf{e}_\phi/s)+s\psi\mathbf{e}_\phi$, $U=s\Omega$, then

$$\frac{\partial\chi}{\partial t}+\mathbf{u}_p\cdot\boldsymbol{\nabla}\chi=\eta(\Delta-\frac{2}{s}\frac{\partial}{\partial s})\chi$$
$$\frac{\partial\psi}{\partial t}+\mathbf{u}_p\cdot\boldsymbol{\nabla}\psi=\mathbf{B}_p\cdot\boldsymbol{\nabla}\Omega+\eta(\Delta+\frac{2}{s}\frac{\partial}{\partial s})\psi$$

with $(\Delta-(2/s)\partial/\partial s)\chi=\psi=0\in{}^c\mathcal{D}$ and $\chi\sim\mathcal{O}(|\mathbf{x}|^{-1})$ as $|\mathbf{x}|\to\infty$. Forming poloidal energy equation

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{D}}\chi^2 d\mathbf{x}=\eta\int_{\mathcal{D}}\chi\left(\Delta-\frac{2}{s}\frac{\partial}{\partial s}\right)\chi d\mathbf{x}=-\eta\int_{\mathbb{R}^3}|\boldsymbol{\nabla}\chi|^2 d\mathbf{x}\le-\eta c_3^2\int_{\mathcal{D}}\chi^2 d\mathbf{x},$$

so $|\mathbf{B}_p|\to 0$ (exponentially). When χ is negligible, can show similarly that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{D}}\psi^2 d\mathbf{x}=\eta\int_{\mathcal{D}}\psi\left(\Delta+\frac{2}{s}\frac{\partial}{\partial s}\right)\psi d\mathbf{x}=-\eta\int_{\mathcal{D}}|\boldsymbol{\nabla}\psi|^2 d\mathbf{x}$$

Similar results hold for flows independent of z.

- More general conditions. For variable compressibility, non-uniform η, etc no really useful results are known. Equation for χ still holds: since $\chi(0,z)=0$ and $\chi\to 0$ as $|\mathbf{x}|\to\infty$, must exist a positive maximum of χ, at $\mathbf{X}(t)$ where $\boldsymbol{\nabla}\chi=0$, $\Delta\chi\le 0$. So growing dynamo impossible. Can then be argued [Hide & Palmer] that if $\Delta\chi(\mathbf{X})=0$ for all time then χ becomes non-differentiable near $\mathbf{X}$ and so $\mathbf{B}_p\to 0$ (hard to rigorize).

  ◦ Steady dynamo not possible since for small meridional circle S_ε, boundary C_ε, radius ε around $\mathbf{X}$, with $B_\varepsilon \equiv \oint_{C_\varepsilon} |\mathbf{B}_p|\cdot d\mathbf{x}/2\pi\varepsilon$,

$$(\max_{\mathcal{D}}|\mathbf{u}|)B_\varepsilon S_\varepsilon \ge \int_{S_\varepsilon}(\mathbf{u}_p\wedge\mathbf{B}_p)\cdot d\mathbf{x} = \int_{S_\varepsilon}\eta(\mathbf{x})\boldsymbol{\nabla}\wedge\mathbf{B}_p\cdot d\mathbf{x}\sim 2\pi\varepsilon B_\varepsilon\eta(\mathbf{X})$$

  which is impossible as $\varepsilon\to 0$.

  ◦ Further use of maximum principles [Ivers & James] shows that both poloidal and toroidal fields tend to zero exponentially, but rate not well bounded.

3. Steady and Time-dependent Velocities

- Smooth, steady **u** not usually efficient as dynamos at large Rm ; not enough stretching. Smooth axisymmetric or 2D flows cannot be fast dynamos if steady (no exponential stretching of material lines). Time-dependent flows can stretch much better, even if Eulerian form very simple. As example consider [G.O.] Roberts flow, Galloway-Proctor flow.

$$\text{Roberts flow:}\quad \mathbf{u}(x,y) \propto \boldsymbol{\nabla} \wedge (\psi(x,y)\mathbf{e}_z) + \gamma\psi(x,y);$$

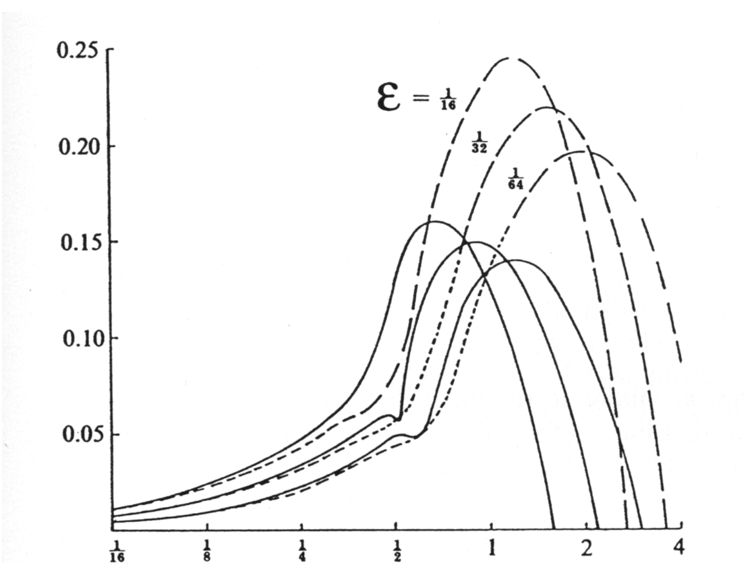
$$\psi = \sin x \sin y.$$

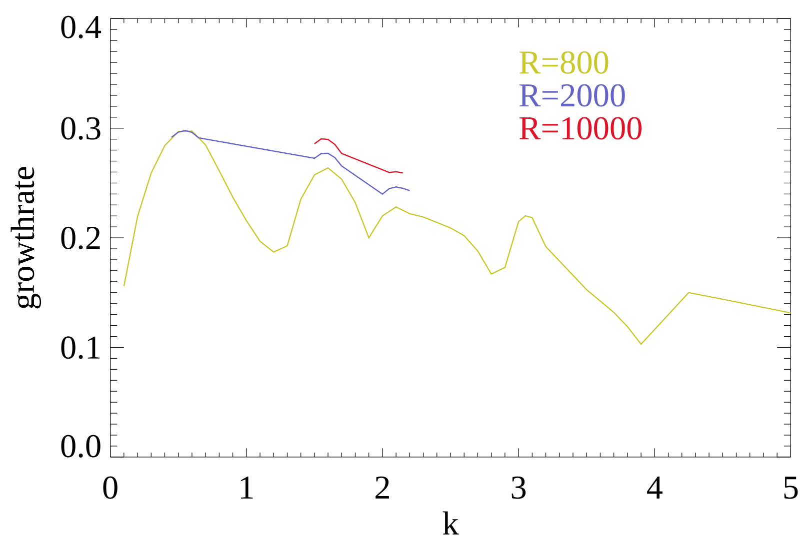
$$\text{GP-flow:}\quad \mathbf{u}(x,y,t) \propto \boldsymbol{\nabla} \wedge (\psi(x,y,t)\mathbf{e}_z) + \gamma\psi(x,y,t);$$

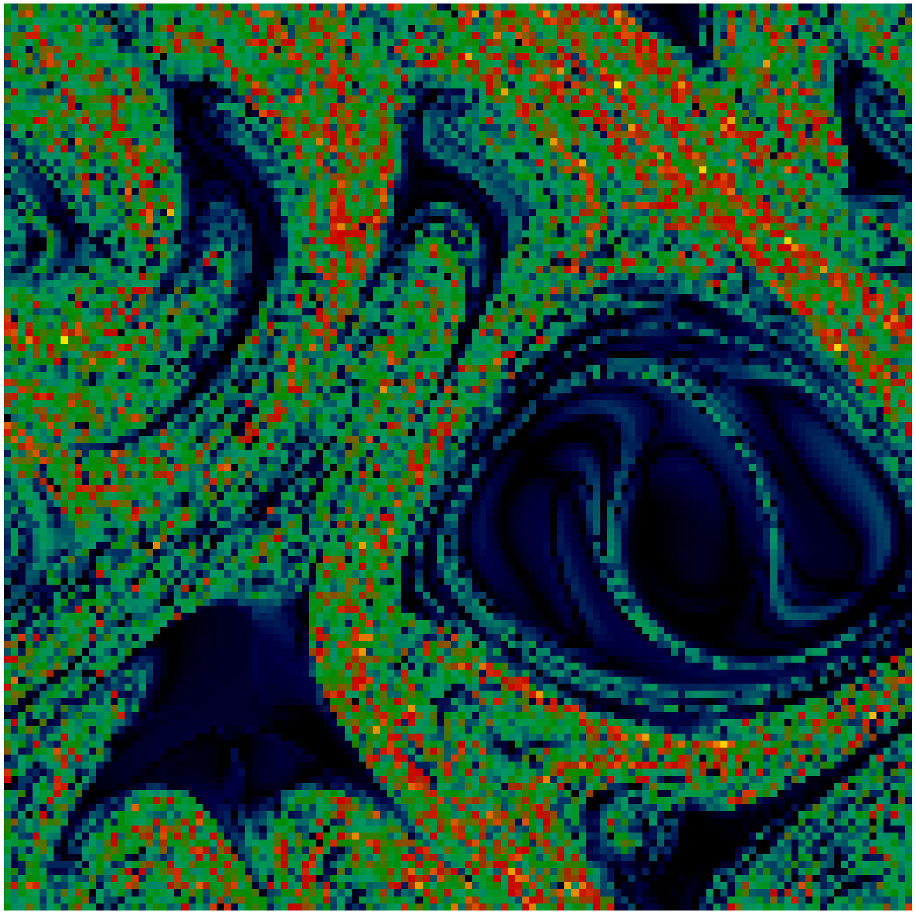
$$\psi = \sin(y + \varepsilon \sin \omega t) + \cos(x + \varepsilon \cos \omega t)$$

◦ Roberts flow has fixed cellular pattern; no stretching except at cell corners. GP-flow has stretching almost everywhere.

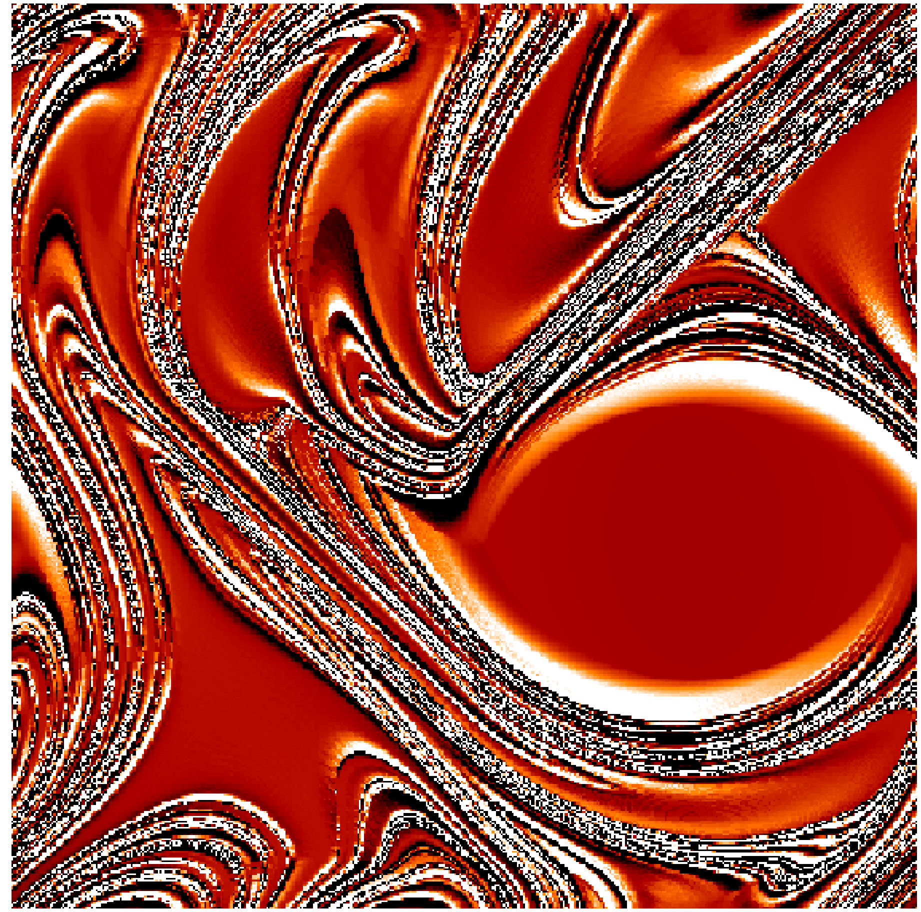
∘ Both flows have fields of form $\mathbf{B} = Re(\tilde{\mathbf{B}}(x,y,t)e^{ikx})$ For Roberts flow optimum growthrate occurs at large k for $\mathrm{Rm} \gg 1$ ($k \sim (\mathrm{Rm}^{\frac{1}{2}}/\ln \mathrm{Rm})$ (N.B. this these scales are long cf. the thin boundary layer scale $\mathrm{Rm}^{-\frac{1}{2}}$ for field near stagnation points. As $\mathrm{Rm} \to \infty$ optimum growthrate $\sim \mathcal{O}(\ln(\ln \mathrm{Rm})/\ln \mathrm{Rm})$. Just a slow dynamo!! For GP-flow growthrate is $\mathcal{O}(1)$ for large Rm, optimum wavenumber also $\mathcal{O}(1)$. Here flow is chaotic, and though there are thin flux structures chaotic regions near the stagnation points do not scale with Rm.







Finite time Liapunov exponents



Field structures for non-diffusive evolution

- Remark: Toroidal and 2D theorems can be got round for time-dependent flows, because of non-normality. Consider the pulsed Beltrami flow

$$\begin{aligned}\mathbf{u} &= (0, \sin x, \cos x) \quad (0 \le t \le \tau) \\ &= (\sin y, 0, \cos y) \quad (\tau \le t \le 2\tau)\end{aligned}$$

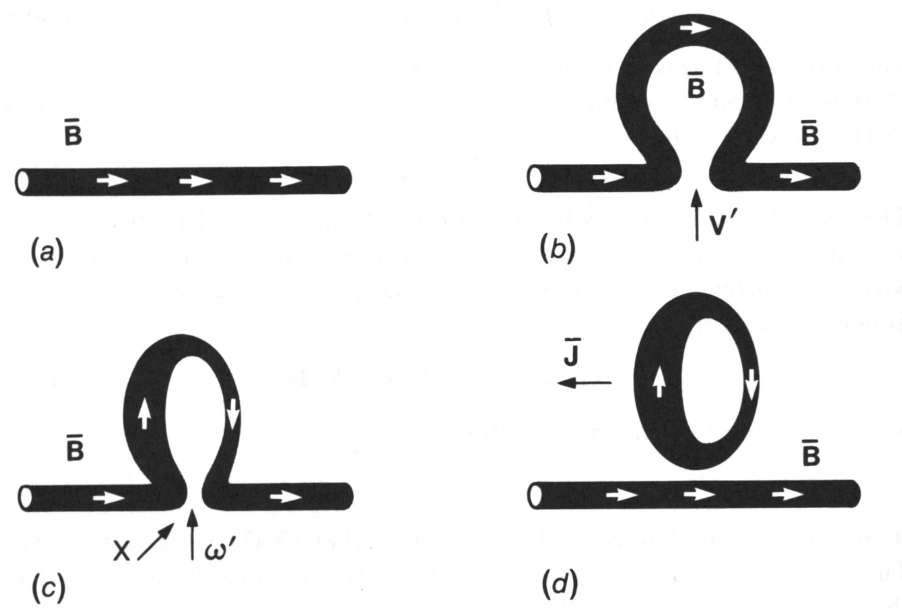
This is planar flow for almost all time! But for each interval get transient growth! Have for $(0 \le t \le \tau)$ with $\eta = 0$, and $\overline{\mathbf{B}} = Re(\tilde{\mathbf{B}}_H exp(ikz))$

$$\overline{\mathbf{B}}_H(\tau) = J_0(k\tau)\overline{\mathbf{B}}_H(0) - i\tau J_1(k\tau)\overline{B_x}(0)\mathbf{e}_y$$

which can be large for large τ, even for small diffusion. Then the second pulse can refold the field and give further growth.

4. Two-scale Dynamos

Roberts, GP and pulsed dynamos (and extensions to 3D flows such as the ABC model) are small scale dynamos. Magnetic field has some scales comparable to that of $\mathbf{u}$. But if $\mathbf{B}$ exists on two distinct scales then dynamo action can be easily verified. Simplest model [Parker]; suppose small scale 'cyclonic events' act on uniform field. If velocity has helicity $(\mathbf{u}\cdot\boldsymbol{\nabla}\wedge\mathbf{u} \neq 0)$ then emf generated (anti-)parallel to field. Sign of α opposite to helicity?? (only true for short-lived events) Thus get extra term $\boldsymbol{\nabla} \wedge \alpha\mathbf{B}$, (the α-effect) in the I.E.:



Parker supposed (for solar field) that toroidal field $\gg$ poloidal field due to strong zonal shears. Modelling large -scale field as axisymmetric, get model system

$$\frac{\partial A}{\partial t} + s^{-1}\mathbf{u}_p \cdot \boldsymbol{\nabla}(sA) = \alpha B + \mathrm{Rm}^{-1}(\Delta - s^{-2})A$$

$$\frac{\partial B}{\partial t} + s\mathbf{u}_p \cdot \boldsymbol{\nabla}(B/s) = [\boldsymbol{\nabla} \wedge (\alpha \boldsymbol{\nabla} \wedge \mathbf{B}_p)] + s\mathbf{B}_p \cdot \boldsymbol{\nabla}(U/s) + \mathrm{Rm}^{-1}(\Delta - s^{-2})B$$

- More general approach. Define some average (denoted by $\overline{\cdots}$) and write $\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}'$, $\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$, etc. Then, taking the average,

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla} \wedge \mathcal{E} + \boldsymbol{\nabla} \wedge (\overline{\mathbf{u}} \wedge \overline{\mathbf{B}}) - \boldsymbol{\nabla} \wedge (\eta \boldsymbol{\nabla} \wedge \overline{\mathbf{B}})$$

where $\mathcal{E} \equiv \overline{\mathbf{u}' \wedge \mathbf{B}'}$. Equation for $\mathbf{B}'$ is

$$\frac{\partial \mathbf{B}'}{\partial t} = \boldsymbol{\nabla} \wedge (\overline{\mathbf{u}} \wedge \mathbf{B}') + \boldsymbol{\nabla} \wedge (\mathbf{u}' \wedge \overline{\mathbf{B}}) + \boldsymbol{\nabla} \wedge (\mathbf{u}' \wedge \mathbf{B}' - \overline{\mathbf{u}' \wedge \mathbf{B}'}) - \boldsymbol{\nabla} \wedge (\eta \boldsymbol{\nabla} \wedge \mathbf{B}')$$

- What is $\mathcal{E}$? Clearly $\mathcal{E}$ (for fixed $\mathbf{u}$) is a linear functional of $\overline{\mathbf{B}}$. Assuming a local relation, get

$$\mathcal{E}_i = \alpha_{ij}\overline{B}_j - \beta_{ijk}\frac{\partial \overline{B}_k}{\partial x_j} + \dots$$

  - α_{ij} is a pseudo-tensor; symmetric part is non-zero only if statistics of $\mathbf{u}$ lack mirror-symmetry. Anti-symmetric part acts like a velocity – only non-zero if no isotropy or homogeneity. If suppose isotropic, then $\alpha_{ij} = \alpha\delta_{ij}$. α can be related to helicity. Similarly $\beta_{ijk} = \beta eps_{ijk}$; "turbulent magnetic diffusivity".

- If $\overline{\mathbf{u}} = 0$ can see that α leads to dynamo action. Consider (with $\eta + \beta \to \eta$)

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \boldsymbol{\nabla} \wedge (\alpha\overline{\mathbf{B}}) - \boldsymbol{\nabla} \wedge (\eta\boldsymbol{\nabla} \wedge \overline{\mathbf{B}})$$

If α, β uniform, get solutions of form $Re(\hat{\mathbf{B}}\exp(i\mathbf{k}\cdot\mathbf{x} + pt))$, with $(p + \eta k^2)^2 = \alpha^2 k^2$, so $p_+ > 0$ for all suff.small k.

- α tensor will take more general forms with lower symmetry of flow statistics. In a sphere, when there are two preferred directions, namely the rotation $\mathbf{\Omega}$ and the radial vector $\mathbf{r}$ get more general form

$$\mathcal{E} = \alpha_1(\mathbf{\Omega}\cdot\mathbf{r})\overline{\mathbf{B}} + \alpha_2\mathbf{r}(\mathbf{\Omega}\cdot\overline{\mathbf{B}}) + \alpha_3\mathbf{\Omega}(\mathbf{r}\cdot\overline{\mathbf{B}}) + \ldots$$

Note that both rotation and preferred direction would seem necessary for an α-effect.

- Calculation of α. This is difficult!! Two approaches:

  - small Rm at scale of $\mathbf{u}'$. To calculate *alpha* approximate $\mathbf{B}'$ equation by $0 = \overline{\mathbf{B}}\cdot\nabla\mathbf{u}' + \eta\Delta\mathbf{B}'$. with $\overline{\mathbf{B}}$ uniform. For fourier mode with wavenumber $\mathbf{k}$ have $B'_i = i\overline{B}_j k_j u'_i/\eta k^2$ so

$$\mathcal{E}_i = \alpha_{ij}\overline{B}_j = i\varepsilon_{ipq}k_j\overline{u'^*_p u'_q}\ \overline{B}_j/\eta k^2$$

Giving $\alpha \propto i\varepsilon_{ijk}k_j\overline{u'^*_i u'_k}$ (helicity) in isotropic case.

  - 'Short-sudden' approximation. Assume short correlation times for $\mathbf{u}'$, ignore diffusion. Then $\partial_t\mathbf{B}' \approx \overline{\mathbf{B}}\cdot\nabla\mathbf{u}'$. If correlation time is τ_c then $B'_i \approx \tau_c\overline{\mathbf{B}}\cdot\nabla\mathbf{u}'$, so in isotropic case

$$\alpha = -\frac{\tau_c}{3}\overline{\mathbf{u}'\cdot\nabla\wedge\mathbf{u}'}$$

.

- General results are difficult in intermediate case.
- An exact result. If we suppose fields and flow statistically steady with uniform imposed field $\overline{\mathbf{B}}$ (and periodic b.c.'s for simplicity), write $\mathbf{B}' = \boldsymbol{\nabla} \wedge \mathbf{A}'$; then have

$$\frac{\partial \mathbf{A}'}{\partial t} = -\boldsymbol{\nabla}\Phi + \overline{\mathbf{u}} \wedge \mathbf{B}' + \mathbf{u}' \wedge \overline{\mathbf{B}} + \mathbf{u}' \wedge \mathbf{B}' - \overline{\mathbf{u}' \wedge \mathbf{B}'} - \eta \boldsymbol{\nabla} \wedge \mathbf{B}'$$

so (ignoring boundary terms from integration by parts)

$$0 = \frac{1}{2}\left(\overline{\mathbf{B}' \cdot \partial_t \mathbf{A}' + \mathbf{A}' \cdot \partial_t \mathbf{B}'}\right) = -\overline{\mathbf{B}} \cdot \mathcal{E} - \eta \overline{\mathbf{B}' \cdot \boldsymbol{\nabla} \wedge \mathbf{B}'}$$

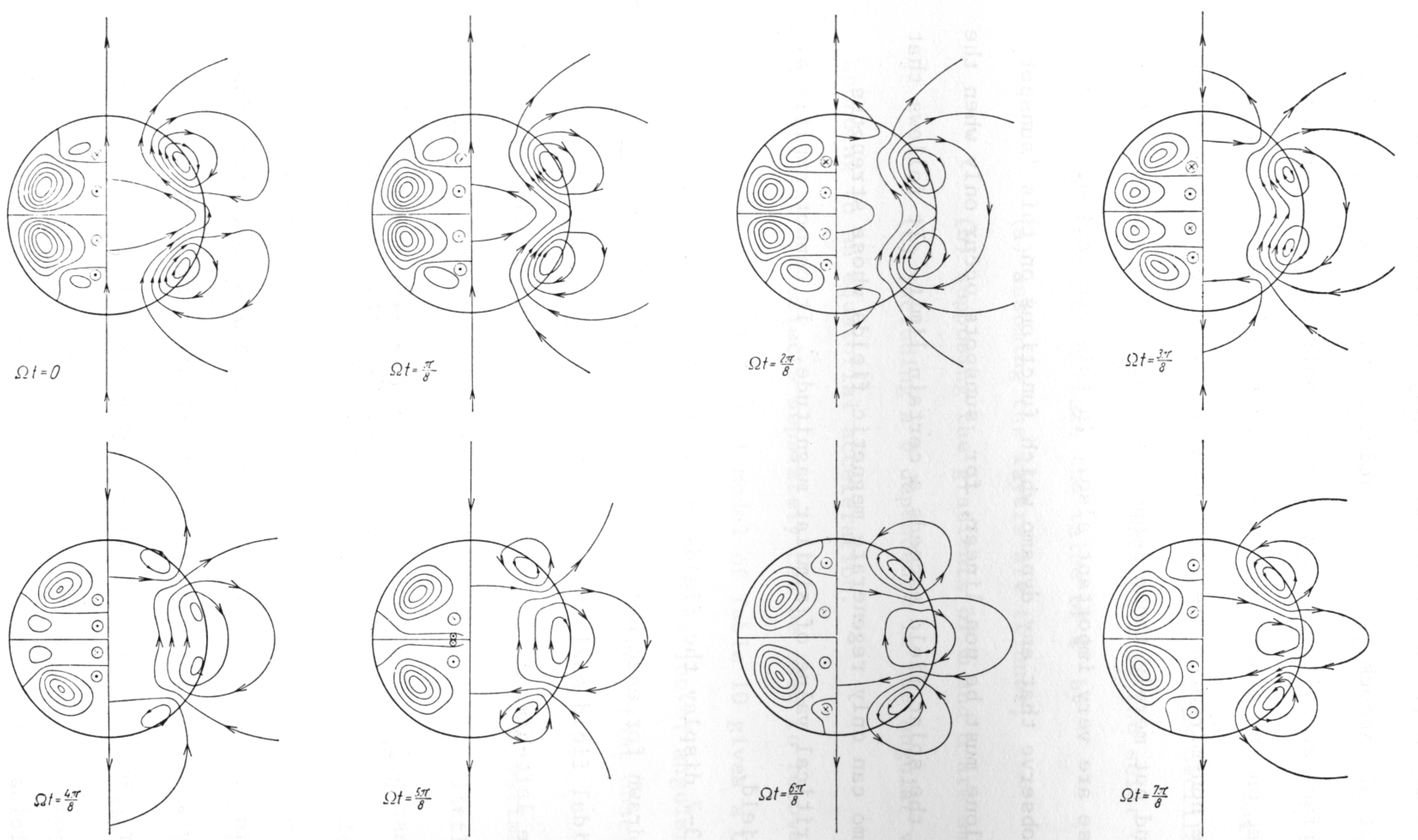
True exactly if boundary terms are ignored. Thus in isotropic case

$$\alpha |\overline{\mathbf{B}}|^2 = -\eta\, \overline{\mathbf{B}' \cdot \boldsymbol{\nabla} \wedge \mathbf{B}'}$$

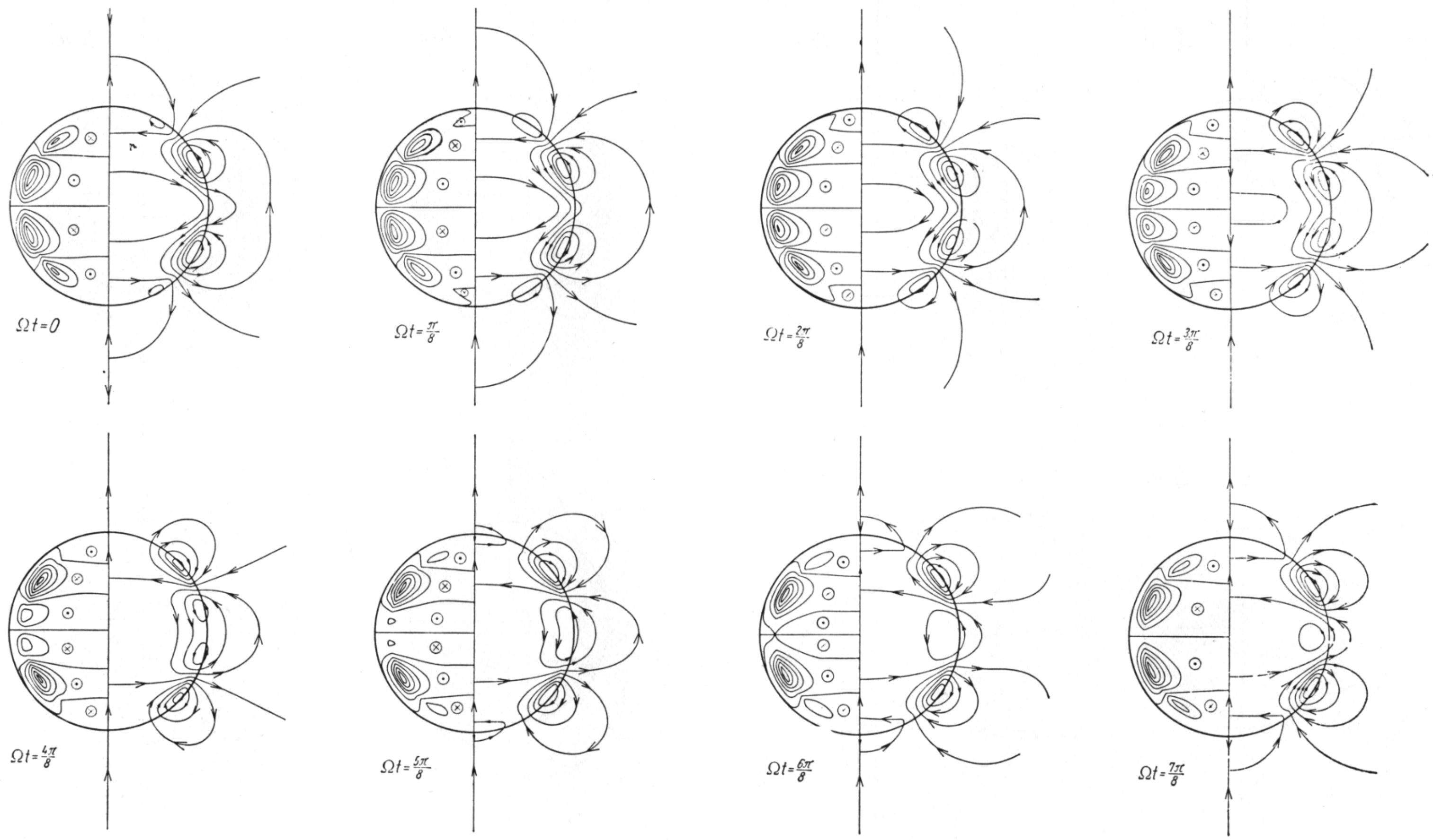
This shows that diffusion must be included in any proper model of α. If α is indept. of η at large Rm, leading to a fast mean field dynamo, and $\mathbf{B}' \sim \eta^a \overline{\mathbf{B}}$, and have filling factor $\sim \eta^b$, then $1 + 2a + b = 0$. $b = 0$, $a = 1/2$ is possible and plausible, but intermittent fields ($b > 0$) demand greater amplification.

- Mean field models. With α-effect term Cowling's theorem does not apply (toroidal field can sustain poloidal). So can investigate axisymmetric models. Physical considerations suggest α odd about equator, U even, so get two types of field structure.

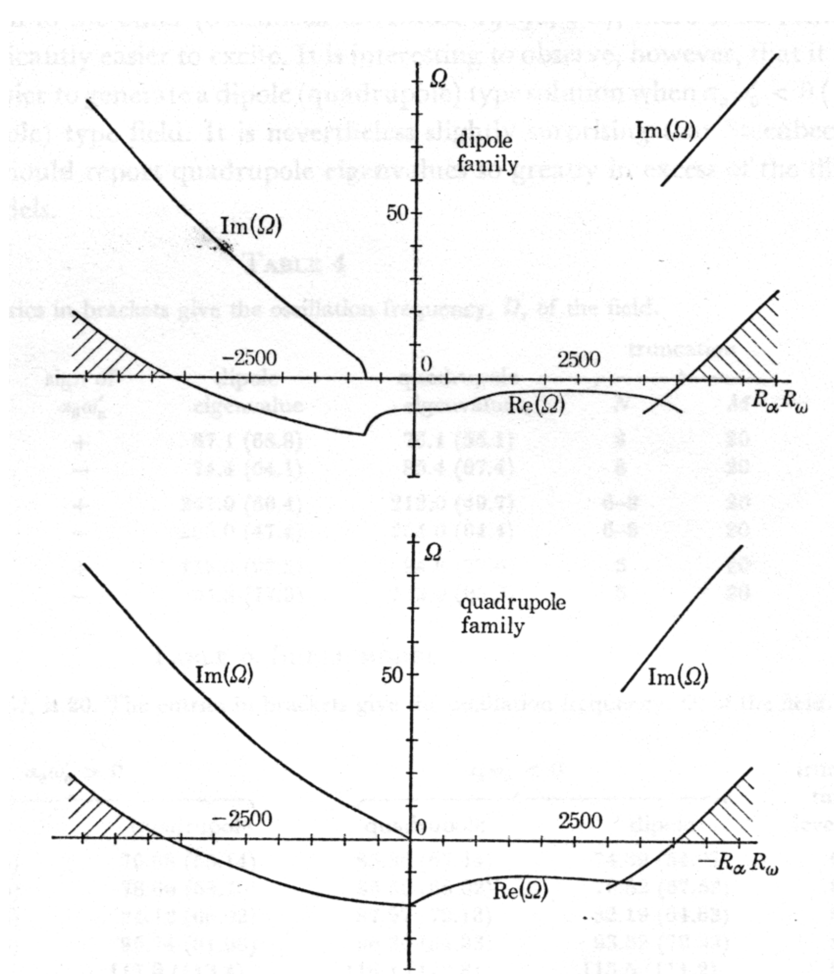
(i) Dipole: B odd, A even



(ii) Quadrupole; A odd, B even.



There is a near symmetry, associated with the adjoint dynamo problem, between dipole(quadrupole) modes with $\alpha, \mathbf{u}$, and quadrupole(dipole) modes with $\alpha, -\mathbf{u}$.



- Most models are of two types: (i) α^2, with U neglected; (ii) $\alpha\omega$, with α term in B equation neglected, as in Parker model. α^2 models typically give steady dynamos (real growthrates) while $\alpha\omega$ models usually give cyclic dynamos (complex growthrates). Can understand latter in terms of dynamo waves. Use cartesian geometry; let $A = A(x,t), B = B(x,t)$, $\mathbf{B}_p \cdot \boldsymbol{\nabla} U \sim \Omega A_x$. then get simplified system.

$$\frac{\partial A}{\partial t} = \alpha B + \eta\left(\frac{\partial^2 A}{\partial x^2} - K^2 A\right); \quad \frac{\partial B}{\partial t} = \omega\frac{\partial A}{\partial x} + \eta\left(\frac{\partial^2 B}{\partial x^2} - K^2 B\right)$$

This has travelling wave solutions with $A, B \propto \exp(ik(x - ct))$ when $\alpha\omega = \pm 2\eta^2(k^2+K^2)^2/k$, $c = -\alpha\omega/(2\eta(k^2+K^2))$. Note definite sign of c. $\alpha\omega$ models used to give models of the solar cycle (butterfly diagram) by identifying large B with regions of sunspot eruption.