Towards an inviscid treatment of Earth's core

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- Taylor, 1963
- Proctor, 1975; Malkus & Proctor, 1975
- 1980s onwards, many

Plan

- Setup of the inviscid problem
- A fully spectral expansion for the full sphere
- Use of the expansion for the inviscid problem — a problem with enumerable constraints
- A toy problem

Forces in the core

$$\text{Ekman number} = \text{E} = \frac{\text{Viscous Force}}{\text{Coriolis Force}} \sim 10^{-15}$$

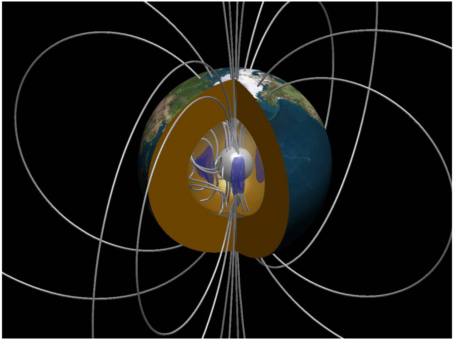


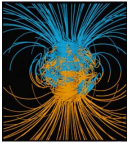
Figure: J. Aubert

Preamble I

Simplified dynamics

10^{-6} 10^{-7} 10^{-15} 0 Ekman No.

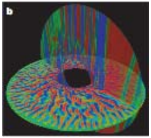
1995



Glatzmaier & Roberts, Nature

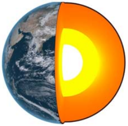
Cray C-90
1 year real time
(USA)

2008



Kageyama et al., Nature

4,096 processors on
Earth Simulator
months real time
(Japan)



Earth
Like
regime

Taylors constraint

Navier-Stokes and Induction

[Non-dimensionalised]

$$
\begin{aligned}
R_o\left(\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u}\right)+\hat{\mathbf{z}}\times\mathbf{u} &= -\,\boldsymbol{\nabla}\Pi+C\hat{\mathbf{r}}+E\nabla^2\mathbf{u} \\
&\quad +[\boldsymbol{\nabla}\times\mathbf{B}]\times\mathbf{B}, \\
\frac{\partial \mathbf{B}}{\partial t} &= \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})+\nabla^2\mathbf{B},
\end{aligned}
$$

[+ equation of heat transfer]

with $\boldsymbol{\nabla}\cdot\mathbf{u}=\boldsymbol{\nabla}\cdot\mathbf{B}=0$, where $\mathbf{u}$ denotes the core flow, Π the modified pressure, $\mathbf{B}$ the magnetic field and C the buoyancy force.
$R_o \sim 10^{-9}$ (Rotation/Magnetic Decay)
$E \sim 10^{-15}$

Magnetostrophic Balance

[Non-dimensionalised]
Slow motions

$$\hat{\mathbf{z}} \times \mathbf{u} = -\nabla\Pi + C\hat{\mathbf{r}} + [\nabla \times \mathbf{B}] \times \mathbf{B},$$

Coriolis Pressure Buoyancy Lorentz

The Taylor State (Taylor [1963])

This truly makes viscosity unimportant (E=Ro=0) [Dimensional]
Integrate

$$\begin{array}{ccccccccc} 2\rho\boldsymbol{\Omega} \wedge \mathbf{v} & = & -\nabla p & + & \mathbf{J} \wedge \mathbf{B} & + & \rho'\mathbf{g} \\ \text{Coriolis force} & = & -\text{Pressure Gradient} & + & \text{Lorentz Force} & + & \text{Buoyancy} \end{array}$$

over cylinders coaxial with rotation axis; find

$$T = \int_{C(s)} [\mathbf{J} \wedge \mathbf{B}]_\phi \, d\phi \, dz = 0 \quad \forall s$$

Applies on *every* cylinder.
Taylor showed that when this condition is satisfied, the flow in the core can be uniquely found. [It is necessary and sufficient].

One difficulty

Spherical: mechanical boundary conditions on v; insulating boundary conditions on B

Cylindrical: integration domain for B field

The need for 2 coordinate systems has historically caused problems

The basic equations [Taylor, 1963]

If

$$\int_{C(s)} [\mathbf{J} \wedge \mathbf{B}]_\phi \, d\phi \, dz = 0 \quad \forall s \tag{1}$$

$\mathbf{B}$ and buoyancy then determines the ageostrophic flow $\mathbf{u}_{\mathrm{mag}}$

$$2\rho\Omega \wedge \mathbf{u}_{\mathrm{mag}} = -\nabla p + \mathbf{J} \wedge \mathbf{B} + \rho' g \hat{\mathbf{r}} \tag{2}$$

but there is an unknown geostrophic flow $u_G(s)$.
This is determined by the requirement that Taylor's Constraint is satisfied *at all times* $\Rightarrow$

$$\mathcal{L} u_G(s) = F(\mathbf{u}_{\mathrm{mag}}, \mathbf{B}) \tag{3}$$

Determination of $u_G(s)$

$$\frac{d}{dt}\int_{C(s)} [\mathbf{J}\wedge\mathbf{B}]_\phi \, d\phi \, dz = 0 \tag{4}$$

leads to

$$\alpha(s)\left(\frac{u_G(s)}{s}\right)'' + \beta(s)\left(\frac{u_G(s)}{s}\right)' = G(s) \tag{5}$$

where $\alpha(s)$ and $\beta(s)$ are given by the magnetic field and $G(s)$ is a known form independent of $u_G(s)$.
In a sphere this 2nd order d.e. can be solved with the 2 constraints (i) regularity on the axis (ii) conservation of angular momentum
The system evolves according to

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla(\mathbf{u}\wedge\mathbf{B}) + \eta\nabla^2\mathbf{B} \tag{6}$$

Two types of problem

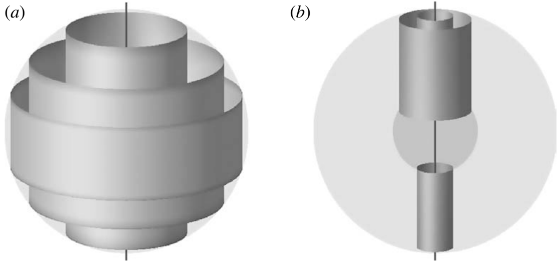


Figure 1. An illustration of cylinders over which Taylor's constraint is defined: (a) in the bulk of the core and (b) inside the tangent cylinder. The outer-core and inner-core spherical boundaries are shown in light grey.

We start with the full-sphere problem

The fully spectral method

- In order that $\nabla \cdot \mathbf{B} = 0$ I write

$$\mathbf{B} = \nabla \wedge \nabla \wedge (S\hat{\mathbf{r}}) + \nabla \wedge (T\hat{\mathbf{r}})$$

- I expand S and T in spherical harmonics

$$S = \sum^{L} S_l^m(r) Y_l^m(\theta, \phi)$$

- In radius I want something akin to a Chebyshev expansion (with spectral convergence)

$$S_l^m(r) = \sum_{n=1}^{N} f_l^n(r)$$

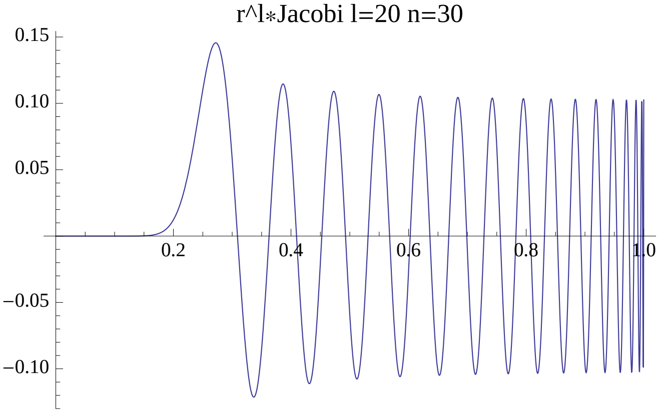
- Consideration of regularity at the origin demands

$$S_l^m(r) \sim r^{l+1}(1 + r^2 + \dots) \quad \text{as} \quad r \to 0$$

The Jones-Worland polynomials

$$f_l^m(r) = r^{l+1} P_n^{(\alpha,\beta)}(2r^2 - 1)$$

where $P_n^{(\alpha,\beta)}$ is a Jacobi polynomial

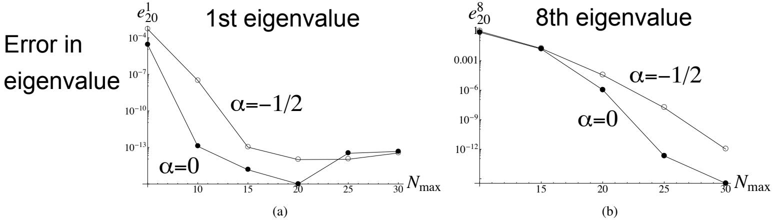


$$\alpha = -1/2, \ \beta = l + 1/2$$

The Jones-Worland polynomials

- $f_l^m(r) = r^{l+1} P_n^{(\alpha,\beta)}(2r^2 - 1)$
- Jones/Worland suggested $\alpha = -1/2$ (Chebycheff type)
- We have used this basis in kinematic dynamo calculations
- We observe spectral convergence
- We find superior performance to any Chebycheff scheme (or finite differences) that do not honour origin regularity
- We find slightly better performance with $\alpha = 0$ (Legendre type)

Test of Magnetic decay problem



Use Jones-Worland polynomials to solve for $l = 20$ eigenvalues.
$\alpha = 0$ slightly superior.

The Taylor torque

$$T(s) = \int_{C(s)} [\mathbf{J} \wedge \mathbf{B}]_\phi s \, d\phi \, dz$$

$$T(s) = \sum [\mathbf{S}_l^m, \mathbf{S}_n^m] + [\mathbf{S}_l^m, \mathbf{T}_n^m] + [\mathbf{T}_l^m, \mathbf{T}_n^m]$$

$\mathbf{S}_l^m$ is poloidal
$\mathbf{T}_l^m$ is toroidal

The interactions

$$[\boldsymbol{T}_l^m, \boldsymbol{T}_n^m] = \oint_{C(s)} \frac{l(l+1)\, T_l^m(r)\, T_n^m(r)}{r^3 \sin\theta} \left(Y_l^m \frac{\partial Y_n^m}{\partial \phi} \right) s \,\mathrm{d}z \,\mathrm{d}\phi + \mathrm{sc},$$

$$[\boldsymbol{S}_l^m, \boldsymbol{S}_n^m] = \oint_{C(s)} \frac{l(l+1) S_l^m \nabla_n^2 S_n^m}{r^3 \sin\theta} \left(Y_l^m \frac{\partial Y_n^m}{\partial \phi} \right) s \,\mathrm{d}z \,\mathrm{d}\phi + \mathrm{sc},$$

$$\begin{aligned}[\boldsymbol{T}_l^m, \boldsymbol{S}_n^m] = \oint_{C(s)} \frac{1}{r^3} \bigg(& l(l+1)\, T_l^m \frac{\mathrm{d} S_n^m}{\mathrm{d}r}\, Y_l^m \frac{\partial Y_n^m}{\partial \theta} \\ & - n(n+1) S_n^m \frac{\mathrm{d}\, T_l^m}{\mathrm{d}r}\, Y_n^m \frac{\partial Y_l^m}{\partial \theta} \bigg) s \,\mathrm{d}z \,\mathrm{d}\phi,\end{aligned}$$

Selection rules

Interactions are zero *unless*

(1) S–S, T–T or T–S: $m_1 = m_2$,
(2) S–S or T–T: $m_1 \neq 0$, $l_1 - l_2 = 0 \pmod 2$; not both sine or cosine,
(3) T–S: $l_1 - l_2 = 1 \pmod 2$; both sine or both cosine,
(4) $T-T$: $l_1 \neq l_2$, and
(5) S–S, $l_1 = l_2$: $S_l^{m\ s}(r)/S_l^{m\ c}(r)$ not constant.

Form of Taylor Torque

- Choose polynomial expansion in radius
- We satisfy insulating boundary conditions and find that the Taylor integral is

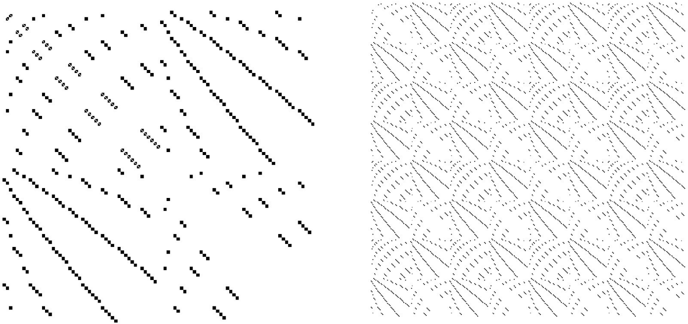
$$T(s) = s^2\sqrt{1-s^2}\sum_{i=1}^{C}\alpha_i s^{2(i-1)}$$

  where s is cylindrical radius
- The problem of $T(s) = 0 \; \forall s$ is now trivial
- Choose every $\alpha_i = 0$ in the expression for $T(s)$

Counting the constraints

- There are exactly $C = L + 2N - 2$ constraints in a sphere.
- Each constraint is quadratic in the magnetic field.
- $$\mathbf{b}^T \mathsf{A}_1 \mathbf{b} = 0$$
  $$\mathbf{b}^T \mathsf{A}_2 \mathbf{b} = 0$$
  $$\vdots$$
  $$\mathbf{b}^T \mathsf{A}_C \mathbf{b} = 0$$
  where **b** are coefficients of the expansion for **B** (magnetic field))
- When our spectral expansion is truncated at L in $Y_l^m(\theta, \phi)$ and N in r, we have $2NL(L+2)$ free parameters.
- In principle, **B**-fields satisfying Taylor's constraint are ubiquitous
- Existence is clear

The sparse matrices



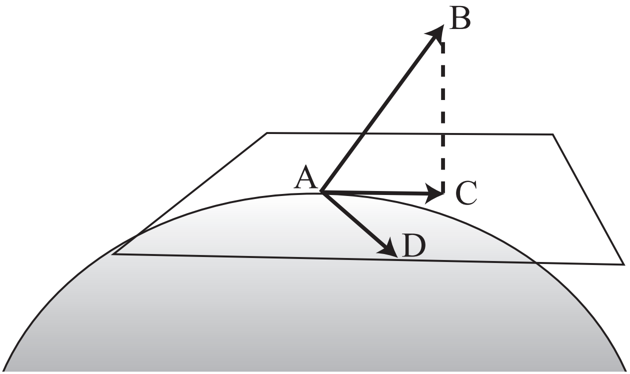
Two different orderings of one of the constraint matrices

Scaling of the constraints

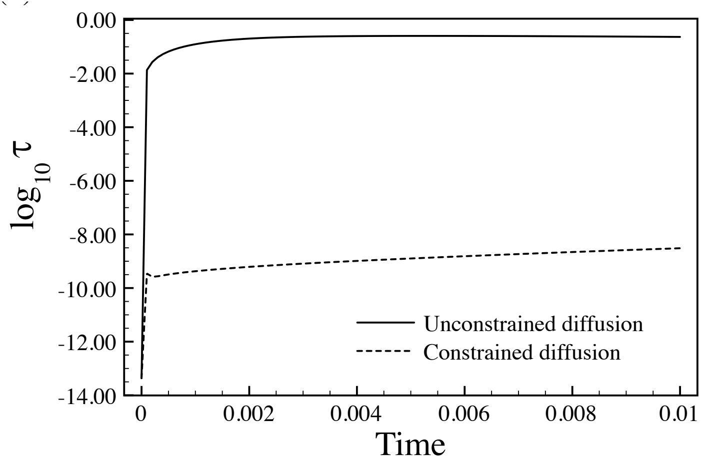
L_{max}	N_{max}	$\mathcal{L}$	$\mathcal{C}$	NNZ$^{\mathrm{M}}$	NNZ$^{\mathrm{C}}$	Density$^{\mathrm{C}}$ (%)	Storage$^{\mathrm{C}}$/Mb
6	4	384	12	20,754	23,414	1.7	0.18
8	6	960	18	146,694	167,358	0.88	1.3
10	6	1440	20	299,406	351,876	0.72	2.7
12	4	1344	18	199,182	248,374	0.61	1.9
12	6	2016	22	542,502	654,042	0.61	5.0
14	8	3584	28	1,867,634	2,242,554	0.52	17
20	10	8800	38	10,722,486	13,236,186	0.36	100
50	20	104000	88	1,373,887,706	1,781,641,206	0.14	13,000

Toy problem: A first try at time-stepping

- Turn off the effect of $\mathbf{u}_{mag}$, allow only diffusion
- The $\mathbf{B}$-field decays
- We use a projection method instead of the action of u_G to ensure Taylor's constraint is satisfied

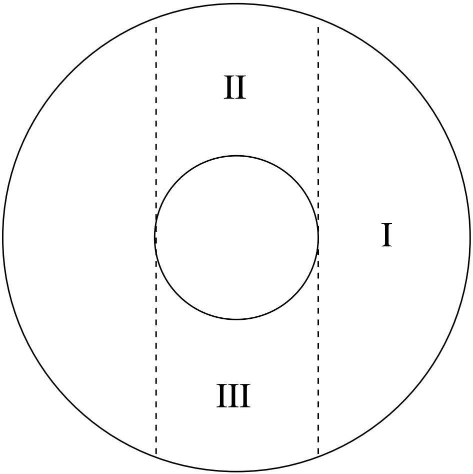


Taylorisation



$$\tau^2 = \frac{(\text{Taylor torques over cylinders})^2}{(\text{Taylor torques})^2 \text{ over cylinders}}$$

Trouble in shells



The Hollerbach/Proctor constraint

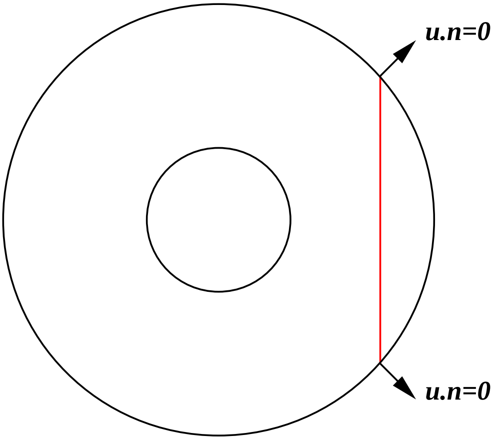
$$2\hat{\mathbf{z}} \wedge \mathbf{u} = -\nabla P + \mathbf{f}$$

$$\frac{d\mathbf{u}}{dz} = -1/2(\nabla \wedge \mathbf{f})$$

$$\mathbf{u}(z) = -1/2 \int_{-z_h}^{z} (\nabla \wedge \mathbf{f})\, dz + \mathbf{c}$$

Need 2 boundary conditions to determine c_s and c_z.

The Hollerbach/Proctor constraint



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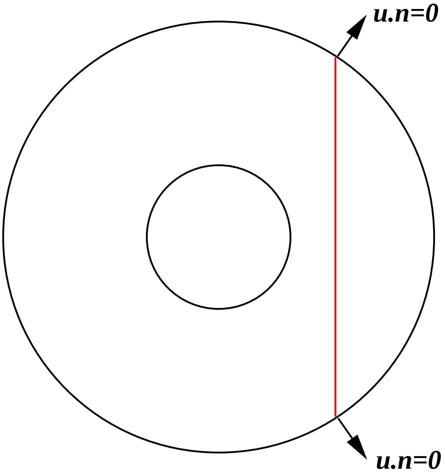

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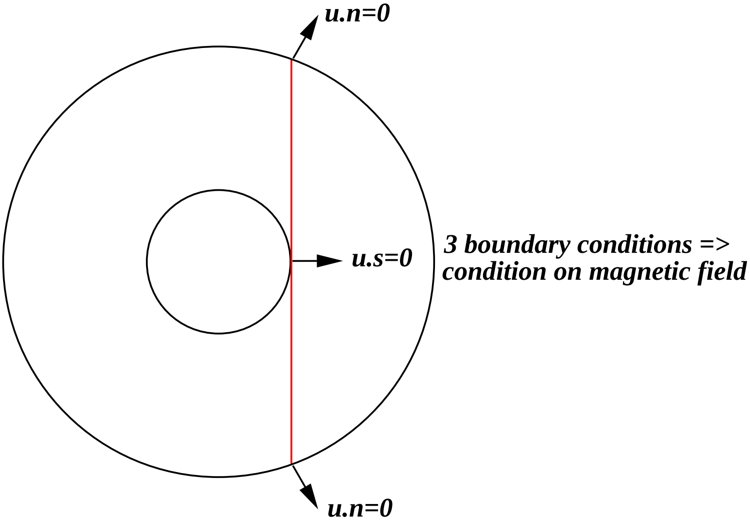
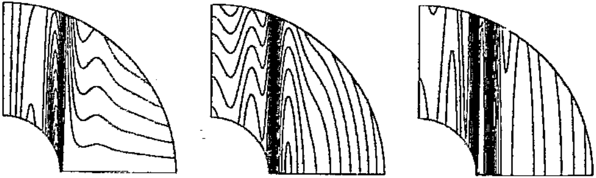
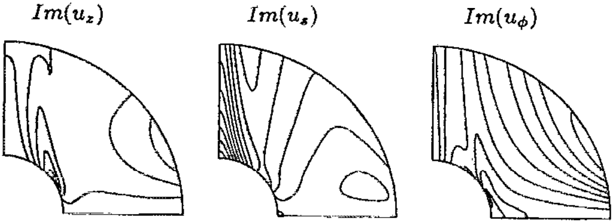


Hollerbach's [1994] numerical simulations



Flow from imposed field that does not satisfy constraint (with small viscosity)



Flow when field allowed to adjust, satisfies the constraint

The form of the constraint

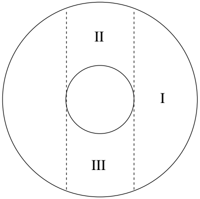
- $\mathbf{f}$ force must satisfy

$$z_h \int_{-z_h}^{z_h} (\nabla \wedge \mathbf{f})_z \, dz - r_i \int_{-z_h}^{0} (\nabla \wedge \mathbf{f})_s \, dz + r_i \int_{0}^{z_h} (\nabla \wedge \mathbf{f})_s \, dz = 0$$

  where $\mathbf{f}$ are the Lorentz+buoyancy forces. [z_h is the half-height of the tangent cylinder]

- This can be written as $2L$ constraints on the field.

Further troubles in a spherical shell



Recall Taylor's recipe for findiing $u_G(s)$.
Solve 2nd order differential equation.

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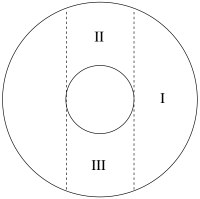

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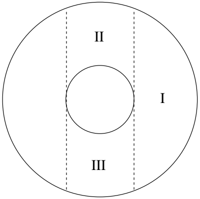

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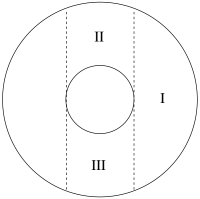
Solve 2nd order differential equation.
In I $\exists$ 2 constants of integration
In II $\exists$ 2 constants of integration

Further troubles in a spherical shell



Recall Taylor's recipe for findiing $u_G(s)$.
Solve 2nd order differential equation.
In I $\exists$ 2 constants of integration
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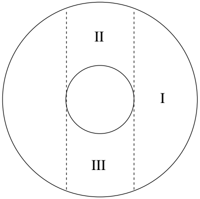
Further troubles in a spherical shell



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Taylor (1963): regularity on axis, and conserve angular momentum
$\Rightarrow$ 3 constraints

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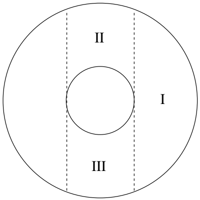

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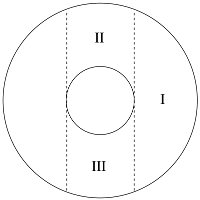
To avoid infinite shear, demand (a là Hollerbach/Proctor)
continuity in $u_G(s)$ at tangent cylinder:

Further troubles in a spherical shell



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continuity in $u_G(s)$ at tangent cylinder:
But now have 5 constraints on 6 unknowns!

Summary

- In a full sphere, a fully spectral method provides a finite characterisation of Taylor's constraint
- Jones/Worland polynomials provide a numerically attractive representation in radius
- A projection operator can be used to make $T(s) = 0$
- We are presently studying geometric integrators for time-marching schemes