ROLE OF COMPRESSIBILITY IN THE EARTH'S CORE CONVECTION

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$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\nabla)\mathbf{V} = -\frac{\nabla p}{\overline{\rho}} + \frac{\mathbf{F}^a}{\overline{\rho}} - 2\Omega\times\mathbf{V} + \frac{\mathbf{F}^b}{\overline{\rho}} + \nu\nabla^2\mathbf{V} \tag{1.1}$$

$$\nabla\cdot\mathbf{V} = 0 \tag{1.2}$$

$$\mathbf{F}^a = \mathbf{1}_r \overline{g}\overline{\rho}\alpha\vartheta \qquad \mathbf{F}^b = \frac{\nabla\times\mathbf{B}\times\mathbf{B}}{\mu_o}$$

$$\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{V}\nabla)\mathbf{B} = (\mathbf{B}\nabla)\mathbf{V} + \eta\nabla^2\mathbf{B} \tag{1.3}$$

$$\frac{\partial \vartheta}{\partial t} + (\mathbf{V}\nabla)\vartheta = \kappa_T\nabla^2\vartheta + \frac{h}{C_p} \tag{1.4}$$

Energy conservation

$$\frac{\partial \varepsilon^k}{\partial t} = -\nabla\cdot[\mathbf{V}[p+\varepsilon^k] + (\mathbf{V}\sigma')] + \mathbf{F^a}\cdot\mathbf{V} + \mathbf{F}^b\cdot\mathbf{V} - q_v \tag{1.5}$$

$$\frac{\partial \varepsilon^m}{\partial t} = -\nabla\cdot\frac{\mathbf{B}\times\mathbf{E}}{\mu_o} - \mathbf{V}\cdot\mathbf{F}^b - q_j \tag{1.6}$$

$$\frac{\partial \varepsilon^\vartheta}{\partial t} = \nabla\cdot\Big[-\mathbf{V}\varepsilon^\vartheta + C_p\overline{\rho}\kappa_T\nabla\vartheta\Big] + \overline{\rho}h \tag{1.7}$$

$$\varepsilon^\vartheta = C_p\overline{\rho}\vartheta \qquad \varepsilon^k = \frac{\overline{\rho}\mathbf{V}^2}{2} \qquad \varepsilon^m = \frac{\mathbf{B}^2}{2\mu_o}$$

$$q_v = \sigma'_{ik}\frac{\partial V_i}{\partial x_k} \qquad q_j = \eta\frac{[\nabla\times\mathbf{B}]^2}{\mu_o}$$

$$\frac{\partial(\varepsilon^k+\varepsilon^m)}{\partial t} = -\nabla\cdot\Big[\mathbf{V}[p+\varepsilon^k]+(\mathbf{V}\sigma')+\frac{\mathbf{B}\times\mathbf{E}}{\mu_\mathbf{o}}\Big]+\mathbf{F^a}\cdot\mathbf{V}-q_v-q_j$$

$$\frac{\partial\varepsilon^\vartheta}{\partial t} = \nabla\cdot\Big[-\mathbf{V}\varepsilon^\vartheta + C_p\overline{\rho}\kappa_T\nabla\vartheta\Big]+\overline{\rho}h$$

Integrating both of the equations over space and then averaging them over long time period we obtain:

$$W^a = O_v + Q_j, \quad \text{and} \quad I^{sa}(r_2) - I^{sa}(r_1) = H, \quad (8,9)$$

where $r_1 = r_{ICB}$, $r_2 = r_{CMB}$ and

$$W^a = \iiint \langle \mathbf{V}\cdot\mathbf{F}^a\rangle d^3\mathbf{r}, \quad Q_{v,j} = \iiint \langle q_{v,j}\rangle d^3\mathbf{r} \quad \text{and}$$

$$I^{sa}(r) = \iint r^2 C_p\overline{\rho}\kappa_T\langle\frac{\partial\vartheta}{\partial r}\rangle sin(\theta)d\theta d\phi, \quad H=\iiint\langle h\overline{\rho}\rangle d^3\mathbf{r}$$

Problems:

1. Adiabatic temperature, $\overline{T}(r)$, creates heat flux

$$I^a(r) = \iint r^2 C_p\overline{\rho}\kappa\frac{\partial\overline{T}}{\partial r} sin(\theta)d\theta d\phi = 4\pi r^2 C_p\overline{\rho}\kappa\frac{\partial\overline{T}}{\partial r},$$

which increases from $\sim 0.25\times10^{12}$ W on ICB to $\sim 3.2\times10^{12}$W on CMB. What is the source compensating this energetic loss?

2. Archimedean work, W^a, cools the superadiabatic heat flux. Where is the term describing this cooling in the heat transport equation?

3. Why is the heating due to the Ohmic and the viscous dissipations not included in the heat transport equation?

These effects are not small: $\sim 70\%$, 20% and 20% of the heat flux at CMB respectively.

$$\frac{D\mathbf{V}}{Dt} = -\frac{\nabla p}{\overline{\rho}} + \frac{p}{\overline{\rho}}\frac{\overline{\rho}'}{\overline{\rho}} + \mathbf{F}^a - 2\Omega\times\mathbf{V} + \mathbf{F}^b + \nu\nabla^2\mathbf{V} \quad (2.1)$$

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B}\nabla)\mathbf{V} + \eta\nabla^2\mathbf{B}, \qquad \nabla\cdot\mathbf{V}\overline{\rho} = 0 \quad (2.3, 2)$$

$$\frac{D\vartheta}{Dt} = \kappa_T\nabla^2\vartheta + \frac{h}{C_p} \quad +$$

$$+ \quad \kappa\nabla^2\overline{T} - D\frac{\vartheta V_r}{d} + \eta\frac{[\nabla\times\mathbf{B}]^2}{\mu_o C_p\overline{\rho}} + \nu\frac{[\nabla\times\mathbf{V}]^2}{C_p} \quad (2.4)$$

where the adiabatic temperature satisfies the equation

$$\frac{d\overline{T}^a}{dr} = -D\frac{\overline{T}^a}{d}, \quad \text{and} \quad D = \frac{d\alpha\overline{g}(r)}{C_p} \quad (2.5)$$

is usually called the compressibility parameter.

These equations have a simple physical meaning. The term $(p/\overline{\rho})(\overline{\rho}'/\overline{\rho})$ in (2.1) describes an additional buoyancy force due to compressibility. Superadiabatic temperature in (2.4) increases due to the additional heating, h, and the heating due to both the Ohmic and the viscous dissipations. It decreases due to the Adiabatic cooling, $\kappa\nabla^2\overline{T} < 0$, maintaining the adiabatic temperature distribution. The Archimedean cooling $D\vartheta V_r/d$ could be positive or negative, depending on location but, averaged over the whole core and time, it is negative (and equal to both dissipations with an opposite sign).

ENERGY BALANCE

The kinetic and the magnetic energy balance have the same form as in the Boussinesq approximation:

$$\frac{\partial(\varepsilon^k+\varepsilon^m)}{\partial t} = -\nabla\cdot\Big[\mathbf{V}[p+\varepsilon^k]+(\mathbf{V}\sigma')+\frac{\mathbf{B}\times\mathbf{E}}{\mu_{\mathbf{o}}}\Big] \quad +$$

$$+ \quad \mathbf{F^a}\cdot\mathbf{V}-q_v-q_j \tag{3.1}$$

but the heat energy equation includes 4 new terms:

$$\frac{\partial\varepsilon^\vartheta}{\partial t} = \nabla\cdot\Big[-\mathbf{V}\varepsilon^\vartheta + C_p\overline{\rho}\kappa_T\nabla\vartheta\Big]+\overline{\rho}h \quad +$$

$$+ \quad \nabla\cdot\Big[\mathbf{1}_r C_p\overline{\rho}\kappa\frac{\partial\overline{T}}{\partial r}\Big]-\mathbf{F^a}\cdot\mathbf{V}+q_v+q_j \tag{3.2}$$

The sum of these two equations gives the whole energy balance:

$$\frac{\partial(\varepsilon^\vartheta+\varepsilon^k+\varepsilon^m)}{\partial t} = \overline{\rho}h-\nabla\cdot\Big[\mathbf{V}(\varepsilon^\vartheta+\varepsilon^k+p)]+$$

$$+\frac{\mathbf{B}\times\mathbf{E}}{\mu_o}+(\mathbf{V}\sigma)-C_p\overline{\rho}\kappa_T\nabla\vartheta-\mathbf{1}_r C_p\overline{\rho}\kappa\frac{\partial\overline{T}}{\partial r}\Big] \tag{3.3}$$

If additional heating is absent then (3.3) gives the whole energy conservation: the energy changes in any volume only due to the energy flux across its walls.

DIMENSIONLESS EQUATIONS

To non-dimensionalise the momentum and the heat transport equations we accept the commonly used units for space, time, velocity, magnetic field and so on (they are listed in Table 1). However, for the additional heating and the superadiabatic temperature we use rather different units, as it is prompted by the heat transport equation:

$$\frac{\partial \vartheta}{\partial t} + (\mathbf{V}\nabla)\vartheta = \kappa_T \nabla^2 \vartheta + \frac{h}{C_p} \quad +$$

$$+ \quad \kappa \nabla^2 \overline{T} - D\frac{\vartheta V_r}{d} + \eta \frac{[\nabla\times\mathbf{B}]^2}{\mu_o C_p \overline{\rho}} + \nu \frac{[\nabla\times\mathbf{V}]^2}{C_p}$$

Temperature changes due to the action of heating with typical rate h_1. This change is proportional to the typical time scale $t_1 = d^2/\eta$. Thus the natural unit of the temperature, ϑ_1, $\sim h_1 t_1/C_p$. What value could be used as h_1? It should be the adiabatic cooling. First, because it is the only source which always exists in convection. Second, it is a standard (unchanged) one. And third, it is a large term eating up 70% of the energy of the superadiabatic heat flux. That is why we choose the typical value of adiabatic cooling, $C_p \kappa \Delta\overline{T}/d^2$ as a unit for h_1. Then the unit of the superadiabatic temperature takes the form:

$$\vartheta_1 = \frac{h_1 t_1}{C_p} = \frac{C_p \kappa \Delta\overline{T}}{d^2 C_p}\frac{d^2}{\eta} = \frac{\kappa}{\eta}\Delta\overline{T} \approx 3\times10^{-3} K$$

To non-dimensionalise the momentum equation, let's divide it by the Coriolis force $2\Omega V_1\rho = 2\Omega\rho(\eta/d)$. Respectively the buoyancy force, $\mathbf{F}^a = \mathbf{1}_r\overline{g}(r)\overline{\rho}\alpha\vartheta$, takes the form:

$$\frac{\mathbf{F}^a}{2\Omega\rho}\frac{d}{\eta} = \mathbf{1}_r\frac{\overline{g}(r)}{\overline{g}_1}\frac{\vartheta(\mathbf{r})}{\vartheta_1}\frac{\overline{g}_1 d\alpha\vartheta_1}{2\Omega\eta} = \mathbf{1}_r\hat{\overline{g}}(r)\hat{\vartheta}(\mathbf{r})Ar$$

where the hats are introduced for dimensionless values and

$$Ar = \frac{\overline{g}_1 d\alpha\vartheta_1}{2\Omega\eta} = \frac{\kappa}{\eta}\frac{\overline{g}_1 d\alpha\Delta\overline{T}}{2\Omega\eta} = \left[\frac{\kappa}{\eta}\right]^2\frac{\overline{g}_1 d\alpha\Delta\overline{T}}{2\Omega\kappa} = q^2 Ra$$

$$Ar = q^2 Ra = \frac{\kappa\overline{g}_1 d\alpha\Delta\overline{T}}{2\Omega\eta^2} \sim 10^3$$

The value of ϑ_1 is rather overestimated since here we have implicitly assumed that the typical space scale in the diffusion of the superadiabatic temperature is the same as it is for the adiabatic one (d), but in reality it several times smaller. Respectively Ar is overestimated as well.

We want to emphasize that the value q in the expression for Ar, being based on the molecular diffusivity κ, differs with approximately 6 orders from the coefficient at the diffusion term, $q_T \sim 1$, in the heat transport equation, based on the turbulent diffusivity κ_T:

$$\frac{\partial\hat{\vartheta}}{\partial t} + (\hat{\mathbf{V}}\nabla)\hat{\vartheta} = q_T\nabla^2\hat{\vartheta} + \frac{\hat{h}}{\hat{\overline{\rho}}} + \nabla^2\hat{\overline{T}} + \quad \ldots$$

$$\frac{E}{Pr_m}\left(\frac{\partial\hat{\mathbf{V}}}{\partial t}+(\hat{\mathbf{V}}\nabla)\hat{\mathbf{V}}-Pr_m\nabla^2\hat{\mathbf{V}}\right)=\nabla\frac{\hat{p}}{\hat{\overline{\rho}}}-\mathbf{1}_z\times\hat{\mathbf{V}}-$$

$$-\frac{\nabla\times\hat{\mathbf{B}}\times\hat{\mathbf{B}}}{\hat{\overline{\rho}}}+Ar\mathbf{1}_r\hat{\overline{g}}\hat{\vartheta},\qquad \nabla\cdot\hat{\overline{\rho}}\hat{\mathbf{V}}=0 \quad (4.1,2)$$

$$\frac{\partial\hat{\mathbf{B}}}{\partial t}=\nabla^2\hat{\mathbf{B}}+\nabla\times[\hat{\mathbf{V}}\times\hat{\mathbf{B}}] \quad (4.3)$$

$$\frac{\partial\hat{\vartheta}}{\partial t}+(\hat{\mathbf{V}}\nabla)\hat{\vartheta}=q_T\nabla^2\hat{\vartheta}+\nabla^2\hat{\overline{T}}+\frac{\hat{h}}{\hat{\overline{\rho}}}\quad -$$

$$-D\left[\hat{\overline{g}}(r)\hat{\vartheta}\hat{V}_r-\frac{[\nabla\times\hat{\mathbf{B}}]^2}{Ar\hat{\overline{\rho}}}-E\frac{[\nabla\times\hat{\mathbf{V}}]^2}{Ar}\right] \quad (4.4)$$

Table 2

q	κ/η	2.5×10^{-6}
q_T	κ_T/η	~ 1.0
Pr_m	ν/η, ν_T/η	5×10^{-7}, ~ 1
D	$d\alpha\overline{g}_1/C_p$	0.11
Ra	$\overline{g}_1\alpha\Delta\overline{T}d/2\Omega\kappa$	1.6×10^{14}
Ar	$q^2\cdot Ra$	10^3

Non-dimensional parameters

AMPLITUDES OF THE SOLUTION

The Archimedean force, $Ar\mathbf{1}_r\hat{\overline{g}}\hat{\vartheta}$, in (4.1) in some locations is balanced by the Coriolis force, but in others - by the Lorentz force. This allows to estimate the amplitudes of the velocity and the magnetic field:

$$\hat{V} \sim Ar \sim 10^3 \qquad \hat{B} \sim \sqrt{Ar} \sim 30$$

So dimensional values equal

$$\vartheta \sim \frac{\kappa}{\eta}\Delta\overline{T} \approx 3{\times}10^{-3}K \tag{4.4}$$

$$V = \hat{V}\frac{\eta}{d} \sim \frac{\kappa}{\eta}\alpha\Delta\overline{T}\frac{\overline{g}_1}{2\Omega} = 10^{-3}ms^{-1} \tag{4.5}$$

$$B = \hat{B}\sqrt{2\Omega\overline{\rho}_1\mu_o\eta} \sim \sqrt{\frac{\kappa\overline{g}_1 d\alpha\Delta\overline{T}}{2\Omega\eta^2}}\sqrt{2\Omega\overline{\rho}_1\mu_o\eta} =$$

$$= \sqrt{\frac{\kappa}{\eta}\alpha\Delta\overline{T}}\sqrt{\mu_o\overline{\rho}_1\overline{g}_1 d} = 0.067T \tag{4.6}$$

The dimensional amplitudes of these values are extremely large: $\overline{g}_1/2\Omega = 2.8{\times}10^4 ms^{-1}$, $\sqrt{\mu_o\overline{\rho}_1\overline{g}_1 d} = 376T$. Nevertheless, the estimation gives acceptable quantities of the values due to the small parameter $(\kappa/\eta)\alpha\Delta\overline{T} = 3.2{\times}10^{-8}$.

In fact all these values including Ar must be multiplied by l/d where l is the typical space scale of the solution. If we accept for example $l/d = 1/5$ this estimation would be in a good quantitative agreement with the results of Glatzmaier and Roberts (1996).

$$\frac{\partial\hat{\vartheta}}{\partial t}+(\hat{\mathbf{V}}\nabla)\hat{\vartheta}=q_T\nabla^2\hat{\vartheta}+\nabla^2\hat{\overline{T}}+\frac{\hat{h}}{\hat{\overline{\rho}}}+$$

$$-D\left[\hat{\overline{g}}(r)\hat{\vartheta}\hat{V}_r-\frac{[\nabla\times\hat{\mathbf{B}}]^2}{Ar\hat{\overline{\rho}}}-E\frac{[\nabla\times\hat{\mathbf{V}}]^2}{Ar}\right] \qquad (4.3)$$

Since $\hat{B}\sim\sqrt{Ar}$, the Ohmic dissipation in the bulk of the core is of the order of Carnot efficiency $D\sim 10^{-1}$ as one may expect. This value is small in comparison with the adiabatic cooling for example, which is of the order of 1. The viscous dissipation, $\sim EAr$, is much smaller due to the small value of E. However, both dissipations are essential in the magnetic and the Ekman layers where their values rise to $DAr\sim 10^2$. Thus the dynamics of the layers is affected essentially by the dissipations.

The whole (integrated over the core) magnetic dissipation is of the order of $D\sim 10^{-1}$ and the whole viscous one is of the order of $DAr\sqrt{E}\sim 5\times10^{-3}$ for the turbulent viscosity. Thus the solution is Ohmic controlled and so the Taylor constraint has to satisfy even for large turbulent viscosity. Both dissipations become of the same order if $E\sim Ar^{-2}\sim 10^{-6}$. Thus we can expect that the solution will satisfy the Taylor constraint at $E<10^{-6}$. This estimation has been obtained at a rather overestimated value of $Ar(\sim 10^3)$. In a real numerical solution if $l/d\sim 1/5$ for example, Ar reduces to 200 and respectively E would increase to 2.5×10^{-5}.

INCOMPRESSIBLE LIMIT

If we decide to neglect all the compressible effects, then in the limit $D \Rightarrow 0$ we will obtain the equations of Boussinesq Approximation:

$$\frac{E}{Pr_m}\left(\frac{\partial\hat{\mathbf{V}}}{\partial t}+(\hat{\mathbf{V}}\nabla)\hat{\mathbf{V}}-Pr_m\nabla^2\hat{\mathbf{V}}\right)=$$

$$-\frac{\nabla\hat{p}}{\hat{\bar{\rho}}}-\mathbf{1}_z\times\hat{\mathbf{V}}-\frac{\nabla\times\hat{\mathbf{B}}\times\hat{\mathbf{B}}}{\hat{\bar{\rho}}}+Ar\mathbf{1}_r\hat{\bar{g}}\hat{\vartheta}, \quad (5.1)$$

$$Ar=\frac{\kappa\bar{g}_1 d\alpha\Delta\overline{T}}{2\Omega\eta^2}, \qquad \nabla\cdot\hat{\mathbf{V}}=0 \quad (5.2)$$

$$\frac{\partial\hat{\vartheta}}{\partial t}+(\hat{\mathbf{V}}\nabla)\hat{\vartheta}=q_T\nabla^2\hat{\vartheta}+\frac{\hat{h}}{\hat{\bar{\rho}}} \quad (5.4)$$

But even in this case some of the features of the compressible study are preserved we know the amplitude of the Archimedean force Ar and the order of amplitudes of the solution $\hat{\vartheta}$, $\hat{\mathbf{V}}$ and $\hat{\mathbf{B}}$.

It is important to emphasize that the amplitude of the Archimedean force Ar is not connected, as it is usually assumed, with the dimensionless diffusivity of the superadiabatic temperature q_T.

SUMMARY

1. Contrary to the equations of the Boussinesq approximation the compressible equations satisfy the law of energy conservation. This law can be used as a test of the solution at any time step.

2. As a consequence, the Ohmic and the viscous dissipations can be taken into consideration. Their role is very important in the layers.

3. These equations take into account the largest cooling term, the adiabatic cooling, which gives a correct estimation for the superadiabatic temperature and as a consequence, the correct amplitude of the Archimedean force.

4. The amplitudes of the solution can be roughly estimated on this base.

5. Due to the new terms the heat transport equation has a different form than the equation for the light constituent. So they cannot be united in one equation for the so called codensity.

6. Finally the compressible approximation is not in fact more complex than the computational point of view than the Boussinesq one and so any Boussinesq code can be easily redone to a compressible one.

7. Non-convective shells in the Earth's core are possible.

Table 1

L_1	$d = r_2 - r_1$	$2.26{\times}10^6 m$
t_1	d^2/η	$8.1{\times}10^4$ years
$\overline{g}_1$	$\overline{g}(r_1)$	$4.1 ms^{-2}$
$\overline{\rho}_1$	$\overline{\rho}(r_1)$	$1.2{\times}10^4 kgm^{-3}$
V_1	η/d	$9{\times}10^{-7} ms^{-1}$
B_1	$\sqrt{2\Omega\overline{\rho}_1\mu_o\eta}$	$2.1{\times}10^{-3} T$
h_1	$C_p\kappa\Delta\overline{T}/d^2$	$1.06{\times}10^{-12} Wkg^{-1}$
ϑ_1	$(\kappa/\eta)\Delta\overline{T}$	$3.2{\times}10^{-3} K$
$\hat{\overline{T}}(r) =$	$[\overline{T}(r) - \overline{T}(r_2)]/\Delta\overline{T}$	$0 \leq \hat{\overline{T}}(r) \leq 1$

Unit used for non-dimensionalisation of the equations

Table 2

q	κ/η	$2.5{\times}10^{-6}$
Pr_m	ν/η	$5{\times}10^{-7}$
q_T	κ_T/η	~ 1
Pr_{mT}	ν_T/η	~ 1
E	$\nu/2\Omega L_1^2$	$1.3{\times}10^{-15}$
E_T	$\nu_T/2\Omega L_1^2$	$2.6{\times}10^{-9}$
D	$d\alpha\overline{g}_1/C_p$	0.11
Ra	$\overline{g}_1\alpha\Delta\overline{T}d/2\Omega\kappa$	$1.6{\times}10^{14}$
Ar	$q^2\cdot Ra$	$1.0{\times}10^{3}$
Rm	Vd/η	$\sim Ar = 1.0{\times}10^{3}$
Pe	Vd/κ_T	$\sim Rm \sim 1.0{\times}10^{3}$

Values of non-dimensional parameters

Table 3

r_1	$1.22{\times}10^6 m$	r_2	$3.48{\times}10^6 m$
$\overline{T}(r_1)$	$5300K$	$\overline{T}(r_2)$	$4016K$
$\overline{\rho}(r_1)$	$12166kg/m^3$	$\overline{\rho}(r_2)$	$9903kg/m^3$
$\overline{g}(r_1)$	$10.66m/s^2$	$\overline{g}(r_2)$	$4.1m/s^2$
$I^a(r_1)$	$0.25{\times}10^{12}W$	$I^a(r_2)$	$3.2{\times}10^{12}W$
κ	$5{\times}10^{-6}m^2s^{-1}$	ν	$10^{-6}m^2s^{-1}$
η	$2m^2s^{-1}$	κ_T	$\sim \eta$
α	$10^{-5}K^{-1}$	$\Delta\overline{T}$	$1.3{\times}10^3 K$
Ω	$7.3{\times}10^{-5}s^{-1}$	C_p	$840Jkg^{-1}K^{-1}$

Parameters of the Earth's core used in Tables 1 and 2