What controls the decay of passive scalars in smooth flows?

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The exponential decay of the variance of a passive scalar released in a homogeneous random two-dimensional flow is examined. Two classes of flows are considered: short-correlation-time (Kraichnan) flows, and renewing flows, with complete decorrelation after a finite time. For these two classes, a closed evolution equation can be derived for the concentration covariance, and the variance decay rate γ_2 is found as the eigenvalue of a linear operator. By analyzing the eigenvalue problem asymptotically in the limit of small diffusivity κ, we establish that γ_2 is either controlled (i) locally, by the stretching characteristics of the flow, or (ii) globally, by the large-scale transport properties of the flow and by the domain geometry. We relate the eigenvalue problem for γ_2 to the Cramer function encoding the large-deviation statistics of the stretching rates; hence we show that the Lagrangian stretching theories developed by Antonsen *et al.* [Phys. Fluids **8**, 3094 (1996)] and others provide a correct estimate for γ_2 as $\kappa \to 0$ in regime (i). However, they fail in regime (ii), which is always the relevant one if the domain scale is significantly larger than the flow scale. Mathematically, the two types of controls are distinguished by the limiting behavior as $\kappa \to 0$ of the eigenvalue identified with γ_2: in the local case (i) it coincides with the lower limit of a continuous spectrum, while in the global case (ii) it is an isolated discrete eigenvalue. The diffusive correction to γ_2 differs between the two regimes, scaling like $1/\log^2 \kappa$ in regime (i), and like κ^σ for some $0 < \sigma < 1$ in regime (ii). We confirm our theoretical results numerically both for Kraichnan and renewing flows. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2033908]

I. INTRODUCTION

We consider the advection–diffusion of a passive scalar by a spatially smooth, nondivergent velocity field. The assumed spatial smoothness of the velocity field makes this problem relevant to a number of applications: Batchelor-regime turbulence, chaotic-advection flows, and more generally flows (such as geophysical flows) dominated by their large-scale component. A topic of recent theoretical investigation and discussion has been the extent to which the evolution of a scalar in such flows can be predicted given quantitative information on stretching histories following fluid particles. We shall use the term "Lagrangian stretching theories" to denote theories[1–5] that develop such a description. A separate line of investigation has introduced the idea of a "strange eigenmode"[6] that dominates the long-time evolution of the advection-diffusion equation and analyzed the properties of the strange eigenmode[7–10] through numerical simulations and analytic techniques. The term "strange" is used since the spatial scales of the eigenmode reduce indefinitely as diffusivity tends to zero.

The relevant governing equation is the advection-diffusion equation

$$\partial_t C = \mathcal{A}C \coloneqq \kappa \Delta C - \mathbf{v} \cdot \nabla C, \tag{1.1}$$

for the concentration C of the scalar. The velocity field is prescribed and satisfies $\nabla \cdot \mathbf{v} = 0$; in the two-dimensional case on which we concentrate, this implies that $\mathbf{v} = (-\psi_y, \psi_x)$ for some streamfunction ψ. We shall follow many previous authors in taking ψ to be a stationary random function of time. We also assume that C has zero spatial average.

If ψ were independent of t then we would expect the decay of C to be determined by the smallest relevant eigenvalue of the time-independent operator $\mathcal{A}$. If ψ were time periodic we would expect it to be determined by the smallest relevant Floquet exponent of the time-periodic operator $\mathcal{A}$. This structure carries over to the case where ψ is a random function of time and there, for almost all initial conditions and realizations of the flow, we expect that C decays exponentially in the long-time limit in the sense that

$$C(\mathbf{x},t) \sim \exp(-\gamma_C t) B(\mathbf{x},t) \quad \text{as } t \to \infty, \tag{1.2}$$

where $B(\mathbf{x},t)$ is a stationary random function of time. Here, $-\gamma_C$ is the largest Liapunov exponent of the operator $\mathcal{A}$ and controls the rate of decay of the concentration; $B(\mathbf{x},t)$ is the corresponding Liapunov (eigen)function and controls the spatial structure of the decaying scalar field. The behavior (1.2), with a deterministic $\gamma_C > 0$ is expected from Oseledec's multiplicative ergodic theory[11] and its infinite-dimensional generalization.[12] Note that, because of the small-scale cutoff imposed by diffusivity, (1.1) can in effect

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be treated as a finite-dimensional system, at least for two-dimensional flows.[8] A major aim of studies of passive-scalar decay is then to determine the decay rate γ_C as well as the statistics of its finite-time counterparts and the statistics of $B(\mathbf{x},t)$, in particular, in the limit $\kappa \to 0$.

We concentrate on flows that are random in time with spatially homogenous statistics. These flows are mixing, so that the decay rate γ_C tends to a nonzero value as $\kappa \to 0$. This contrasts with flows in which particle trajectories do not explore the entirety of the fluid domain. In such flows, the scalar decay is limited by diffusion, and $\gamma_C \to 0$ as $\kappa \to 0$. This is the relevant situation for many time-periodic flows in the presence of invariant tori.[13] The assumption of spatial homogeneity also rules out the application of no-slip boundary conditions to the flow (as would be appropriate for viscous flows in bounded domains). Such boundary conditions lead to the suppression of mixing near the boundary and to a $\kappa^{1/2}$ scaling for the scalar-decay rate.[14,15] Including the effects of the no-slip boundary condition is essential to explain many laboratory experiments.[16] The work we present here applies most directly to the periodic domains employed in numerous recent numerical simulations of passive-scalar decay, but the general conclusions are relevant to many applications in which the no-slip boundary condition is not appropriate, particularly geophysical and astrophysical flows and flows confined by free surfaces.

An *a priori* expectation is that the decay rate γ_C of the random operator $\mathcal{A}$, like the eigenvalue and Floquet multiplier in the time-independent and time-periodic cases, respectively, is determined by the global structure of the flow and by the domain for (1.1). If this holds it limits the usefulness of the Lagrangian stretching theories, which take account only of local information. Indeed several recent papers have demonstrated numerically[9,10,17] that the predictions of the Lagrangian stretching theories are quantitatively incorrect. The limitations of Lagrangian stretching theories are perhaps best demonstrated by their inconsistency with the results of homogenization theory,[18] valid for domain sizes large compared to the velocity scale, which predict decay rates inversely proportional to the square of the domain size. Voth *et al.*[19] (in experiments) and Fereday and Haynes[7] and Schekochihin *et al.*[20] (using theoretical and numerical approaches) all demonstrate clear differences in this regime between the actual decay rate and that predicted by the Lagrangian stretching theories.

Nonetheless, when the velocity scale and domain size for the advection-diffusion problem are similar, there seems to be some agreement between the decay rate predicted by the Lagrangian stretching theories, in particular, the decay rate of the scalar variance first derived by Antonsen *et al.*,[1] and the actual decay rate.[5,7] Firm conclusions have proved elusive, however, partly because of the apparent slow approach of decay rates to their limiting value as κ tends to zero.

In this paper we analyze further the scalar-decay problem and the relevance of the Lagrangian stretching theories, considering, in particular, how the decay rate changes with the ratio of velocity scale to domain size. We consider spatially homogeneous random flows for which a closed evolution equation can be obtained for the covariance

$$\Gamma(\mathbf{x},t) := \langle C(\mathbf{x}+\mathbf{y},t)C(\mathbf{y},t)\rangle$$

of the scalar field, where $\langle\cdot\rangle$ denotes the average over all realizations of the random flow. We focus our attention on the decay rate γ_2 of $\langle C^2\rangle = \Gamma(\mathbf{0},t)$, which satisfies

$$\langle C^2\rangle \sim \exp(-\gamma_2 t) \quad \text{as } t \to \infty.$$

This provides a first characterization of the decay of the scalar-field fluctuations.

For the flows that we consider, the decay rate γ_2 is obtained as the eigenvalue of the "transfer operator" that governs the time evolution of $\Gamma(\mathbf{x},t)$. We relate this operator in the limit $\kappa \to 0$ to the equation governing the probability density function of line elements (or interparticle separation). This makes it possible to derive the conditions under which the Lagrangian stretching theories provide a valid approximation to γ_2 in that limit. For $\kappa=0$, the spectrum of the transfer operator has a continuous part and, sometimes, depending on the flow and domain size, isolated discrete eigenvalues. We show that Lagrangian stretching theories correctly predict the decay rate γ_2 for $\kappa \to 0$ if there are no such isolated eigenvalues. This situation is what we refer to as the "locally controlled" case. If there are isolated eigenvalues, however, the Lagrangian stretching theories fail in their prediction of γ_2; this we refer to as the "globally controlled" case. In each of these two cases we obtain the κ dependence of γ_2 using matched asymptotics or similar methods.

The structure of the paper is as follows. In Sec. II we briefly summarize the aspects of the Lagrangian stretching theories that are important for the detailed comparison to follow. In Sec. III we consider the Kraichnan (-Kazantsev) limit of random flows with a very short correlation time. In Sec. IV we consider flows with finite correlation time, specifically "renewing" flows in which time is divided into equal intervals and the velocity field within a given interval is assumed statistically independent of that within any other interval. In both Secs. III and IV, we verify the theoretical predictions against numerical simulations. The paper concludes in Sec. V with a discussion.

Throughout the paper we make reference to some of the literature on the kinematic dynamo (e.g., Ref. 21 and references therein). This concerns the transport and diffusion by a random velocity field of a vector (the magnetic field) rather than a scalar as is the case here, but the close analogy between the two problems means that our results have direct analogs, some obtained previously, in the kinematic dynamo theory.

II. LAGRANGIAN STRETCHING THEORIES

The Lagrangian stretching theories of Antonsen *et al.*,[1] Son,[4] and Balkovsky and Fouxon,[2] relate the decay rate of the tracer variance (and higher-order moments) in the limit $\kappa \to 0$ to the stretching statistics of the flow. These statistics concern the line elements $\mathbf{d}$, which evolve according to

$$\dot{\mathbf{d}} = \mathbf{d} \cdot \nabla \mathbf{v}, \tag{2.1}$$

where $\dot{}$ denotes the time derivative. The line elements typically grow exponentially with time, making it convenient to introduce the finite-time stretching factors

$$h \coloneqq \frac{1}{t} \log \frac{d(t)}{d(0)} \quad \text{where } d = |\mathbf{d}|. \tag{2.2}$$

The statistics of the stretching factors can be captured in the framework of large-deviation theory: for large t, h behaves as the average of a large number of independent random variables, and its probability density function $p(h;t)$ can be approximated as

$$p(h;t) \propto \exp[-tG(h)], \tag{2.3}$$

where $G(h)$ is the so-called Cramer function (or entropy, using the thermodynamic analogy[22]), a non-negative convex function.[2,3] The Liapunov exponent for the line elements $\mathbf{d}$ is then simply the average $\bar{h}$ of the stretching factors in the limit $t \to \infty$ and corresponds to the minimum of the Cramer function,

$$G(\bar{h}) = G'(\bar{h}) = 0.$$

The Cramer function depends on the details of the stretching accomplished by the flow, and can be determined analytically only for very special flows (e.g., isotropic flows in the Kraichnan limit of zero correlation time). Its usefulness stems from the fact that, according to the Lagrangian stretching theories, it encapsulates all the information about the flow that is necessary to describe the scalar decay in the limit $\kappa \to 0$.

In place of the Cramer function, it is sometimes convenient to use its Legendre transform, the so-called free energy

$$F(m) = \sup_h [mh - G(h)]. \tag{2.4}$$

This function estimates the growth rate of the mth moment of the line elements: indeed, approximating

$$\langle d^m \rangle = \int_0^\infty e^{mht} p(h;t) dh,$$

using (2.3) and Laplace's method gives

$$\langle d^m \rangle \propto \exp[F(m)t] \quad \text{as } t \to \infty. \tag{2.5}$$

From this, it is clear that $F(0)=0$; it can also be shown that $F(-2)=0$ and $F'(0)=\bar{h}$.[23]

The Lagrangian stretching theories of scalar decay relate the decay rate of $\langle C^2 \rangle$ to $G(h)$ or, equivalently, to $F(m)$. To achieve this, the scale separation between scalar and velocity fields is exploited to replace the velocity $\mathbf{v}$ by its linear approximation $\mathbf{x} \cdot \nabla \mathbf{v}$, where $\nabla \mathbf{v}$ is evaluated along a trajectory and is a function of time only. The scalar field $C(\mathbf{x},t)$ may then be written as a superposition of elementary solutions (sine functions for Antonsen *et al.*;[1] Gaussian functions for Balkovsky and Fouxon[2]), and its moments are derived by averaging such elementary solutions over an ensemble of trajectories. The approaches of both Antonsen *et al.*[1] and Balkovsky and Fouxon[2] lead to an expression for the scalar variance of the form

$$\langle C^2 \rangle \propto \int_0^\infty e^{-ht} p(h;t) dh \propto \int_0^\infty e^{-t[h+G(h)]} dh.$$

Approximating the integral for $t \to \infty$ implies that the predicted decay rate γ_2^L of $\langle C^2 \rangle$ should be either $h_* + G(h_*)$, where h_* is the solution of $G'(h_*)=-1$, if $h_* \geq 0$, or $G(0)$ if $h_* \leq 0$. In terms of the free energy (2.4), this prediction can be rewritten as

$$\gamma_2^L = \begin{cases} -F(-1) & \text{if } F'(-1) \geq 0, \\ G(0) & \text{if } F'(-1) \leq 0. \end{cases} \tag{2.6}$$

III. DECAY RATE IN THE KRAICHNAN LIMIT

We first consider flows in the Kraichnan limit corresponding to velocity fields with infinitesimal correlation time. These flows can be regarded as limiting cases of the more general renewing flows examined in Sec. IV. It is nonetheless worthwhile providing a specific treatment of the Kraichnan limit: the calculations in this case are particularly straightforward and lead to a number of results which, as will be seen, carry over to the general renewing flows.

In the Kraichnan limit, the velocity field $\mathbf{v}$ formally satisfies

$$\langle \mathbf{v}(\mathbf{x},t) \rangle = 0 \text{ and } \langle v_i(\mathbf{x},t) v_j(\mathbf{x}',t') \rangle = 2B_{ij}(\mathbf{x}-\mathbf{x}')\delta(t-t'),$$

assuming homogeneity. For such a velocity field, closed equations can be derived for the multipoint correlation functions of C.[3,24] In particular, the covariance satisfies[25]

$$\partial_t \Gamma = 2\kappa \Delta \Gamma + 2D_{ij}(\mathbf{x}) \partial_{ij}^2 \Gamma, \tag{3.1}$$

where

$$D_{ij}(\mathbf{x}) \coloneqq B_{ij}(0) - B_{ij}(\mathbf{x}).$$

Note that, in contrast with most previous work,[4,14,20] we do not make an assumption of isotropy so that (3.1) does not reduce to one spatial dimension.

We now consider the spectrum of (3.1), i.e., the values λ satisfying

$$\mathcal{L}_\kappa \Gamma \coloneqq -2[\kappa \Delta + D_{ij}(\mathbf{x}) \partial_{ij}^2] \Gamma = \lambda \Gamma, \tag{3.2}$$

with suitable boundary conditions. In what follows we assume that the velocity field, and hence $D_{ij}(\mathbf{x})$, are $2\pi \times 2\pi$ periodic. We take C to be also periodic but with period $2\pi P \times 2\pi P$ for some integer P; this provides boundary conditions for Γ. For $\kappa \neq 0$, the spectrum of (3.2) is purely discrete. Clearly, the decay rate γ_2 of the variance will be the smallest positive eigenvalue λ of $\mathcal{L}_\kappa$.

In general, the eigenvalues λ of $\mathcal{L}_\kappa$ are determined by the global behavior of $D_{ij}(\mathbf{x})$ and by the domain period P. However, in the limit $\kappa \to 0$, the leading-order approximation of at least some of the eigenvalues is entirely determined by the local form of $D_{ij}(\mathbf{x})$ near $x=y=0 \pmod{2\pi}$. As we now show, this is because they emerge for $\kappa \neq 0$ from the continuous part of the spectrum of the operator $\mathcal{L}_0 = -2D_{ij}(\mathbf{x})\partial_{ij}^2$ and because this part of the spectrum depends only on the local form of $D_{ij}(\mathbf{x})$ where it vanishes. The local control of the variance decay occurs when these eigenvalues associated with the continuous spectrum of $\mathcal{L}_0$ are the only eigenvalues of $\mathcal{L}_\kappa$. In this case, γ_2 is predicted correctly Lagrangian stretching theories. The global control of the variance decay occurs when $\mathcal{L}_\kappa$ has other eigenvalues, related to isolated discrete eigenvalues of $\mathcal{L}_0$; this case is treated in Sec. III B.

A. Continuous spectrum of $\mathcal{L}_\kappa$: Local control

The theory of singular elliptic partial differential equations[26,27] indicates that the spectrum of $\mathcal{L}_0$ (in L^2) has a continuous part on the half line $[\ell,\infty)$. Its lower limit $\ell>0$ depends only on the local form of $D_{ij}(\mathbf{x})$ near its singularity $x=y=0 \pmod{2\pi}$. Specifically, ℓ is the smallest value of λ such that $\mathcal{L}_0\Gamma=\lambda\Gamma$, or equivalently the local version of this equation, namely,

$$S_{ijkl}x_kx_l\partial^2_{ij}\Gamma=\lambda\Gamma \tag{3.3}$$

admit oscillatory solutions as $\mathbf{x}\to 0$. (Oscillatory means that there are zero-level curves for $|\mathbf{x}|$ arbitrarily small; see Dunford and Schwartz,[26] Glazman,[27] and Piepenbrink.[28]) Here, the constant tensor

$$S_{ijkl}:=-\left.\partial^2_{kl}D_{ij}(\mathbf{x})\right|_{\mathbf{x}=0}=\left.\partial^2_{kl}B_{ij}(\mathbf{x})\right|_{\mathbf{x}=0}$$

satisfies $S_{ijkl}=S_{jikl}=S_{ijlk}$ and $S_{ijil}=S_{ijkj}=0$. This provides a relatively straightforward procedure for calculating ℓ which we now describe.

We consider solutions of the form

$$\Gamma(\mathbf{x})=r^{\sigma-1}f_\sigma(\theta), \tag{3.4}$$

where (r,θ) are polar coordinates, and σ is a parameter to be determined. Introducing into (3.3) leads to an ordinary differential equation for $f_\sigma(\theta)$:

$$\mathcal{T}_\sigma f_\sigma(\theta):=S_{ijkl}x_kx_lr^{1-\sigma}\partial^2_{ij}[r^{\sigma-1}f_\sigma(\theta)]=\lambda f_\sigma(\theta). \tag{3.5}$$

With the periodicity condition for $f_\sigma(\theta)$, this is an eigenvalue problem whose solution provides a relationship between λ and σ. Let $\lambda=\Lambda(\sigma)$ denote the branch of this relationship corresponding the smallest value of λ at given $\sigma\in\mathbb{R}$. Then, ℓ is obtained as the turning value of λ (since larger values correspond to at least one complex value of σ and thus to an oscillatory solution). Denoting the turning point by σ_*, we have a first characterization of ℓ,

$$\ell=\Lambda(\sigma_*) \quad \text{with } \Lambda'(\sigma_*)=0. \tag{3.6}$$

A second characterization is deduced from the fact, established in Appendix A, that $\Lambda(\sigma)$ is an even function. It follows that $\sigma_*=0$, and therefore that $\Lambda'(0)=0$ and

$$\ell=\Lambda(0). \tag{3.7}$$

In other words, ℓ is the smallest eigenvalue of $\mathcal{T}_0$.

The continuous spectrum $[\ell,\infty)$ of $\mathcal{L}_0$ is relevant to the scalar-decay problem because it is the limit as $\kappa\to 0$ of (part of) the discrete spectrum of $\mathcal{L}_\kappa$. In fact, for our purpose it is best to interpret the continuous spectrum as such a limit and regard the points in this spectrum as approximate eigenvalues. The corresponding approximate eigenfunctions can then be viewed as the nonsmooth limits of the (necessarily smooth) eigenfunctions of $\mathcal{L}_\kappa$ with $\kappa\neq 0$. In Appendix B, we use matched asymptotics to shed some light on the limiting process. We show, in particular, that $\mathcal{L}_\kappa$ has an eigenvalue near ℓ given by

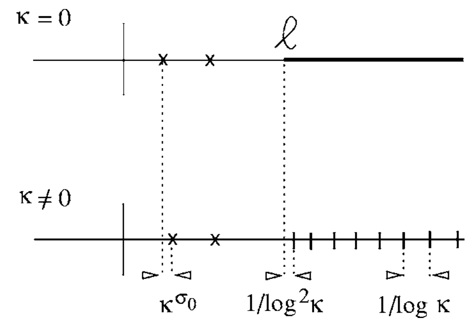

FIG. 1. Schematic of the spectra of $\mathcal{L}_0$ and $\mathcal{L}_\kappa$ both for flows the Kraichnan limit and for renewing flows. For $\kappa=0$, the spectrum of $\mathcal{L}_0$ consists of the continuous part $[\ell,\infty)$ and, possibly, of discrete eigenvalues ($\times$). For $\kappa\neq 0$, discrete eigenvalues (|) replace the continuous spectrum. The scaling of distances between points in the two spectra as $\kappa\to 0$ is indicated.

$$\lambda=\ell-\frac{2\pi^2\Lambda''(0)}{\log^2\kappa}+o(1/\log^2\kappa) \quad \text{as } \kappa\to 0. \tag{3.8}$$

We also show that to the right of ℓ, $\mathcal{L}_\kappa$ has an infinite number of eigenvalues separated by gaps whose size scales like $1/\log\kappa$. The continuous spectrum of $\mathcal{L}_0$ and the corresponding part of the spectrum of $\mathcal{L}_\kappa$ are therefore as illustrated in Fig. 1. [A result equivalent to (3.8) has previously been deduced for the growth rate of the magnetic field in the corresponding dynamo problem by Schekochihin *et al.*,[29] but making the assumption of spatial isotropy of the flow statistics, which reduces $\mathcal{L}_\kappa$ to an ordinary differential operator.]

The eigenvalue (3.8) is the smallest eigenvalue that emerges from the continuous spectrum for nonzero κ. Potentially, it can be identified with the variance decay rate γ_2; whether this is the case (and hence whether γ_2 is controlled locally) depends on whether $\mathcal{L}_\kappa$ has other, globally controlled, smaller eigenvalues, not associated with the continuous spectrum of $\mathcal{L}_0$. The existence of such eigenvalues is discussed in the next section.

B. Discrete spectrum of $\mathcal{L}_0$: Global control

In addition to its continuous part $[\ell,\infty)$, the spectrum of $\mathcal{L}_0$ can have a discrete part, with isolated eigenvalues $\lambda<\ell$ that depend on the global structure of $\mathcal{L}_0$ and on the domain period P. As mentioned before, it is clear, in particular, that for large P such isolated eigenvalues always exist: in this limit, homogenization theory can be used to approximate $\mathcal{L}_0$ by its spatial average, and deduce that $\mathcal{L}_0$ has eigenvalues that are $O(1/P^2)$ as $P\to\infty$.

To define the eigenvalue problem for $\mathcal{L}_0$ properly, the behavior of potential eigenfunctions at the origin needs to be specified. This is best done by requiring that the eigenfunctions of $\mathcal{L}_0$ be the limit as $\kappa\to 0$ of eigenfunctions of $\mathcal{L}_\kappa$. Matched asymptotic calculations detailed in Appendix B show that this is the case if they satisfy

$$\lim_{r\to 0} r\Gamma = 0. \tag{3.9}$$

Together with the periodicity of Γ, (3.9) specifies the eigenvalue problem for $\mathcal{L}_0$ completely. If this problem admits eigenvalues, λ_0 say, with $0<\lambda_0<\ell$, the corresponding eigenfunctions behave like $\Gamma \propto r^{-1+\sigma_0} f_{\sigma_0}(\theta)$, where σ_0 is the positive solution of $\Lambda(\sigma_0)=\lambda_0$. We show in Appendix B that $\mathcal{L}_\kappa$ then has eigenvalues near each λ_0 given by

$$\lambda = \lambda_0 + c\kappa^{\sigma_0} + o(\kappa^{\sigma_0}) \quad \text{as } \kappa \to 0, \tag{3.10}$$

for some constant c (see Fig. 1). The smallest of these then gives the variance decay rate γ_2.

C. Connection with Lagrangian stretching theories

We have shown that the variance decay rate γ_2 is given by (3.10) if a discrete eigenvalue $\lambda_0<\ell$ of $\mathcal{L}_0$ exists, or by (3.8) if not. We now relate these results to the Lagrangian prediction (2.6). Our main result is that

$$\ell = G(0) = -F(-1). \tag{3.11}$$

Thus, the Lagrangian stretching theory predicts the correct variance decay rate (in the limit $\kappa\to 0$), provided that there are no discrete eigenvalues $\lambda_0<\ell$. Furthermore the predicted decay rate lies precisely on the boundary, corresponding to $F'(-1)=0$, between the regimes of validity of the two expressions (2.6) given by the theory.

The key to deriving (3.11) is the close connection that exists between the covariance equation (3.1) and the Fokker-Planck equation for the line elements $\mathbf{d}$, which evolve according to (2.1).[18,24] In the Kraichnan limit, $\nabla \mathbf{v}$ is a white noise and the corresponding Fokker-Planck equation for $p(\mathbf{d};t)$ follows as

$$\partial_t p = -S_{ijkl} d_k d_l \partial^2_{ij} p. \tag{3.12}$$

This is identical to the covariance equation (3.1) in the local limit $\mathbf{x}\to 0$ and for $\kappa=0$. This connection leads to a direct relation between the lower bound of the continuous spectrum ℓ on the one hand, and the distribution of finite-time Liapunov exponents as described by $G(h)$ or $F(m)$ on the other. To see this, we look for an asymptotic solution of (3.12) of the large-deviation form

$$p(\mathbf{d},t) \sim \frac{1}{d^2 t} e^{-G(h)t} q(\theta;h,t), \tag{3.13}$$

consistent with (2.3). Here, $\mathbf{d}=(d\cos\theta, d\sin\theta)$, and q is assumed to depend weakly on h and t. Introducing (3.13) into (3.12) leads, at leading order, to

$$[hG'(h) - G]q \sim -S_{ijkl} d_k d_l d^{2+G'(h)} \partial^2_{ij} \{ d^{-[2+G'(h)]} q \}.$$

This differential equation, which governs the θ dependence of q, is identical to (3.5), with the correspondence

$$\lambda = G(h) - hG'(h) \quad \text{and } \sigma = -[1+G'(h)]. \tag{3.14}$$

This relates the Cramer function $G(h)$ to the function $\lambda=\Lambda(\sigma)$ which can be obtained by solving the eigenvalue problem (3.5). In fact, up to a shift and a change of sign, the function $\lambda=\Lambda(\sigma)$ is nothing other than the Legendre transform of $G(h)$, i.e., the free energy $F(m)$:

$$F(m) = -\Lambda(-m-1). \tag{3.15}$$

Now, differentiating (3.14), we find that

$$\frac{d\lambda}{dh} = -hG''(h) = \Lambda'(\sigma)\frac{d\sigma}{dh} = -\Lambda'(\sigma)G''(h),$$

and hence that $h=\Lambda'(\sigma)$. Noting that, by definition, ℓ is the maximum value of $\Lambda(\sigma)$, it follows from that $\ell=G(0)$ as announced. The evenness of $\Lambda(\sigma)$ can then be used: as we have seen, it implies that $\ell=\Lambda(0)$ and hence, using (3.15), that $\ell=-F(-1)$. This completes the proof of (3.11).

We remark that the evenness of $\Lambda(\sigma)$, on which the second equality in (3.11) rests, also implies that

$$F(m) = F(-m-2). \tag{3.16}$$

In view of (2.5), this reflects an interesting property of the line-element dynamics: the moments $\langle d^m\rangle$ and $\langle d^{-m-2}\rangle$ grow at the same rate. In the Discussion we examine how this property generalizes for flows in dimensions higher than 2 and with finite correlation time.

D. An example

In this section, we illustrate our theoretical results for a particular flow. We choose the flow to be a limiting form of the alternating sinusoidal shear flow considered by several previous authors, which will be used in Sec. IV. We take the streamfunction

$$\psi = a[\cos(x+\phi_1(t)) + \cos(y+\phi_2(t))], \tag{3.17}$$

where a is a constant, and $\phi_1(t)$ and $\phi_2(t)$ are independent random functions identically and uniformly distributed in $[0,2\pi]$ with specified correlation time τ_a, say. It follows that $a^2\langle\cos\phi_1(t)\cos\phi_1(t')\rangle = a^2 f(t/\tau_a)$, where the function f satisfies $f(0)=1$ and $f(s)\to 0$ as $s\to\infty$. Hence a suitable limit $\tau_a\to 0$, $a\to\infty$ is possible with $a^2\tau_a$ fixed and $a^2\langle\cos\phi_1(t)\cos\phi_1(t')\rangle \to 2\delta(t-t')$. Under these choices, (3.2) becomes

$$\mathcal{L}_\kappa\Gamma = -2\kappa\Delta\Gamma - 4[\sin^2(y/2)\partial^2_{xx} + \sin^2(x/2)\partial^2_{yy}]\Gamma = \lambda\Gamma, \tag{3.18}$$

and its local version (3.3) becomes

$$-(y^2\partial^2_{xx} + x^2\partial^2_{yy})\Gamma = \lambda\Gamma. \tag{3.19}$$

Equation (3.5) defining the relationship $\lambda=\Lambda(\sigma)$ takes a convenient form when the substitution

$$f_\sigma(\theta) = (3+\cos\chi)^{\sigma/4} g_\sigma(\chi) \quad \text{with } \chi = 4\theta,$$

is performed. This leads to the self-adjoint eigenvalue problem

$$[(3+\cos\chi)g'_\sigma]' + \left[\frac{\sigma^2(1-\cos\chi)}{8(3+\cos\chi)} - \frac{3\cos\chi+1}{16}\right] g_\sigma = -\lambda g_\sigma, \tag{3.20}$$

where g_σ is 2π periodic. This can be solved numerically for fixed σ, e.g., using a shooting method, leading to the relation $\lambda=\Lambda(\sigma)$ displayed in Fig. 2. As expected, $\Lambda(\sigma)$ is even, with $\Lambda(1)=\Lambda(-1)=0$; it is not parabolic, as would be the case for

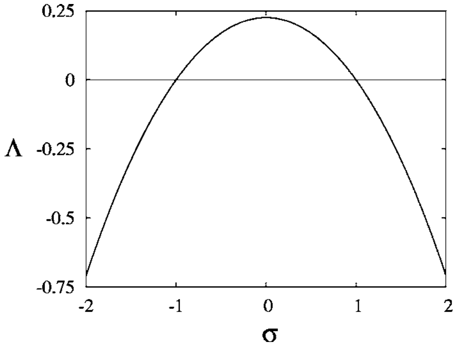

FIG. 2. Function $\Lambda(\sigma)$ for the flow with streamfunction given in (3.17) in the Kraichnan limit.

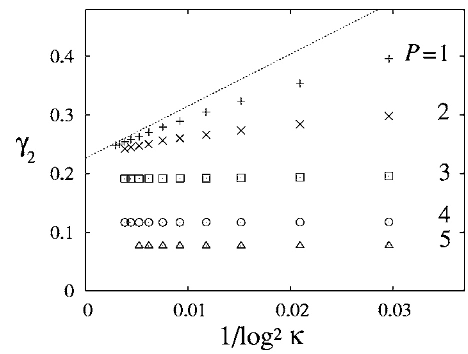

FIG. 3. Decay rate γ_2 as a function of $1/\log^2 \kappa$ for the flow with streamfunction (3.17) in the Kraichnan limit. Numerical results obtained for the domain sizes $P=1$ (+), 2 (×), 3 (□), 4 (○), and 5 (△) are compared with the asymptotic formula (3.8) for the locally controlled decay rate (straight line).

an isotropic flow, but it is nevertheless very well approximated by a parabola for $-1<\sigma<1$. For $\sigma=0$, we find

$$\ell = \Lambda(0) = 0.22578138 \cdots .$$

Estimating the second derivative $\Lambda''(0) \approx 2\Lambda(0) \approx 0.45$, we derive from (3.8) the estimate

$$\gamma_2 = 0.226 + 8.88/\log^2 \kappa + o(1/\log^2 \kappa) \tag{3.21}$$

for the variance decay rate.

This is only a relevant estimate if $\mathcal{L}_0$ does not have discrete eigenvalues $\lambda_0 < \ell$. For the operator in (3.18), it is easy to show that there are such eigenvalues when the domain size satisfies $P \geqslant 3$. This follows from the upper bound on the lowest eigenvalue of $\mathcal{L}_0$ afforded by the Rayleigh quotient:

$$\begin{aligned}\lambda_0 &\leqslant \int\int u\mathcal{L}_0 u dxdy \Big/ \int\int u^2 dxdy \\ &= 4\int\int [\sin^2(y/2)(\partial_x u)^2 \\ &\quad + \sin^2(x/2)(\partial_y u)^2] dxdy \Big/ \int\int u^2 dxdy,\end{aligned}$$

where the test function $u(x,y)$ integrates to zero over the periodic domain $2\pi P \times 2\pi P$. Taking $u(x,y)=\cos(x/P)$ leads to the bound

$$\lambda_0 \leqslant 2/P^2,$$

which is smaller than ℓ for $P \geqslant 3$. Note also that in the limit of large P, homogenization can be used to approximate $\mathcal{L}_\kappa$ by $-2(\kappa+1)\Delta$, leading to the globally controlled decay rate $\gamma_2 \sim 2(\kappa+1)/P^2$ as $P \to \infty$.

We have solved the eigenvalue problem (3.18) numerically. The smallest eigenvalue, i.e., the decay rate γ_2, obtained for several values of P is shown in Fig. 3 as a function of $1/\log^2 \kappa$ in accordance with asymptotic formula (3.8) for the locally controlled decay rate. For $P=1$ and $P=2$, the results are consistent with a local control of the decay, and the predicted slow approach to the asymptotic value ℓ. For $P=1$, the numerical points convincingly line up with the straight line predicted by (3.8) as $\kappa \to 0$. The smallest values of κ that we were able to achieve were too large to demonstrate a comparable agreement for $P=2$, although the convergence to ℓ as $\kappa \to 0$ can reasonably be inferred in this case too.

For $P \geqslant 3$, the decay is globally controlled and the approach of γ_2 to the asymptotic value λ_0 is much more rapid than the $1/\log^2 \kappa$ dependence of the locally controlled case, in agreement with the asymptotic result (3.10). The validity of this result is demonstrated in Fig. 4 which shows γ_2 for $P=3$, 4, and 5 as a function of κ^{σ_0}. For each P, the value of σ_0 has been estimated by extrapolating λ_0 from the numerical results, then solving $\Lambda(\sigma_0)=\lambda_0$. The solutions are $\sigma_0=0.40$, 0.70, and 0.81 for $P=3$, 4, and 5, respectively. The linear dependence of γ_2 on κ^{σ_0} predicted by (3.10) is clear on the figure. Note also that as P increases, γ_2 for $\kappa=0$ tends to the limit $2/P^2$ given by homogenization theory.

IV. RENEWING FLOWS

We now examine how the results obtained in the previous sections for the Kraichnan limit extend to finite-correlation-time flows. As in previous sections we continue

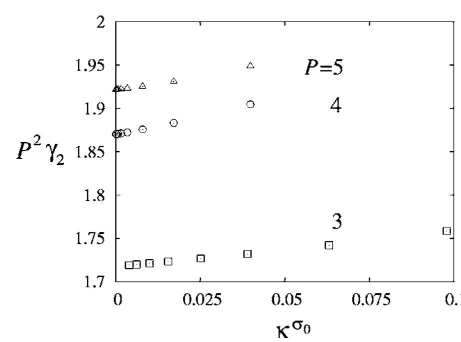

FIG. 4. Scaled decay rate $P^2\gamma_2$ as a function of κ^{σ_0} for the flow with streamfunction (3.17) in the Kraichnan limit. The numerical results obtained for the domain sizes $P=3$ (□, $\sigma_0=0.40$), 4 (○, $\sigma_0=0.70$), and 5 (△, $\sigma_0=0.81$) are shown to be consistent with the linear dependence on κ_0^σ predicted by the asymptotic formula (3.10) for the globally controlled decay rate. Homogenization theory predicts that $P^2\gamma_2 \sim 2$ as $P \to \infty$.

to consider flows that are spatially homogeneous. We extend the time behavior to flows that become completely uncorrelated after some finite time τ; more precisely, the velocity fields in each time interval of length τ are assumed to be independent, identically distributed processes. Such flows have been considered previously in the dynamo literature, first by Zel'dovich *et al.*,[23] who refer to the associated random process as an "innovating process." Childress and Gilbert[21] review relevant dynamo results and use the alternative terms "renovating" or "renewing flows." We use the latter term.

The advantage of the assumption of a renewing flow is that it leads to a closed dynamics for the covariance $\Gamma(\mathbf{x},t)$, as is also the case in the Kraichnan limit. In this present case, this dynamics is governed by the time-τ map of the variance which we now derive. Let $G_\kappa(\mathbf{x},\mathbf{x}',t)$ denote the Green's function of the advection-diffusion equation (1.1). We relate $\Gamma(\mathbf{x},\tau)$ to $\Gamma(\mathbf{x},0)$ as follows:

$$\Gamma(\mathbf{x},\tau) = \left\langle \int\int G_\kappa(\mathbf{x},\mathbf{x}_1,\tau)G_\kappa(\mathbf{0},\mathbf{x}_2,\tau)C(\mathbf{x}_1,0)C(\mathbf{x}_2,0)d\mathbf{x}_1 d\mathbf{x}_2 \right\rangle = \int\int \langle G_\kappa(\mathbf{x},\mathbf{x}_1,\tau)G_\kappa(\mathbf{0},\mathbf{x}_2,\tau)\rangle \Gamma(\mathbf{x}_1-\mathbf{x}_{2,0})d\mathbf{x}_1 d\mathbf{x}_2,$$

where we have used spatial homogeneity and the statistical independence between $G_\kappa(\cdot,\cdot,\tau)$ and $C(\cdot,0)$ which follows from the assumed statistical independence of the velocity field between different time intervals. Defining

$$\mathcal{G}_\kappa(\mathbf{x},\mathbf{x}',\tau) = \int \langle G_\kappa(\mathbf{x},\mathbf{x}'+\mathbf{y},\tau)G_\kappa(\mathbf{0},\mathbf{y},\tau)\rangle d\mathbf{y}, \qquad (4.1)$$

we obtain a linear map between $\Gamma(\mathbf{x},\tau)$ and $\Gamma(\mathbf{x},0)$ in the form

$$\Gamma(\mathbf{x},\tau) = \int \mathcal{G}_\kappa(\mathbf{x},\mathbf{x}',\tau)\Gamma(\mathbf{x}',0)d\mathbf{x}'.$$

The large-time decay of the tracer covariance and, in particular, of the variance, are clearly controlled by the spectrum of this map. Using ν as the spectral parameter, we consider the eigenvalue problem

$$\mathcal{L}_\kappa\Gamma(\mathbf{x}) := \int \mathcal{G}_\kappa(\mathbf{x},\mathbf{x}',\tau)\Gamma(\mathbf{x}')d\mathbf{x}' = \nu\Gamma(\mathbf{x}). \qquad (4.2)$$

Again, the boundary conditions impose that $\Gamma(\mathbf{x})$ be $2\pi P \times 2\pi P$ periodic. The variance decay rate γ_2 is related to the largest value of ν according to $\tau\gamma_2 = -\log \nu$.

In the Kraichnan case, the eigenvalue problem analogous to (4.2), i.e., (3.2), is a straightforward (elliptic) differential eigenvalue problem. There is a well-developed theory for the spectrum of such operators, and we relied, in particular, on the established relationship between the location of the continuous spectrum and the oscillatory nature of solutions. Here, in contrast, we need to determine the spectrum of an integral operator. We are not aware of theoretical results analogous to those we invoked in the Kraichnan case, although such results can be expected to extend to integral operators.

However, we showed in the Kraichnan case that interpreting the spectrum of the singular operator $\mathcal{L}_0$ as the limit of the spectrum of $\mathcal{L}_\kappa$ as $\kappa \to 0$ made the reliance on the established theory of singular differential operators inessential. We will use this interpretation for renewing flows and we will show that the qualitative properties of the spectrum deduced in the Kraichnan case continue to hold for these flows. Specifically, the variance decay rate can be either locally controlled and given by (3.8), or globally controlled and given by (3.10). The equalities (3.11) relating the predictions of Lagrangian stretching theories to the locally controlled decay rate also continue to hold.

A. Local control and global control

For $\kappa=0$, the Green's function $G_\kappa(\mathbf{x},\mathbf{x}',\tau)$ reduces to

$$G_0(\mathbf{x},\mathbf{x}',\tau) = \delta(\mathbf{x}-\phi_\tau\mathbf{x}'),$$

where ϕ_τ denotes the time-τ flow of the velocity field. Introducing into (4.1), and using incompressibility in the form $\det \nabla\phi_\tau = 1$ leads to

$$\mathcal{G}_0(\mathbf{x},\mathbf{x}') = \langle \delta(\mathbf{x}' - \phi_{-\tau}\mathbf{x} + \phi_{-\tau}\mathbf{0})\rangle. \qquad (4.3)$$

Note that the function $\mathcal{G}_0(\mathbf{x},\mathbf{x}')$ is singular [and equal to $\delta(\mathbf{x})$] at $\mathbf{x}'=0$. It is this singularity that allows the possibility of a continuous spectrum for $\mathcal{L}_0$.

We now consider the limit $\mathbf{x}\to 0$ which controls the continuous spectrum. In this limit,

$$\phi_{-\tau}\mathbf{x} - \phi_{-\tau}\mathbf{0} \simeq S_{-\tau}\mathbf{x},$$

where

$$S_\mu = \nabla\phi_\mu|_{\mathbf{x}=0}$$

is a random matrix with $\det S_\mu = 1$, and hence $\mathcal{G}_0(\mathbf{x},\mathbf{x}',\tau) \simeq \langle \delta(\mathbf{x}' - S_{-\tau}\mathbf{x})\rangle$.

The spectral problem (4.2) thus reduces to

$$\langle \Gamma(S_{-\tau}\mathbf{x})\rangle = \nu\Gamma(\mathbf{x}). \qquad (4.4)$$

(Note that the operator on the left-hand side is self-adjoint if $S_{-\tau}$ and $S_{-\tau}^{-1}$ have the same statistics, i.e., if the random flow is time reversible.) The scale invariance of this problem suggests considering solutions of the form

$$\Gamma(\mathbf{x}) = r^{\sigma-1}f_\sigma(\theta),$$

where f_σ is 2π periodic. This leads to the one-dimensional eigenvalue problem

$$\mathcal{T}_\sigma f_\sigma(\theta) := \langle r'^{\sigma-1} f_\sigma(\theta')\rangle = \nu f_\sigma(\theta), \tag{4.5}$$

where r' and θ' are functions of θ given by

$$r' \begin{pmatrix} \cos\theta' \\ \sin\theta' \end{pmatrix} = S_{-\tau} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}. \tag{4.6}$$

The eigenvalue problem (4.5) defines a relation between ν and σ. For fixed σ, the branch corresponding to the largest value of ν is of interest; we denote it by

$$\nu = N(\sigma). \tag{4.7}$$

Correspondingly, we can define a function $\Lambda(\sigma)$ by

$$\Lambda(\sigma) = -\frac{1}{\tau}\log N(\sigma),$$

which generalizes that defined in the Kraichnan limit. At this stage, the reasoning performed in the Kraichnan limit can be repeated: in terms of $\lambda = -\log(\nu)/\tau$, the lower limit ℓ to the continuous spectrum is given by $\Lambda(\sigma_*)$, where $\Lambda'(\sigma_*)=0$. We show in Appendix A that $\Lambda(\sigma)$ is again an even function; hence the characterization (3.7) of ℓ continues to hold.

For $\kappa \neq 0$, the continuous spectrum $[\ell,\infty)$ turns into a discrete spectrum, with gaps that become vanishingly small as $\kappa \to 0$. The matched-asymptotic calculation reported in Appendix B, which allowed us to describe this in the Kraichnan limit, may be extended to apply to the renewing flows of this section. Details are given in Appendix C.

The important point is that the leading-order solution is determined by the solution of the local eigenvalue problem [(3.5) in the Kraichnan limit; (4.5) in the general case] and other details of the operator $\mathcal{L}_\sigma$ are important only in determining higher-order corrections. Thus, the approximation (3.8) for the smallest eigenvalue of $\mathcal{L}_\kappa$ emerging from the continuous spectrum of $\mathcal{L}_0$ remains valid. Again, this eigenvalue is a useful approximation to the variance decay rate only if there are no discrete eigenvalues $\lambda_0 < \ell$ of $\mathcal{L}_0$. If there are such eigenvalues, with the smallest being λ_0 then the decay rate is approximated by (3.10), with σ_0 defined by $\Lambda(\sigma_0) = \log N(\sigma_0)/\tau = \lambda_0$. Continuity with the Kraichnan case and with corresponding homogenization results in the non-Kraichnan case make it clear that such isolated eigenvalues can exist for renewing flows.

B. Connection with Lagrangian stretching theories

The connection discussed in Sec. III C between $\Lambda(\sigma)$ and the Cramer function $G(h)$ for the stretching factors h [and hence with the free energy of the Liapunov exponent $F(m)$] extends from the Kraichnan limit to renewing flows. To see this, we note that line elements $\mathbf{d}(t)$ satisfy

$$\mathbf{d}(n\tau) = \nabla\phi_\tau \mathbf{d}((n-1)\tau) = S_{-\tau}^{-1}\mathbf{d}((n-1)\tau).$$

Correspondingly, their probability density function obeys the recurrence relation

$$p(\mathbf{d}, n\tau) = \int \langle \delta(S_{-\tau}\mathbf{d} - \mathbf{d}')\rangle p(\mathbf{d}', (n-1)\tau) d\mathbf{d}'.$$

Introducing the large-deviation form (3.13) with $t = n\tau$ gives at leading order

$$q(\theta; h) = e^{\tau[G(h) - hG'(h)]} \int q(\theta'; h')(d'/d)^{-[2+G'(h)]} \times \langle \delta(S_{-\tau}\mathbf{d} - \mathbf{d}')\rangle d\mathbf{d}'.$$

Rewriting this expression as

$$\langle (d'/d)^{-[2+G'(h)]} q(\theta'; h)\rangle = e^{-\tau[G(h) - hG'(h)]} q(\theta; h),$$

where

$$d' \begin{pmatrix} \cos\theta' \\ \sin\theta' \end{pmatrix} = d S_{-\tau} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix},$$

we recognize that q satisfies the same eigenvalue problem (4.5) as f_σ, with the relations

$$\sigma = -[1 + G'(h)] \quad \text{and} \quad \nu = \exp\{-\tau[G(h) - hG'(h)]\}.$$

In terms of $\lambda = -\log(\nu)/\tau$, we thus extend (3.14) from the Kraichnan case to the more general case of renewing flows. It follows that (3.15) and (3.16) remain valid. Therefore the equalities (3.11) which encapsulate the connection between Lagrangian stretching theories and the spectrum of $\mathcal{L}_0$ (and hence the decay rate γ_2 in the locally controlled case) extend to renewing flows.

C. An example

As an example, we consider the alternating sine flow whose velocity field for $t \in [0,\tau)$ is given by

$$u = \begin{cases} U\sin(y + \phi_1) & \text{for } 0 \leqslant t < \tau/2, \\ 0 & \text{for } \tau/2 \leqslant t < \tau, \end{cases}$$
$$v = \begin{cases} 0 & \text{for } 0 \leqslant t < \tau/2, \\ U\sin(x + \phi_2) & \text{for } \tau/2 \leqslant t < \tau, \end{cases} \tag{4.8}$$

where ϕ_1 and ϕ_2 are independent random phases uniformly distributed in $[0, 2\pi]$. The flow in successive time intervals $[n\tau, (n+1)\tau), n = 1, 2, \ldots$, is of the same form, with phases independent from those in previous time intervals. The decay of a scalar in this type of flow has been studied in many of the papers already referenced.

For this flow, the time-τ map ϕ_τ is given by

$$\phi_\tau \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + \alpha\sin(y + \phi_1) \\ y + \alpha\sin[x + \alpha\sin(y + \phi_1) + \phi_2] \end{pmatrix}, \tag{4.9}$$

where $\alpha := U\tau/2$. Correspondingly,

$$S_{-\tau} = \nabla\phi_{-\tau}|_{\mathbf{x}=\mathbf{0}} = (\nabla\phi_\tau|_{\mathbf{x}=\phi_{-\tau}\mathbf{0}})^{-1} = \begin{pmatrix} 1 + \alpha^2\cos(\phi_1 - \alpha\sin\phi_2)\cos\phi_2 & -\alpha\cos(\phi_1 - \alpha\sin\phi_2) \\ -\alpha\cos(\phi_2) & 1 \end{pmatrix}.$$

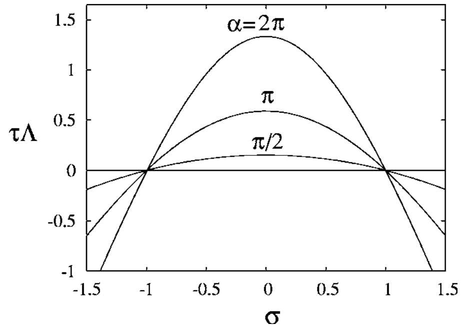

FIG. 5. Function $\Lambda(\sigma)$ for the alternating sine flow defined by (4.8), for different values of α.

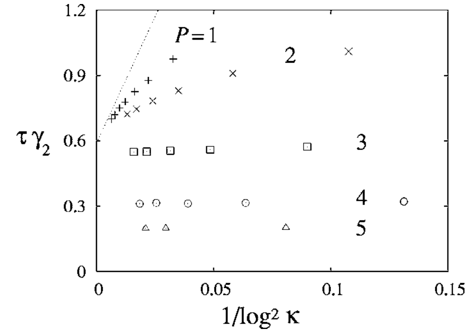

FIG. 6. Decay rate γ_2 as a function of $1/\log^2 \kappa$ for the alternating sine flow defined by (4.8), with $\alpha=\pi$. Numerical results obtained for the domain sizes $P=1$ (+), 2 (×), 3 (□), 4 (○), and 5 (△) are compared with the asymptotic formula (3.8) for the locally controlled decay rate (straight line).

For this ensemble of matrices $S_{-\tau}$, we have solved the eigenvalue problem numerically (4.5). The resulting curve $\nu = N(\sigma)$ or rather $\lambda = \Lambda(\sigma)$ is displayed for several values of α in Fig. 5.

We test our asymptotic results (3.8) and (3.10) against numerical simulations obtained not by solving Eq. (4.2) for the covariance, but by performing many realizations of the evolution of the scalar under a random flow and taking a suitable average of the results. The reason is that, while the kernel appearing in the integral operator could be evaluated numerically in principle, the memory required to store the information defining the kernel increases very rapidly with spatial resolution. Rather than solving the full advection-diffusion equation for the velocity field (4.8) in each time interval $[n\tau,(n+1)\tau)$, we follow Ref. 30 and alternate applications of the random map

$$C(\mathbf{x}) \mapsto C(\phi_{-\tau}\mathbf{x}),$$

corresponding to pure advection, with applications of the diffusion map

$$C(\mathbf{x}) \mapsto \exp(\kappa\tau\Delta)C(\mathbf{x}),$$

the latter being carried out efficiently in Fourier space. The asymptotic behaviors (3.8) and (3.10) for the decay rate γ_2 hold for these maps alternating advection and diffusion as well as for the maps corresponding to continuous advection diffusion.

The decay rate of the variance is shown as a function of diffusivity κ and spatial periodicity P in Fig. 6. The results shown are based on averages over 500 realizations for each values of κ and P and the value of α is taken to be π (cf. Fig. 5). Again the results for each P are shown as a function of $1/\log^2 \kappa$ in accordance with asymptotic formula (3.8) for the locally controlled decay rate. As in the Kraichnan case, for $P=1$ and $P=2$ the results are consistent with a local control of the decay and the predicted slow approach to the asymptotic value ℓ. Also as in the Kraichnan case, for $P=1$ the numerical points convincingly match the asympotic prediction (3.8), here given by

$$\tau\gamma_2 \approx 0.589 + 24.9/\log^2 \kappa,$$

as $\kappa \to 0$, and indicated by the straight line. Once again, the smallest values of κ that we were able to achieve were too large to demonstrate a comparable agreement for $P=2$, although the behavior shown is consistent with the asymptotic prediction. For $P=3,4,5$ the facts that the values of the decay rate are below the value predicted by the local theory and that the dependence on κ is weak are strong evidence for global control.

This evidence is strengthened when the form of the dependence on κ is examined more closely. Figure 7 shows γ_2 for $P=3$, 4, and 5 as a function of κ^{σ_0}. For each P, the value of σ_0 has been estimated by extrapolating λ_0 from the numerical results, then solving $\Lambda(\sigma_0)=\lambda_0$. The solutions are $\sigma_0=0.25$, 0.68, and 0.80 for $P=3$, 4, and 5, respectively. There is a good agreement with the linear dependence of γ_2 on κ^{σ} predicted by (3.10). Note also that as P increases, there is apparently a good agreement between the limiting value of γ_2 for $\kappa=0$ and the estimate $\pi^2/(2\tau P^2) = 4.94/(\tau P^2)$ given by a simple homogenization theory (which approximates the effective diffusivity by the mean-square displacement divided by τ).

V. DISCUSSION

In this paper we have analyzed the advective-diffusive decay of a scalar in a two-dimensional velocity field that is a stationary random function of time. For simplicity we have assumed that the flow is (statistically) spatially homogeneous. In principle the analysis could be repeated without

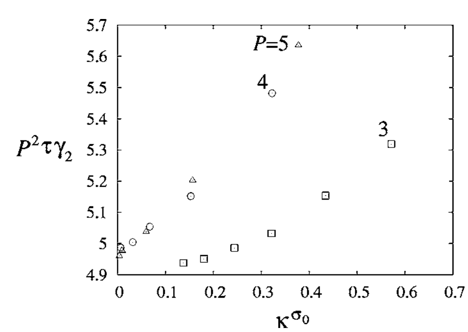

FIG. 7. Scaled decay rate $P^2\gamma_2$ as a function of κ^{σ_0} for for the alternating sine flow defined by (4.8), with $\alpha=\pi$. The numerical results obtained for the domain sizes $P=3$ (□, σ_0=0.25), 4 (○, σ_0=0.68), and 5 (△, σ_0=0.80) are shown to be consistent with the linear dependence on κ_0^{σ} predicted by the asymptotic formula (3.10) for the globally controlled decay rate. Homogenization theory predicts that $P^2\tau\gamma_2 \sim \pi^2/2$ as $P\to\infty$.

this assumption, but the number of space variables would double. We have also focused on the behavior of the spatial covariance $\Gamma(\mathbf{x},t)$. We have identified a crucial property of the operator $\mathcal{L}_0$ which is the limit as $\kappa \to 0$ of the operator $\mathcal{L}_\kappa$ describing the time evolution of $\Gamma(\mathbf{x},t)$.

If $\mathcal{L}_0$ has no isolated eigenvalues then, as $\kappa \to 0$, the leading eigenvalue ν of $\mathcal{L}_\kappa$ tends to ℓ, the upper limit of the continuous spectrum of $\mathcal{L}_0$, with the difference tending to zero as $(\log \kappa)^{-2}$. In this case the limiting value of the decay rate of the variance is precisely that predicted by the Lagrangian stretching theories of Antonsen *et al.*,[1] Son,[4] Balkovsky and Fouxon,[2] and others. We term this case "locally controlled decay." If $\mathcal{L}_0$ has isolated eigenvalues then, as $\kappa \to 0$, the leading eigenvalue of $\mathcal{L}_\kappa$ tends to the smallest isolated eigenvalue of $\mathcal{L}_0$, with the difference tending to zero as κ^σ, where $0<\sigma<1$. We call this case "globally controlled decay." σ may be determined from knowledge of the globally controlled decay rate and from the Lagrangian stretching properties, as encoded by the Cramer function $G(h)$ or equivalently the free energy $F(p)$.

Of course, whether or not the decay is locally or globally controlled can be determined only by global analysis of the operator $\mathcal{L}_0$.

The globally controlled regime is relevant, in particular, when the spatial scale of the scalar is significantly larger than the spatial scale of the velocity field (large P in our notation). The decay of the scalar is then well captured by homogenization theories and is a diffusive decay. Voth *et al.*[19] have noted the need for uniting such an approach with a stretching-based model that might apply when the spatial scale of the scalar is smaller. Our analysis has gone some way towards that: it shows how the variance decay rate predicted by the Lagrangian stretching theories fits within the framework provided by the idea of a strange eigenmode (itself simply a manifestation of the ergodic multiplicative theory of Oseledec[11] applied to the advection-diffusion equation) and gives a correct estimate in the locally controlled case.

Note that this agreement in the predictions of the variance decay rate does not contradict the conclusions of Sukhatme and Pierrehumbert[9] and Fereday and Haynes[7] concerning the predictions about higher-order spatial moments of some of the Lagrangian stretching theories.[2,4] [By spatial moments, we mean the L^p norms $\overline{C^p}$, where the average — is over space but not realizations of the random flow; see comment (vi) below.] Because at sufficiently large times the decay is dominated by the slowest decaying strange eigenmode, the decay rate of the spatial moments $\overline{C^p}$ will be linear in the moment index p and, equivalently, the probability density function of the scalar concentration approaches a self-similar form. The important point seems to be that the Lagrangian stretching theories are valid only for a finite time. But this allows those theories to represent the structure of the advection-diffusion operator acting over finite time and hence to approximate part of the spectrum of that operator.

We make further remarks.

(i) Our results are derived first under the short-correlation-time approximation (the Kraichnan-Kazantsev regime) and then under the assumption of renewing flows. Childress and Gilbert[21] note, in the context of the dynamo problem, that there appears to be *no* evidence that renewing flows are qualitatively different from more general classes of random flows with finite correlation times. One way to prove this might be to approximate a given random flow by a renewing flow with time interval τ. If the correlations of the given flow decay sufficiently fast, then any statistical measure of the given flow (or its advective-diffusive action on a scalar) should be approximated arbitrarily well by taking τ large enough. Therefore we expect our results to apply to a large class of random flows.

(ii) A mechanistic description of the difference between locally controlled and globally controlled cases is as follows. Consider the action of the advection-diffusion operator over some finite (but sufficiently large) time, as described by Fereday and Haynes.[7] In a first regime described by a local picture, an initially localized scalar anomaly is stretched out to the scale of the flow (this covers stages I and II of Fereday and Haynes[7]). In a second regime (stages III and IV), the global structure of the flow is important, and the anomaly is dispersed across the flow domain by repeated stretching and folding on the flow scale. The long-time decay is determined by the repeated action of the operator, and either the local process (first regime) or the global process (second regime) may be rate determining.

(iii) The slow $(\log \kappa)^{-2}$ approach to the limiting value of decay rate in the locally controlled case explains the difficulty of direct numerical verification or falsification of the Lagrangian stretching prediction of variance decay.[1,5,7] The much weaker κ^σ dependence of decay rate on diffusivity in the globally controlled case is consistent with the argument that it is the global dispersion properties of the flow that are rate controlling in that case. As noted previously, similar dependence of the small-scale dynamo growth rate on magnetic diffusivity in the locally controlled case has been found by Schekochihin *et al.*,[29] who note the possibility of global control as an alternative. Presumably, as for the scalar decay problem, a globally controlled kinematic dynamo would exhibit a much weaker dependence on diffusivity than the locally controlled case.

(iv) In the specific examples that we have considered (alternating sinusoidal shear flows) global control appears only when the scale of spatial periodicity of the scalar is larger than that of the flow—three times larger ($P=3$). However, it is straightforward to show that there are flows for which global control occurs when the spatial periodicities of scalar and flow are the same. A trivial example is generated by modifying the definition of the flow given in Sec. IV C, replacing $\sin(\cdot)$ by $\epsilon \sin(\cdot)+(1-\epsilon)\sin 3(\cdot)$. The decay in such a flow will be globally controlled for small ϵ but locally controlled for ϵ near 1. The transition between the two regimes corresponds to coalescence of the isolated discrete eigenvalue that exists for small ϵ with the continuous spectrum. In the limit $\kappa \to 0$, the decay rate γ_2 is a continuous function of ϵ with discontinuous derivative.

(v) Our analysis, so far limited to two dimension (2D), shows that when the (global) eigenvalue problem for the variance decay is locally controlled, the predicted decay rate, equal to the lower limit ℓ of the continuous spectrum, is $\ell = G(0) = -F(-1)$. We emphasize that the first equality is general and holds both for Kraichnan and renewing flows in any dimension. The second equality, by contrast, generalizes in the case of arbitrary dimension D to $G(0) = -F(-D/2)$, but only for certain flows (which include the Kraichnan case). To see this, let us briefly describe how our analysis extends to the D-dimensional case. We first note that the solutions of the local eigenvalue problem should be sought in the form

$$\Gamma(\mathbf{x}) = r^{\sigma - D/2} f_\sigma(\mathbf{n}),$$

where $\mathbf{n}$ denotes a generalized angle. The corresponding function $\Lambda(\sigma)$ is then related to the free energy $F(m)$ according to

$$F(m) = -\Lambda(-m - D/2), \tag{5.1}$$

cf. (3.15). The general result $\ell = G(0)$ follows from this and from the characterization (3.6) (which holds for Kraichan and renewing flows in any dimension). The generalization of $\ell = -F(-1)$, to dimension D, on the other hand, reads $\ell = -F(-D/2)$ and follows from (5.1) only if $\Lambda(\sigma)$ is even or, equivalently, if $F(m) = F(-m-D)$. This always holds true in the Kraichan limit as a direct extension of the proof in Appendix A 1 shows. For renewing flows, however, it only holds if the stretching statistics are time reversible (see Appendix A 2). This reversibility is guaranteed in 2D, because incompressibility implies that the matrices S_τ and S_τ^{-1} have the same eigenvalues, but not in higher dimensions.

(vi) We have considered the decay rate of the ensemble-averaged variance $\langle C^2 \rangle$. It has been previously noted, again in the context of the kinematic dynamo problem, that this need not be the decay rate of the (spatially averaged) variance $\overline{C^2}$ that is observed over long times in any single realization of the flow (e.g., Ref. 21 and references therein). The problem here is analogous to that of the stretching rates of line elements. While there is a well-defined Liapunov exponent $\bar{h}$, to which the stretching rate of a line element tends at large times in any single flow realization, the manner in which the probability density function for finite-time stretching factors approaches its limiting form as $t \to \infty$ implies, for example, that ensemble-averaged second moments of lengths $\langle d^2 \rangle$ does not increase asymptotically at twice the Liapunov exponent. The difference between ensemble-averaged behavior and single-realization behavior (the intermittency) is encoded in the line-element Cramer function $G(h)$, or equivalently in the free energy $F(m)$: specifically, the growth rate of the ensemble-averaged mth moment $\langle d^m \rangle$ is $F(m)$, that of d^m in a single realization is $m\bar{h}$, and generally $F(m) \neq m\bar{h}$. As noted previously for the advection-diffusion problem we expect a well-defined Liapunov exponent, with the decay of the scalar field matching that in the large-time limit in any single flow realization. But we also expect that the decay rates estimated after a finite time, in other words the finite time Liapunov exponents for the advection-diffusion problem, have a nontrivial distribution. The large-time collapse of this distribution towards the (infinite-time) Liapunov exponent is presumably described by a Cramer function, and a free energy can be defined as its Legendre transform. This free energy gives the decay rate of arbitrary (ensemble-averaged) moments, and there is no obvious reason why it should be linear in the moment index. However, at present, numerical simulations for the flow of Sec. IV C over a wide range of κ and P show that the difference between the decay rate of the ensemble-averaged variance and twice the Liapunov exponent (for the scalar advection-diffusion problem) is small (less than 5%). Therefore, it appears that our results for the decay rate of the ensemble-averaged scalar variance also provide an estimate for the true Liapunov exponents of the advection-diffusion operator.

(vii) The covariance $\Gamma(\mathbf{x},t)$ gives information on the spatial structure of the scalar field as well as the decay rate. The Fourier transform of $\Gamma(\mathbf{x},t)$ implies a spatial power spectrum, but not necessarily a power spectrum that will be observed in any single realization, since the covariance is an average over all realizations. Nonetheless it is interesting to note that the fact that $\Gamma(\mathbf{x},t) \sim |\mathbf{x}|^{\sigma-1}$ implies that the wave-number power spectral (integrated over angle) density $E(k,t) \sim k^{-\sigma}$, where k is wavenumber. In the locally controlled case, $\sigma = 0$, and in the globally controlled case it is determined by $\gamma_2 = \Lambda(\sigma)$, i.e., by (3.14). This prediction for the power-law decay of the power spectrum is precisely that obtained by more heuristic means by Fereday and Haynes,[7] who show a good agreement with the power spectra evaluated from single-flow realizations (see also Ref. 5).

(viii) The predictions for probability density function for scalar concentration

$$\mathcal{P}(C,t) \sim C^{-\beta} \quad \text{with } \beta = 1 + \frac{2G(0)}{\gamma_2},$$

for $|C| \gg 1$ given by Fereday and Haynes[7] may be reinterpreted in the light of the locally versus globally controlled regimes that we have described here. In the locally controlled case we have shown that $\gamma_2 \simeq G(0)$ [with the need for approximation discussed in (vi) above] and hence $\beta \simeq 3$. In the globally controlled case, on the other hand, $\gamma_2 < G(0)$ and hence $\beta > 3$ and the probability density function therefore has

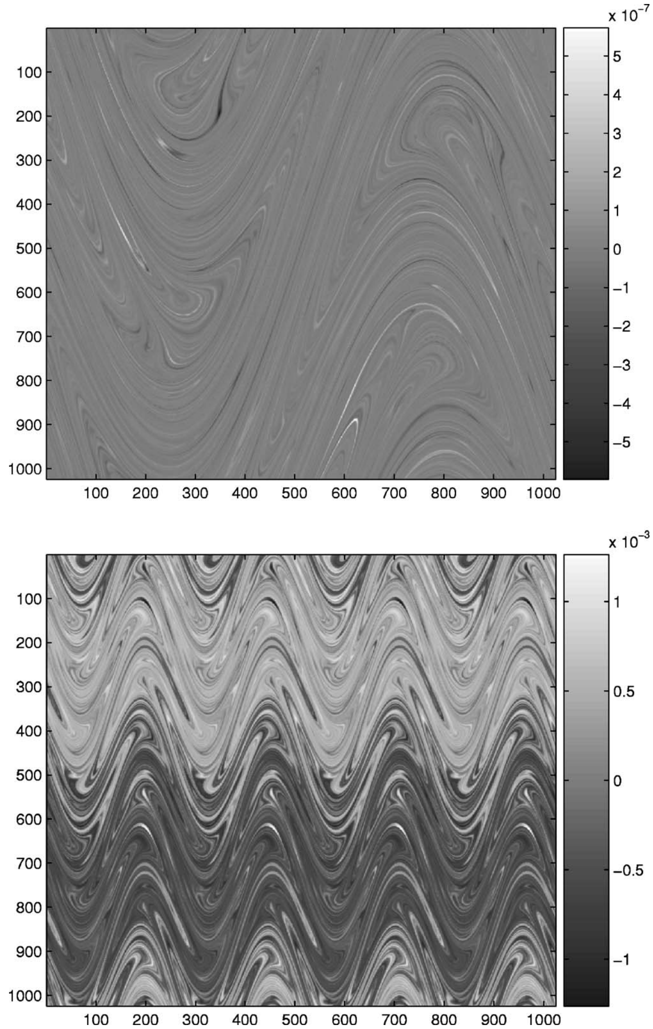

FIG. 8. Typical concentration field for a scalar decaying in the alternating sine flow (4.8) with $\alpha=\pi$. The domain size is $P=1$ (top panel) and $P=4$ (bottom panel).

more rapidly decaying tails. This difference between locally and globally controlled cases is manifested in the scalar fields. Two examples are shown in Fig. 8, for $P=1$ and $P=4$, respectively (using the same parameter values as in Sec. IV C). For $P=1$ there are small regions where the scalar concentration is relatively high. These are associated with fluid elements that have recently experienced weak stretching and account for the shallow tails of the probability density function (see discussion in Fereday and Haynes[7]). For $P=4$ the overall structure corresponds to the gravest mode of a diffusion operator (here $\cos y$), and this structure is modulated on small scales.

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APPENDIX A: EVENNESS OF $\Lambda(\sigma)$

1. Kraichnan limit

We start with the integral

$$I := \int\int_{A_{12}} r^{-1-\sigma} g(\theta) S_{ijkl} x_k x_l \partial_{ij}^2 [r^{-1+\sigma} f(\theta)] dx dy, \qquad \text{(A1)}$$

where A_{12} denotes the annulus $r_1 \leqslant r \leqslant r_2$, and $f(\theta)$ and $g(\theta)$ are arbitrary functions. Integrating by parts using $S_{ijil}=S_{ijkj}=0$ and Green's theorem, this can be rewritten as

$$\begin{aligned} I = & \int\int_{A_{12}} r^{-1+\sigma} f(\theta) S_{ijkl} x_k x_l \partial_{ij}^2 [r^{-1-\sigma} g(\theta)] dx dy \\ & + \int_0^{2\pi} [n_i r^{-\sigma} g(\theta) S_{ijkl} x_k x_l \partial_j [r^{-1+\sigma} f(\theta)]]_{r_1}^{r_2} d\theta \\ & - \int_0^{2\pi} [n_j r^{\sigma} f(\theta) S_{ijkl} x_k x_l \partial_i [r^{-1-\sigma} g(\theta)]]_{r_1}^{r_2} d\theta, \end{aligned} \qquad \text{(A2)}$$

where n_i denotes the components of the unit radial vector. The two integrals in θ vanish identically: this is because their integrands are variations between r_1 and r_2 of functions of θ only. Using the definition of $\mathcal{T}_\sigma$, (A1) and (A2) thus lead to

$$\int_{r_1}^{r_2} r^{-1} dr \int_0^{2\pi} g(\theta) \mathcal{T}_\sigma f(\theta) d\theta = \int_{r_1}^{r_2} r^{-1} dr \int_0^{2\pi} f(\theta) \mathcal{T}_{-\sigma} g(\theta) d\theta.$$

We conclude from this result that $\mathcal{T}_\sigma$ and $\mathcal{T}_{-\sigma}$ are adjoint of one another and, therefore, have the same spectrum. Hence if λ is an eigenvalue of $\mathcal{T}_\sigma$ so that $\lambda=\Lambda(\sigma)$, it is also an eigenvalue of $\mathcal{T}_{-\sigma}$ and $\lambda=\Lambda(-\sigma)$. This proves the evenness of $\Lambda(\sigma)$.

2. Renewing flows

Let us denote $\mathcal{T}_\sigma$ the operator defined in (4.5) by $\mathcal{T}_\sigma^S$, to emphasize the dependence on the ensemble of matrices S (from which the $S_{-\tau}$ are drawn). We start by showing that

$$\mathcal{T}_\sigma^S = (\mathcal{T}_{-\sigma}^{S^{-1}})^\dagger, \qquad \text{(A3)}$$

where $\dagger$ denotes the adjoint. For two arbitrary periodic functions $f(\theta)$ and $g(\theta)$, we compute from the definition (4.5)

$$\int_0^{2\pi} g(\theta)(\mathcal{T}_\sigma^S f)(\theta) d\theta = \int_0^{2\pi} g(\theta) r'^{\sigma-1} f(\theta') d\theta,$$

where r' and θ' are related to θ according to (4.6). Now, we change the integration variable from θ to θ', noting that $\det S=1$ (i.e., area preservation) implies that[23]

$$d\theta' = r'^{-2} d\theta.$$

This leads to

$$\int_0^{2\pi} g(\theta)(\mathcal{T}_\sigma^S f)(\theta)d\theta = \int_0^{2\pi} g(\theta)r'^{\sigma+1}f(\theta')d\theta' = \int_0^{2\pi} g(\theta)r^{-\sigma-1}f(\theta')d\theta',$$

where r and θ are related to θ' according to

$$r\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} = S^{-1}\begin{pmatrix}\cos\theta'\\ \sin\theta'\end{pmatrix}.$$

Using (4.5) and (4.6), we therefore find that

$$\int_0^{2\pi} g(\theta)(\mathcal{T}_\sigma^S f)(\theta)d\theta = \int_0^{2\pi} f(\theta)(\mathcal{T}_{-\sigma}^{S^{-1}} g)(\theta)d\theta.$$

This proves (A3).

Now, (A3) implies that the spectra of $\mathcal{T}_\sigma^S$ and $\mathcal{T}_{-\sigma}^{S^{-1}}$ coincide, that is, $\Lambda^S(\sigma)=\Lambda^{S^{-1}}(-\sigma)$ or, equivalently,

$$F^S(m) = F^{S^{-1}}(-m-2). \tag{A4}$$

On the other hand, in two dimensions, the area-preservation condition $\det S=1$ implies that

$$F^S(m) = F^{S^{-1}}(m). \tag{A5}$$

In other words the stretching properties of the flow are time reversible. This is because the products of n random matrices $S_nS_{n-1}\cdots S_2S_1$, where the S_i are independent realizations of the matrix $S_{-\tau}$, and their inverses $S_1^{-1}S_2^{-1}\cdots S_{n-1}^{-1}S_n^{-1}$ have the same eigenvalues. Combining (A4) and (A5) leads to the symmetry property (3.16) and hence to the evenness of $\Lambda(\sigma)$ for renewing flows in two dimensions.

For completeness, we give an alternative derivation of (A4) which makes no reference to the operator $\mathcal{T}_\sigma^{\mathrm{S}}$. After a time $n\tau$, a line element $\mathbf{d}_0$ with $d_0=|\mathbf{d}_0|=1$ is stretched into

$$\mathbf{d}_n = S_nS_{n-1}\cdots S_1\mathbf{d}_0.$$

The corresponding free energy obeys

$$\exp[nF^S(m)] \propto \langle d_n^m\rangle = \frac{1}{2\pi}\left\langle \int_0^{2\pi} d_n^m d\theta_0\right\rangle \quad \text{as } n\to\infty,$$

where θ_0 denotes the angle between $\mathbf{d}_0$ and a fixed direction. We now change the integration variable to θ_n, satisfying

$$d_n\begin{pmatrix}\cos\theta_n\\ \sin\theta_n\end{pmatrix} = S_nS_{n-1}\cdots\begin{pmatrix}\cos\theta_0\\ \sin\theta_0\end{pmatrix},$$

and find

$$\exp[nF^S(m)] \propto \left\langle \int_0^{2\pi} d_n^{m+2}d\theta_n\right\rangle.$$

Consider now the stretching of the initial line element $\hat{\mathbf{d}}_0=\mathbf{d}_n/d_n$ by the random independent random matrices S^{-1}. We have

$$\hat{\mathbf{d}}_n = S_1^{-1}S_2^{-1}\cdots S_n^{-1}\hat{\mathbf{d}}_0 = \frac{\mathbf{d}_0}{d_n},$$

so that $\hat{d}_n=1/d_n$. Therefore, denoting $\hat{\theta}_0=\theta_n$ the angle between $\hat{\mathbf{d}}_0$ and a fixed direction we find that

$$\exp[nF^S(m)] \propto \left\langle \int_0^{2\pi} \hat{d}_n^{-m-2}d\hat{\theta}_0\right\rangle \propto \exp[nF^{S^{-1}}(-m-2)],$$

thus establishing (A4).

APPENDIX B: ASYMPTOTICS: KRAICHNAN LIMIT

In this appendix, we use matched asymptotics to determine the nature of the spectrum of $\mathcal{L}_\kappa$ for $\kappa\to 0$. This spectrum contains two types of eigenvalues: locally controlled ones, which emerge from the continuous spectrum of $\mathcal{L}_0$, and (possibly) globally controlled ones which are perturbations of isolated eigenvalues of $\mathcal{L}_0$. The locally and globally controlled eigenvalues are separated (they are, respectively, $>\ell$ and $<\ell$); the behavior of their eigenfunctions as $\mathbf{x}\to 0$ is different (oscillatory versus nonoscillatory); and as a result, they have very different asymptotic behaviors which we now examine.

1. Locally controlled eigenvalues

Among the locally controlled eigenvalues, the most important is the one nearest ℓ: in the absence of globally controlled eigenvalues, this is the smallest eigenvalue, and hence it gives the variance decay rate. We start by estimating the location of this eigenvalue.

Let $\ell+\mu$ be the sought eigenvalue. Here $\mu\ll 1$ is a small parameter to be determined; we expect $\mu\to 0$ and $\mu\gg\kappa^{1/2}$ as $\kappa\to 0$. Outside a neighborhood of $x=y=0 (\mathrm{mod}\, 2\pi)$, the corresponding eigenfunction is approximated at leading order by Γ_o (with subscript "o" indicating "outer") satisfying

$$-D_{ij}(\mathbf{x})\partial_{ij}^2\Gamma_\mathrm{o} = \ell\Gamma_\mathrm{o}$$

and the boundary conditions $\partial_x\Gamma_\mathrm{o}=0$ at $x=\pi$ and $\partial_y\Gamma_\mathrm{o}=0$ at $y=0$ which follow from the symmetry of the problem. For $\kappa^{1/2}\ll|\mathbf{x}|\ll 1$, the behavior of Γ_o is governed by the local form (3.3) of this equation; because ℓ corresponds to the double root $\sigma=0$ of $\lambda=\Lambda(\sigma)$, this behavior takes the form

$$\Gamma_\mathrm{o} \sim r^{-1}[(a+\log r)f_0(\theta)+\tilde{f}_0(\theta)] \quad \text{for } r\ll 1. \tag{B1}$$

Here, a is a constant, and $\tilde{f}_0$ is the derivative of $f_\sigma(\theta)$ with respect to σ evaluated at $\sigma=0$. In a neighborhood of the origin of size $\kappa^{1/2}$, the diffusive term in (3.2) cannot be neglected. Rescaling the independent variables according to $(x,y)=\kappa^{1/2}(X,Y)$, we find that the leading-order approximation Γ_i (with subscript "i" indicating "inner") to Γ in this region satisfies

$$-2(\partial_{XX}^2+\partial_{YY}^2)\Gamma_i + S_{IJKL}X_KX_L\partial_{IJ}^2\Gamma_i = \ell\Gamma_i,$$

where $\partial_I=\partial/\partial_{X_I}$. The corresponding asymptotic behavior for $|\mathbf{X}|\gg 1$ is found to be

$$\Gamma_i \sim R^{-1}[(A+B\log R)f_0(\theta)+B\tilde{f}_0(\theta)] \quad \text{for } R\gg 1, \tag{B2}$$

where $R:=\kappa^{-1/2}r$, and A and B are constants.

The correction μ to the eigenvalue ℓ is then derived by matching (B1) and (B2) with the solution in an intermediate region where μ appears in the leading-order approximation to Γ. In this region, where both $|\sigma\log r|\gg 1$ and $|\sigma\log R|\gg 1$, the approximation

$$\Gamma \sim r^{-1}[\gamma_+ r^{\sigma} f_{\sigma}(\theta) + \gamma_- r^{-\sigma} f_{-\sigma}(\theta)] \tag{B3}$$

holds, with

$$\mu \sim \frac{1}{2}\Lambda''(0)\sigma^2, \tag{B4}$$

and where $\gamma_\pm$ are constants. Expanding (B3) for $\sigma \ll 1$ and matching with (B1) gives

$$\gamma_+ + \gamma_- = a \text{ and } \sigma(\gamma_+ - \gamma_-) = 1.$$

Matching with (B2) gives

$$\kappa^{-1/2}(\kappa^{\sigma/2}\gamma_+ + \kappa^{-\sigma/2}\gamma_-) = A$$

and

$$\kappa^{-1/2}\sigma(\kappa^{\sigma/2}\gamma_+ - \kappa^{-\sigma/2}\gamma_-) = B.$$

Solving for $\gamma_\pm$ and μ gives

$$\kappa^{\sigma} = -\frac{1 - 1/\sigma}{a + 1/\sigma}\frac{B + A\sigma}{B - A\sigma} \sim 1$$

since $\sigma \ll 1$. Hence, the correction to the eigenvalue ℓ is found from (B4) to be

$$\mu \sim -\frac{2\pi^2\Lambda''(0)}{\log^2\kappa} \quad \text{as } \kappa \to 0. \tag{B5}$$

This gives the estimate (3.8) for the variance decay rate. We note that the dissipative correction decreases rather slowly to 0 as $\kappa \to \infty$; this implies that ℓ can be a poor approximation to an eigenvalue of $\mathcal{L}_\kappa$ unless κ is exceedingly small.

Away from ℓ, the continuous spectrum of $\mathcal{L}_0$ turns for $\kappa \neq 0$ into a set of isolated eigenvalues, separated by small gaps. To estimate the size of these gaps, we consider an eigenvalue $\lambda > \ell$, i.e., any interior point of the continuous spectrum. Solving the local eigenvalue problem (3.3) or (4.5) gives solutions $f_\sigma(\theta)$ with $\sigma = \pm is$ purely imaginary (since for a given real λ, σ, $-\sigma$, and σ^* are all solutions). For $|\mathbf{x}| \ll 1$, the outer solution of the full eigenvalue problem (3.2) or (4.4) has the asymptotics

$$\Gamma_o \sim a r^{-1} \sin(s \log r + \phi),$$

where a and ϕ are constants, the latter being fixed by the boundary conditions. Similarly, the inner solution (of the local problem with diffusion) behaves like

$$\Gamma_o \sim A R^{-1} \sin(s \log R + \Phi),$$

with A and Φ two constants. Matching these two solutions gives the condition

$$s = 2\frac{\Phi - \phi + n\pi}{\log \kappa}, \quad n \in \mathbb{Z}.$$

Since away from ℓ, $d\lambda/ds = O(1)$, this translates into a set of eigenvalues λ separated by gaps whose size scales like $1/\log \kappa$.

2. Globally controlled eigenvalues

We now consider the asymptotic behavior of possible isolated eigenvalues $\lambda \sim \lambda_0 < \ell$ as $\kappa \to 0$. Their asymptotic study clarifies in what sense their limits λ_0 are eigenvalues of $\mathcal{L}_0$.

In the limit $\kappa \to 0$, and away from $(x,y) = 0 (\text{mod } 2\pi)$, the eigenfunctions of $\mathcal{L}_\kappa$ are well approximated by the solution Γ_o of $\mathcal{L}_0\Gamma = \lambda_0\Gamma$. For $\kappa^{1/2} \ll |\mathbf{x}| \ll 1$, this has the asymptotic behavior

$$\Gamma_o \sim r^{-1}[a_+ r^{\sigma_0} f_{\sigma_0}(\theta) + a_- r^{-\sigma_0} f_{-\sigma_0}(\theta)], \tag{B6}$$

where $a_\pm$ are constants and σ_0 is the positive solution to $\lambda = \Lambda(\sigma_0)$ [thus σ_0, λ, and f_{σ_0} satisfy the eigenproblem (3.20)].

As $|\mathbf{x}| \to 0$, Γ is regularized by the diffusion, which is significant in an $O(\kappa^{1/2})$ neighborhood of the origin. In terms of the scaled coordinates $(X,Y) = \kappa^{-1/2}(x,y)$, the approximation to Γ in this inner region satisfies

$$-2(\partial_{XX}^2 + \partial_{YY}^2)\Gamma_i + S_{IJKL}X_K X_L \partial_{IJ}^2 \Gamma_i = \lambda_0 \Gamma_i.$$

The asymptotic behavior of the inner solution for $R = |\mathbf{X}| \gg 1$ is

$$\Gamma_i \sim R^{-1}[A_+ R^{\sigma_0} f_{\sigma_0}(\theta) + A_- R^{-\sigma_0} f_{-\sigma_0}(\theta)],$$

where $A_\pm$ are constants. Since the inner equation is regular, we expect both these constants to have the same order; the second term in the solution is therefore negligible for $R \gg 1$, and

$$\Gamma_i \sim A_+ R^{-1+\sigma_0} f_{\sigma_0}(\theta).$$

Matching with the outer solution (B6) imposes the conditions

$$a_+ = \kappa^{(1-\sigma_0)/2} A_+ \quad \text{and } a_- = 0.$$

The latter condition provides the boundary condition needed to complete the definition of the (generalized) eigenvalues λ_0 of $\mathcal{L}_0$: to impose it, it is sufficient to demand that

$$\lim_{r \to 0} r\Gamma = 0.$$

This gives eigensolutions of $\mathcal{L}_0\Gamma = \lambda_0\Gamma$ where Γ is periodic in (x,y) and behaves like $r^{\sigma_0 - 1} f_{\sigma_0}(\theta)$ as $(x,y) \to (0,0) \times (\text{mod } 2\pi)$. It is near these generalized eigenvalues that the eigenvalues of $\mathcal{L}_\kappa$ lie. The dependence on κ can be estimated by noting that for $\kappa \neq 0$ the contribution of the inner solution proportional to A_- requires a correction to the leading-order inner solution whose relative magnitude is $O(\kappa_0^{\sigma})$. This can be accounted for by a correction to the eigenvalue of a similar order of magnitude. Thus, we expect eigenvalues of $\mathcal{L}_\kappa$ to behave like

$$\lambda \sim \lambda_0 + c\kappa^{\sigma_0} \quad \text{as } \kappa \to 0, \tag{B7}$$

for some constant c, near a discrete eigenvalue of $\mathcal{L}_0$.

APPENDIX C: ASYMPTOTICS: RENEWING FLOWS

In this appendix we give further details of the analysis leading to (3.8) and (3.10) for renewing flows.

We first note that if κ is sufficiently small then the Green's function $G_\kappa(\mathbf{x}, \mathbf{x}', t)$ is well approximated by solving the advection–diffusion equation (or, more strictly, its adjoint) along a backwards trajectory starting at $\mathbf{x}$ and approximating the flow as a linear function of space. This gives the expression

$$G_\kappa(\mathbf{x},\mathbf{x}',t) = \frac{[\det B(\mathbf{x},\tau)]^{1/2}}{\pi} \times \exp[-(\mathbf{x}'-\phi_{-\tau}\mathbf{x})B(\mathbf{x},\tau)(\mathbf{x}'-\phi_{-\tau}\mathbf{x})],$$

where

$$B(\mathbf{x},\tau)^{-1} = \kappa\int_0^\tau S_{-\tau}S_{-\tau'}^{-1}S_{-\tau'}^{-T}S_{-\tau}^{T}d\tau',$$

closely related to the results derived previously in the works of Balkovsky and Fouxon,[2] Son[4] and Falkovich *et al.*[3] and to earlier work on the magnetic-field problem (Vishik;[31] see also Childress and Gilbert[21]).

Substituting into (4.1), it follows that

$$\mathcal{G}_\kappa(\mathbf{x},\mathbf{x}',\tau) = \Bigg\langle \frac{[\det B(\mathbf{0},\tau)\det B(\mathbf{x},\tau)]^{1/2}}{[\det[B(\mathbf{0},\tau)+B(\mathbf{x},\tau)]]^{1/2}\pi} \times\exp\{-(\mathbf{x}'-\phi_{-\tau}\mathbf{x}+\phi_{-\tau}\mathbf{0})B(\mathbf{0},\tau)[B(\mathbf{0},\tau)+B(\mathbf{x},\tau)]^{-1} \times B(\mathbf{x},\tau)(\mathbf{x}'-\phi_{-\tau}\mathbf{x}+\phi_{-\tau}\mathbf{0})\}\Bigg\rangle. \quad (C1)$$

Note that when κ is small the Gaussian function in (C1) may for most choices of $\mathbf{x}$ and $\mathbf{x}'$ be approximated by a delta function to give (4.3). However, it turns out to be crucial to account for the finite spread in the Gaussian when $\mathbf{x}$ and $\mathbf{x}'$ are close to $\mathbf{0}$, in particular, when $|\mathbf{x}|\sim|\mathbf{x}'|\sim\kappa^{1/2}$. In this case $\mathcal{G}_\kappa(\mathbf{x},\mathbf{x}',\tau)$ may be approximated by

$$\mathcal{G}_\kappa(\mathbf{x},\mathbf{x}',\tau) \simeq \Bigg\langle \frac{[\det B(\mathbf{0},\tau)]^{1/2}}{2\pi} \times\exp\left[-\frac{1}{2}(\mathbf{x}'-S_{-\tau}\mathbf{x})B(\mathbf{0},\tau)(\mathbf{x}'-S_{-\tau}\mathbf{x})\right]\Bigg\rangle. \quad (C2)$$

The approximate expressions (4.1) and (C2) may now be used as a basis for investigation of the κ dependence of the eigenvalues of the operator $\mathcal{L}_\kappa$ defined in (4.2).

1. Locally controlled eigenvalues

The locally controlled eigenvalues of $\mathcal{L}_\kappa$ emerge from the continuous spectrum $[\ell,\infty)$ of $\mathcal{L}_0$ which is obtained in Sec. IV A by approximating the spectral problem (4.2) by its local version for $\kappa=0$. Two approximations are involved in the local version: the Green's function $\mathcal{G}_0$ is approximated by $\langle\delta(\mathbf{x}'-S_{-\tau}\mathbf{x})\rangle$, and the domain of integration is extended from $[0,2P\pi]\times[0,2P\pi]$ to the whole plane. With $\kappa\neq 0$, these remain valid approximations in the range $\kappa^{1/2}\ll|\mathbf{x}|\ll 1$. Outside this range, however, the more complicated expressions (C2) (for $|\mathbf{x}|\ll\kappa^{1/2}$) and (4.3) [for $|\mathbf{x}|=O(1)$] must be employed.

We treat the eigenvalue problem for $\mathcal{L}_\kappa$ as $\kappa\to 0$ and $\lambda\approx\ell$ in two steps: (i) we first consider a modified eigenvalue problem, where the Green's function $\mathcal{G}_\kappa$ is replaced by the regularized, scale-invariant function

$$\tilde{\mathcal{G}}_\kappa(\mathbf{x},\mathbf{x}',\tau) = \langle\delta(\tilde{\mathbf{x}}'-S_{-\tau}\tilde{\mathbf{x}})\rangle|\det\nabla\tilde{\mathbf{x}}'| \quad \text{with} \quad \tilde{\mathbf{x}} = \frac{|\mathbf{x}|+\kappa^{1/2}}{|\mathbf{x}|}\mathbf{x}, \quad (C3)$$

and the domain is $[-\pi,\pi]\times[-\pi,\pi]$; (ii) we regard the exact eigenvalue problem as a regular perturbation to this modified problem. It turns out that the second step leads to a negligible correction to the eigenvalue.

The modified eigenvalue problem

$$\int_\pi^\pi\int_{-\pi}^\pi \tilde{\mathcal{G}}_\kappa(\mathbf{x},\mathbf{x}',\tau)\Gamma(\mathbf{x}')d\mathbf{x}' = \tilde{\nu}\Gamma(\mathbf{x}), \quad (C4)$$

where $\tilde{\nu}$ denotes the eigenvalue, can be rewritten as

$$\langle\tilde{\Gamma}(S_{-\tau}\tilde{\mathbf{x}})\rangle - \tilde{\nu}\tilde{\Gamma}(\tilde{\mathbf{x}}) = \int_0^{2\pi}d\tilde{\theta}'\int_0^{\kappa^{1/2}}\langle\delta(\tilde{\mathbf{x}}'-S_{-\tau}\tilde{\mathbf{x}})\rangle\tilde{\Gamma}(\tilde{\mathbf{x}}')\tilde{r}'d\tilde{r}' + \int_0^{2\pi}d\tilde{\theta}'\int_{\rho(\tilde{\theta}')}^{\infty}\langle\delta(\tilde{\mathbf{x}}'-S_{-\tau}\tilde{\mathbf{x}})\rangle\tilde{\Gamma}(\tilde{\mathbf{x}}')\tilde{r}'d\tilde{r}', \quad (C5)$$

where $\tilde{\Gamma}(\tilde{\mathbf{x}})=\Gamma(\mathbf{x})$, $(\tilde{r}',\tilde{\theta}')$ are polar coordinates for $\tilde{\mathbf{x}}'$, and $\rho(\theta')$ describes the boundary of $\mathbf{x}'\in[-\pi,\pi]\times[-\pi,\pi]$ in these coordinates. Following Sec. IV A, we consider a solution of the form

$$\tilde{\Gamma}(\tilde{\mathbf{x}}) \simeq \tilde{r}^{-1}[\gamma_+\tilde{r}^{\sigma}f_\sigma(\tilde{\theta})+\gamma_-\tilde{r}^{-\sigma}f_{-\sigma}(\tilde{\theta})], \quad (C6)$$

where $\sigma>0$, $\tilde{\nu}$ and $f_{\pm\sigma}$ satisfy the eigenvalue problem (4.5). The choice of the modified Green's function (C3) ensures that this solution is well behaved as $|\mathbf{x}|\to 0$.

We are interested, in particular, in the eigenvalue $\tilde{\lambda}=-\log\tilde{\nu}/\tau=\ell+\mu$ for $\mu\ll 1$, so that (B4) also holds. By construction, the solution (C6) cancels the left-hand side of (C5); it provides an approximate solution to the modified eigenvalue problem (C4) provided that the two integrals on the right-hand side of (C5) also vanish at leading order in κ. These are significant for $|\tilde{\mathbf{x}}|=O(\kappa^{1/2})$ and $|\tilde{\mathbf{x}}|=O(1)$, respectively. Using the smallness of σ, which allows the approximation of $f_{\pm\sigma}(\tilde{\theta})$ by $f_0(\tilde{\theta})$ and $\tilde{r}^{\pm\sigma}$ by 1 for $\tilde{r}=O(1)$, the conditions for the required vanishing are found to be of the form

$$\kappa^{-1/2}[\gamma_+\kappa^{\sigma/2}+\gamma_-\kappa^{-\sigma/2}]C(\tilde{\mathbf{x}})=0 \text{ and } (\gamma_++\gamma_-)D(\tilde{\mathbf{x}})=0, \quad (C7)$$

for two σ-independent functions $C(\tilde{\mathbf{x}})$ and $D(\tilde{\mathbf{x}})$. This leads to the condition $\kappa^\sigma=1$ and, on using (B4), to the same expression (B5) for the diffusive correction to the eigenvalue ℓ as in the Kraichnan case. It follows that

$$\tilde{\lambda} = \ell - \frac{2\pi^2\Lambda''(0)}{\log^2\kappa} + o(1/\log^2\kappa). \quad (C8)$$

We finally show that the exact eigenvalue λ (or equivalently ν) differs from $\tilde{\lambda}(\tilde{\nu})$ by $o(1/\log^2 \kappa)$ terms, so that (C8) holds for λ as well as for $\tilde{\lambda}$. Standard perturbation theory gives the leading-order estimate

$$\nu - \tilde{\nu} \sim \frac{\int\int \Gamma^\dagger(\mathbf{x})[\mathcal{G}_\kappa(\mathbf{x},\mathbf{x}',\tau) - \tilde{\mathcal{G}}_\kappa(\mathbf{x},\mathbf{x}',\tau)]\Gamma(\mathbf{x}')d\mathbf{x}d\mathbf{x}'}{\int \Gamma^\dagger(\mathbf{x})\Gamma(\mathbf{x})d\mathbf{x}},$$

where $\Gamma(\mathbf{x})$ is the approximate solution (C6), $\Gamma^\dagger(\mathbf{x})$ is its adjoint, with a similar dependence on $\tilde{r}$, and the integration domains can be taken to be $[-\pi,\pi]\times[-\pi,\pi]$. The dominator is the integral of a function proportional to $1/(r+\kappa^{1/2})$ for small r and hence scales like $\log \kappa$. To estimate the numerator, we note that the difference between Green's functions is proportional to $\kappa^{1/2}$ in the range $\kappa^{1/2} \ll |\mathbf{x}|, |\mathbf{x}'| \ll 1$. Outside this range, the difference between the Green's functions is $O(1)$, but $\tilde{\Gamma}(\mathbf{x})$ and $\tilde{\Gamma}^\dagger(\mathbf{x})$ are both small, $O(\sigma) = O(1/\log \kappa)$, as follows from (C6) and (C7). This implies that the numerator is $O(1/\log^2 \kappa)$ and hence leads to an $O(1/\log^3 \kappa)$ estimate for $\nu - \tilde{\nu}$ and $\lambda - \tilde{\lambda}$.

2. Globally controlled eigenvalues

In the globally controlled case $\mathcal{L}_0$ has an eigenvalue ν_0 with a corresponding eigenfunction $\Gamma_0(\mathbf{x})$ under the regularity condition $\lim_{r\to 0} r\Gamma = 0$ (by analogy with Appendix B 2). Suppose that for nonzero κ the corresponding eigenvalue is ν_κ and the corresponding eigenfunction is Γ_κ. Then standard perturbation theory implies that

$$\nu_\kappa - \nu_0 \sim \frac{\int \Gamma_0^\dagger(\mathbf{x})[\mathcal{L}_\kappa \Gamma_0(\mathbf{x}) - \mathcal{L}_0\Gamma_0(\mathbf{x})]d\mathbf{x}}{\int \Gamma_0^\dagger(\mathbf{x})\Gamma_0(\mathbf{x})d\mathbf{x}}, \qquad \text{(C9)}$$

where $\Gamma_0^\dagger$ is the eigenfunction of the operator adjoint to $\mathcal{L}_0$ with eigenvalue ν_0. The analysis in Sec. IV A indicates that $\Gamma(\mathbf{x}) \sim |\mathbf{x}|^{\sigma_0 - 1}$ as $|\mathbf{x}| \to 0$, with σ_0 the positive root of $N(\sigma_0) = \nu_0$ (positive to satisfy the regularity condition). The dominant contribution to the integral in the numerator of the right-hand side of (C9) is the integral of a function that is $O(|\mathbf{x}|^{2\sigma_0 - 2})$ over a region that is of linear dimension $\kappa^{1/2}$. It follows that the estimated correction to the eigenvalue is $O(\kappa^{\sigma_0})$, corresponding to the result (3.10) obtained in the Kraichnan limit.

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