Theoretical aspects of the dynamo effect

- linear transport of the magnetic field $\boldsymbol{B}$ by a (turbulent) flow $\boldsymbol{v}$

  $\partial_t \boldsymbol{B} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \kappa \nabla^2 \boldsymbol{B}$

  $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$

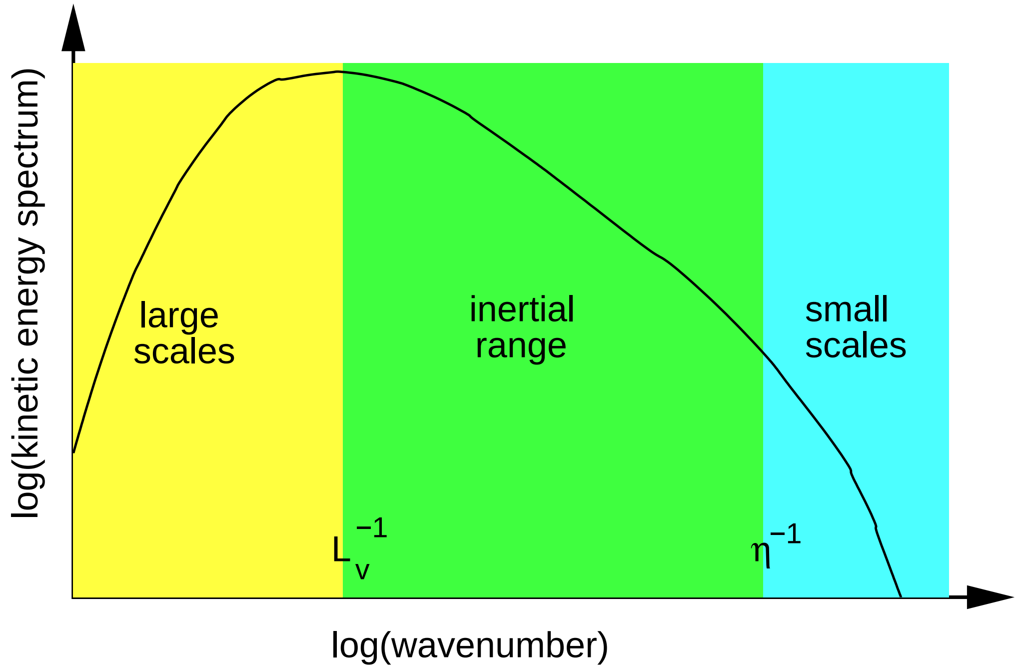
- $\boldsymbol{v}$ incompressible, statistically stationary, homogeneous and isotropic, but not necessarily parity-invariant.

The dynamo effect:
exponential growth in time
of the magnetic field

$$\sigma(\kappa) = \lim_{t \to \infty} \frac{1}{2t} \langle \ln \boldsymbol{B}^2 \rangle > 0$$

Fast dynamo: $\lim_{\kappa \to 0} \sigma(\kappa) > 0$

Characteristic lengthscales. 1



Velocity lengthscales

L_v correlation length of the flow

$\eta = (\nu^3/\epsilon)^{1/4}$ Kolmogorov scale

Magnetic field lengthscales

L correlation length

r_d resistive lengthscale ($\ll L$)

Adimensional numbers

Reynolds number $Re = (L_v/\eta)^{4/3}$

Prandtl number $Pr = \nu/\kappa$

Magnetic Reynolds number $Re_m = RePr$

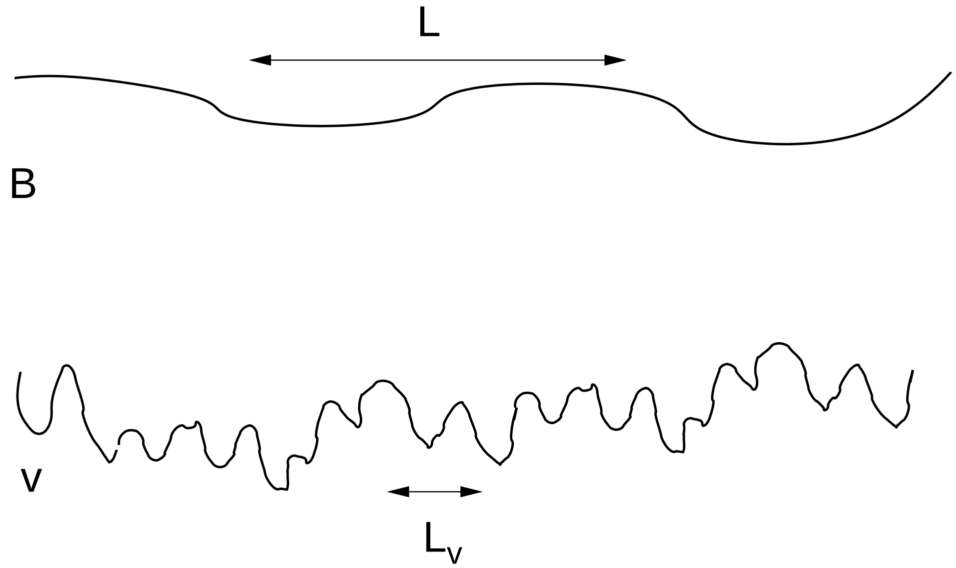
Characteristic lengthscales. 2

- Large-scale dynamo: $L \gg L_v$
  Methods: multiscale asymptotic expansions
  Results: α-effect, effective magnetic resistivity

- Small-scale dynamo: $L \ll \eta$
  Methods: Lagrangian approach
  Results: growth-rates and Lyapunov exponents, geometrical aspects

- Dynamo at all scales (including inertial ones)
  Methods: Kazantsev-Kraichnan model, Schrödinger equation
  Results: Re_m critical, Pr effect.

Large-scale dynamo. 1



Singular perturbation theory in $\epsilon = L_v/L$

Asymptotic methods: multiple scale analysis

Lanotte *et al*, Geophys. Astrophys. Fluid Dynamics **91** (1999)

"Fast" scales $\boldsymbol{x}$, fast times t

"Slow" scales $\tilde{\boldsymbol{x}} = \epsilon \boldsymbol{x}$, slow times $\tilde{t} = \epsilon t$

Very slow times $\tilde{\tilde{t}} = \epsilon^2 t$

$$\begin{aligned}\boldsymbol{B}(\boldsymbol{x}, t, \tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\tilde{t}}) &= \boldsymbol{B}^{(0)}(\boldsymbol{x}, t, \tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\tilde{t}}) \\ &+ \epsilon\, \boldsymbol{B}^{(1)}(\boldsymbol{x}, t, \tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\tilde{t}}) \\ &+ \epsilon^2\, \boldsymbol{B}^{(2)}(\boldsymbol{x}, t, \tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\tilde{t}}) + \ldots\end{aligned}$$

$$\nabla \to \nabla + \epsilon \tilde{\nabla} \qquad \partial_t \to \partial_t + \epsilon \tilde{\partial}_t + \epsilon^2 \tilde{\tilde{\partial}}_t$$

Large-scale dynamo. 2

At order ϵ^0

decompose $\boldsymbol{B}^{(0)} = \langle \boldsymbol{B}^{(0)} \rangle + \boldsymbol{b}^{(0)}$

$$\partial_t \boldsymbol{b}^{(0)} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{b}^{(0)}) - \kappa \nabla^2 \boldsymbol{b}^{(0)} = (\langle \boldsymbol{B}^{0)} \rangle \cdot \boldsymbol{\nabla}) \boldsymbol{v} \; ,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}^{(0)} = 0 \; .$$

Remark:

The *statistics* of $\boldsymbol{v}$ depends neither on fast variables (homogeneity) nor on slow ones (small-scale velocity)

The *statistics* of $\boldsymbol{B}$ depends on slow scales and times

Averaging over $\boldsymbol{v}$ all terms vanish → solve

$$b_i^{(0)}(\boldsymbol{x}, t, \tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\tilde{t}}\,) = S_{ij}(\boldsymbol{x}, t) \langle B_j^{(0)} \rangle (\tilde{\boldsymbol{x}}, \tilde{t}, \tilde{\tilde{t}}\,)$$

where S_{ij} is the zero-average tensor that solves the auxiliary equations

$$\partial_t S_{ij} - \varepsilon_{ikl} \nabla_k (\varepsilon_{lmn} v_m S_{nj}) - \kappa \nabla^2 S_{ij} = \nabla_j v_i$$

$$\nabla_i S_{ij} = 0$$

Large-scale dynamo. 3

At order ϵ^1

$$\partial_t \boldsymbol{B}^{(1)} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}^{(1)}) - \kappa \nabla^2 \boldsymbol{B}^{(1)} =$$
$$-\tilde{\partial}_t \boldsymbol{B}^{(0)} - (\boldsymbol{v} \cdot \tilde{\boldsymbol{\nabla}}) \boldsymbol{B}^{(0)} + 2\kappa \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\nabla}} \boldsymbol{b}^{(0)} + (\tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{b}^{(0)}) \boldsymbol{v} \ ,$$
$$\boldsymbol{\nabla} \cdot \boldsymbol{B}^{(1)} + \tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{B}^{(0)} = 0 \ .$$

Averaging and inserting the solution for $\boldsymbol{b}^{(0)}$:

$$\tilde{\partial}_t \boldsymbol{B}^{(0)} = \tilde{\boldsymbol{\nabla}} \times \left[\langle \boldsymbol{v} \times \boldsymbol{S} \rangle \cdot \langle \boldsymbol{B}^{(0)} \rangle \right]$$

For statistically isotropic $\boldsymbol{v}$ we have

$$\alpha_{ij} = \langle \boldsymbol{v} \times \boldsymbol{S} \rangle_{ij} = \alpha \delta_{ij}$$

$$\boxed{\tilde{\partial}_t \langle \boldsymbol{B}^{(0)} \rangle = \alpha \tilde{\boldsymbol{\nabla}} \times \langle \boldsymbol{B}^{(0)} \rangle}$$

The alpha effect

Remarks:

- For short-correlated flows ($\tau \ll L_v/v$): $\alpha = -\frac{\tau}{3} \langle \boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{v} \rangle$
- For statistically parity-invariant flows: $\alpha = 0$.
  Solve at order ϵ^1: $B_i^{(1)} = \Gamma_{ijl} \tilde{\nabla}_j \langle B_l^{(0)} \rangle$. Go to $O(\epsilon^2)$.

Large-scale dynamo. 4

At order ϵ^2 (for parity invariant flows):

$$\partial_t \boldsymbol{B}^{(2)} - \boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{B}^{(2)}) - \kappa \nabla^2 \boldsymbol{B}^{(2)} =$$
$$-\tilde{\tilde{\partial}}_t \boldsymbol{B}^{(0)} + \kappa \tilde{\nabla}^2 \boldsymbol{B}^{(0)} - (\boldsymbol{v} \cdot \tilde{\boldsymbol{\nabla}}) \boldsymbol{B}^{(1)} + 2\kappa \boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\nabla}} \boldsymbol{B}^{(1)} + (\tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{B}^{(1)}) \boldsymbol{v} ,$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}^{(2)} + \tilde{\boldsymbol{\nabla}} \cdot \boldsymbol{B}^{(1)} = 0 .$$

Averaging and inserting lower-order solutions:

$$\tilde{\tilde{\partial}}_t \langle B_i^{(0)} \rangle = \kappa^e_{ijlm} \tilde{\nabla}_j \tilde{\nabla}_l \langle B_m^{(0)} \rangle$$

For isotropic flows: $\kappa^e_{ijlm} = \kappa^e \delta_{im} \delta_{jl}$

$$\boxed{\tilde{\tilde{\partial}}_t \langle \boldsymbol{B}^{(0)} \rangle = \kappa^e \tilde{\nabla}^2 \langle \boldsymbol{B}^{(0)} \rangle}$$

Magnetic eddy-diffusivity

Remarks:

- For short-correlated flows ($\tau \ll L_v/v$): $\kappa^e = \frac{\tau}{3} \langle v^2 \rangle$.
- κ^e can be negative: large-scale dynamo effect for parity-invariant flows
- Negative κ^e are quite common

  Zheligovsky *et al*, Geophys. Astrophys. Fluid Dyn. **95** (2001)

Small-scale dynamo. 1

The Lagrangian approach

Chertkov *et al*, PRL **83** (1999)

$$\partial_t \boldsymbol{B} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \kappa\nabla^2\boldsymbol{B}$$

$$\Updownarrow$$

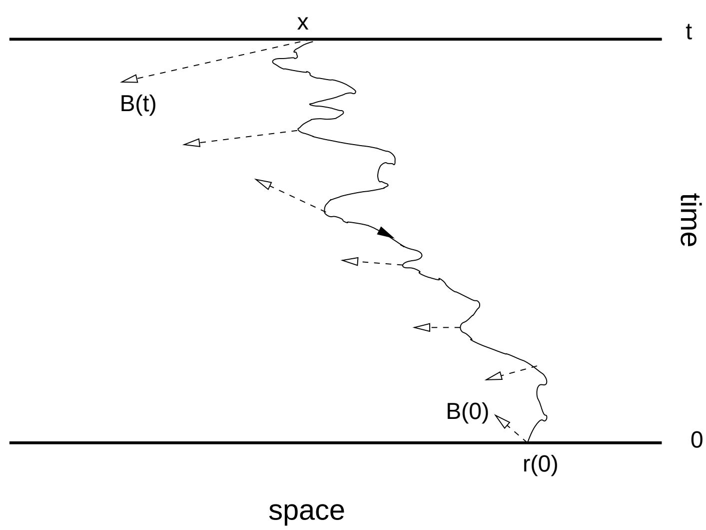
$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}(\boldsymbol{r}(t),t) + \sqrt{2\kappa}\,\boldsymbol{\beta}(t)$$

$$\frac{d\boldsymbol{B}}{dt} = [\boldsymbol{\nabla}\boldsymbol{v}(\boldsymbol{r}(t),t)]^T\cdot\boldsymbol{B}(t)$$

Solve following particle trajectories backward in time

$$\boldsymbol{B}(\boldsymbol{x},t) = \langle \mathsf{W}(t)\cdot\boldsymbol{B}(\boldsymbol{r}(0),0)\rangle_\beta$$

$$d\mathsf{W}/dt = (\boldsymbol{\nabla}\boldsymbol{v})^T\cdot\mathsf{W} \quad \mathsf{W}(0)=1 \quad \boldsymbol{r}(t)=\boldsymbol{x}$$



Small-scale dynamo. 2

Stretching matrices and Lyapunov exponents

Singular value decomposition:

$$\mathsf{W}(t) = \mathsf{U}^T(t)\Lambda(t)\mathsf{V}(t)$$

U and V orthonormal $\qquad \Lambda = \mathrm{diag}(e^{\gamma_1 t}, e^{\gamma_2 t}, e^{\gamma_3 t})$

$$\nabla \cdot \boldsymbol{v} = 0 \Rightarrow \det \mathsf{W} = 1 \Rightarrow \gamma_1 + \gamma_2 + \gamma_3 = 0$$

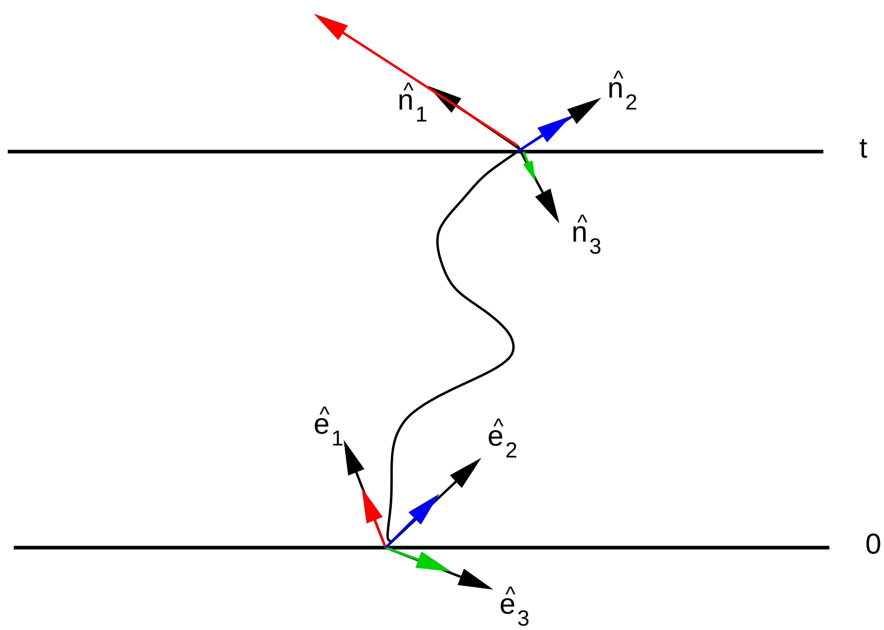
$$\lim_{t\to\infty} \mathsf{V}(t) = \mathsf{V}_0 \qquad \lim_{t\to\infty} \gamma_i = \lambda_i$$

$$P(\gamma_1, \gamma_2, t) \propto \exp[-tS(\gamma_1, \gamma_2)]$$

$S(\gamma_1, \gamma_2)$ Cramér function

S is everywhere nonnegative

S has a parabolic minimum $S(\lambda_1, \lambda_2) = 0$



$$\mathsf{V}_{ij} = [\hat{e}_i]_j \qquad \mathsf{U}_{ij} = [\hat{n}_i]_j$$

Small-scale dynamo. 3

Magnetic field and particle trajectories

Averaging over initial conditions ($\boldsymbol{\nabla}\cdot\boldsymbol{B}=0$)

$$\langle B_i(\boldsymbol{r}_1,0)B_j(\boldsymbol{r}_2,0)\rangle = (\delta_{ij}\nabla^2-\nabla_i\nabla_j)F(|\boldsymbol{r}_1-\boldsymbol{r}_2|/L)$$

$$|\boldsymbol{B}(\boldsymbol{x},t)|^2 = \langle \boldsymbol{B}^T(\boldsymbol{r}_1(0),0)\mathsf{W}^T(t)\mathsf{W}(t)\boldsymbol{B}(\boldsymbol{r}_2(0),0)\rangle_{\beta_1,\beta_2}$$

$$= \mathsf{W}_{ki}(t)\mathsf{W}_{kj}(t)\int p(\boldsymbol{R},0|\boldsymbol{0},t)(\delta_{ij}\nabla^2-\nabla_i\nabla_j)F(R/L)\,d\boldsymbol{R}$$

$p(\boldsymbol{R},0|\boldsymbol{0},t)$ probability of having two particles at separation $\boldsymbol{R}$ at time 0 given that they were at coincident points at time t

$$\mathsf{W}(t)\approx e^{\gamma_1 t}\,\hat{\boldsymbol{n}}_1\otimes\hat{\boldsymbol{e}}_1$$

$$\mathsf{W}_{ki}(t)\mathsf{W}_{kj}(t)\approx e^{2\gamma_1 t}\,\hat{\boldsymbol{e}}_1\otimes\hat{\boldsymbol{e}}_1$$

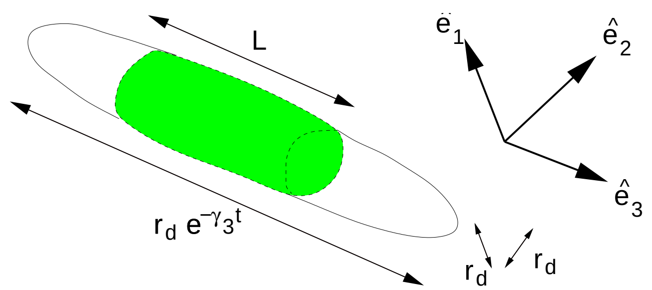
$$\boxed{|\boldsymbol{B}(t)|^2\approx e^{2\gamma_1 t}\int F(\tfrac{R}{L})\left[(\hat{\boldsymbol{e}}_2\cdot\nabla)^2+(\hat{\boldsymbol{e}}_3\cdot\nabla)^2\right]p(\boldsymbol{R},0|\boldsymbol{0},t)\,d\boldsymbol{R}}$$

Small-scale dynamo. 4

Dynamo and Lyapunov exponents. 1

$$|\boldsymbol{B}(t)|^2 \approx e^{2\gamma_1 t} \int F(\tfrac{R}{L}) \left[(\hat{\boldsymbol{e}}_2\cdot\nabla)^2 + (\hat{\boldsymbol{e}}_3\cdot\nabla)^2\right] p(\boldsymbol{R},0|\boldsymbol{0},t)\, d\boldsymbol{R}$$

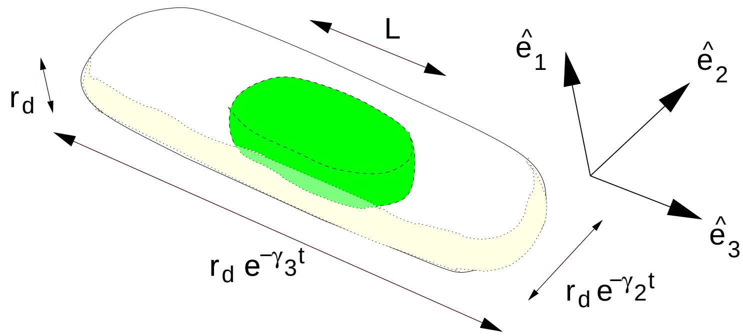
Case 1: $\gamma_2 > 0$



$p \approx 1/(r_d e^{-\gamma_3 t} \cdot r_d^2)$, $\quad (\hat{\boldsymbol{e}}_2\cdot\nabla)^2 \approx 1/r_d^2$, $\quad F\, d\boldsymbol{R} \approx L^2 \cdot L r_d^2$

$$\boxed{|\boldsymbol{B}(t)|^2 \approx (L/r_d)^3\, e^{(2\gamma_1+\gamma_3)t} = (L/r_d)^3\, e^{(\gamma_1-\gamma_2)t}}$$

Case 2: $\gamma_2 < 0$



$p \approx 1/(r_d e^{-\gamma_3 t} \cdot r_d e^{-\gamma_2 t} \cdot r_d)$, $(\hat{\boldsymbol{e}}_2\cdot\nabla)^2 \approx 1/(r_d e^{-\gamma_2 t})^2$, $F\, d\boldsymbol{R} \approx L^2 \cdot (L^2 r_d)$

$$\boxed{|\boldsymbol{B}(t)|^2 \approx (L/r_d)^4 e^{(2\gamma_1+3\gamma_2+\gamma_3)t} = (L/r_d)^4 e^{(\gamma_2-\gamma_3)t}}$$

Small-scale dynamo. 5

Dynamo and Lyapunov exponents. 2

$$\sigma = \lim_{t\to\infty} \frac{1}{2t} \langle \ln |\boldsymbol{B}(t)|^2 \rangle$$

$$\boxed{\sigma = \min\left(\frac{\lambda_1 - \lambda_2}{2}, \frac{\lambda_2 - \lambda_3}{2}\right)}$$

Remarks:

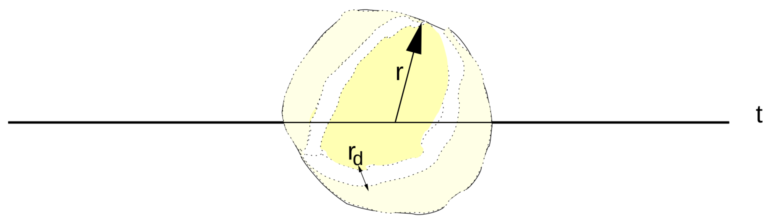
- dynamo if $\lambda_1 > \lambda_2 > \lambda_3$
- statistical antidynamo: for axially symmetric flows $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$, then $\sigma = 0$
- fast dynamo: σ does not depend on κ
- diffusive anomaly: $\sigma < \lambda_1$
- in homogeneous isotropic turbulence $\lambda_2 \approx \lambda_1/4$, then $\sigma \approx 3\lambda_1/8$
- intermittency: $\langle |\boldsymbol{B}(t)|^n \rangle \sim e^{E_n t}$ with $E_n > n\sigma$
- nonuniversality: E_n depends on the detailed form of the Cramér function

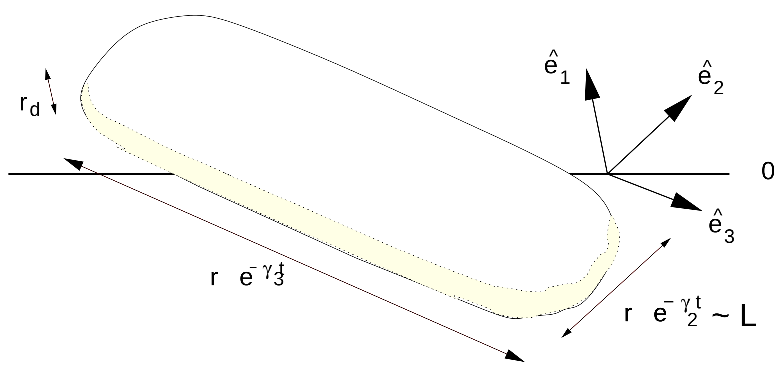
Small-scale dynamo. 6

The spatial structure of the magnetic field

$$C(r,t) = \langle \boldsymbol{B}(\boldsymbol{x}+\boldsymbol{r},t)\cdot \boldsymbol{B}(\boldsymbol{x},t)\rangle =$$

$$= \mathsf{W}_{ki}(t)\mathsf{W}_{kj}(t)\int p(\boldsymbol{R},0|\boldsymbol{r},t)(\delta_{ij}\nabla^2 - \nabla_i\nabla_j)F(R/L)\,d\boldsymbol{R}$$





$$C(r,t) \approx (L/r)\int e^{(\gamma_1-\bar{\gamma}_2)t - tS(\gamma_1,\bar{\gamma}_2)}d\gamma_1 \;\text{ with } \bar{\gamma}_2 = (1/t)\ln(r/L)$$

$$\boxed{C(r,t) \sim r^{-2-h}\,e^{E_2 t}}$$

$h = \partial S/\partial\gamma_2(\gamma_1^\star, 0)$ nonuniversal ($h = 1/2$ for time-reversible flows)

Dynamo at all scales. 1

The Kraichnan-Kazantsev model

$$\partial_t \boldsymbol{B} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \kappa \nabla^2 \boldsymbol{B}$$

The velocity field $\boldsymbol{v}$ is chosen

- Gaussian with zero-mean
- homogeneous and isotropic
- δ-correlated in time

This allows to write a closed p.d.e. for
$$C(r,t) = \langle \boldsymbol{B}(\boldsymbol{x}+\boldsymbol{r},t) \cdot \boldsymbol{B}(\boldsymbol{x},t) \rangle$$

A change of variables maps the p.d.e. into an (imaginary-time) Schrödinger equation
$$\partial_t \psi(r,t) = \frac{1}{m(r)}[\partial_r^2 - V(r)]\psi(r,t)$$

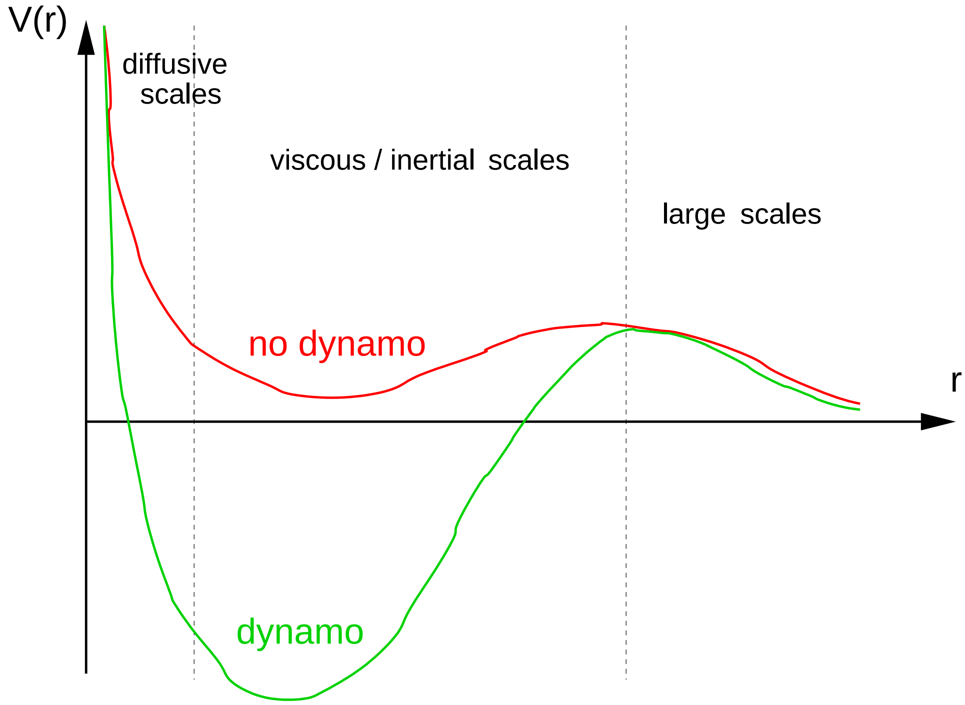
where ψ depends on C while m and U depend on the spectrum of $\boldsymbol{v}$ and on κ

Look at "eigenstates" $\psi(r,t) = \psi_E(r)e^{-Et}$

Dynamo at all scales. 2

Ground states, energy levels, and dynamo

Negative energies correspond to exponential growth in time



When are there bound states ?

What are the typical time- and lengthscales ?

Dynamo at all scales. 3

Analytical results

Vergassola PRE **53** (1996)

Self-similar velocity $\langle|\delta_r v|^2\rangle \sim r^\xi$

Large-scale limit: $\xi = 0$

Small-scale limit: $\xi = 2$

Kolmogorov value: $\xi = 4/3$

$$V(r) = m(r)U(r) = \begin{cases} \dfrac{2}{r^2} & r \ll r_d \\ \dfrac{(2 - \frac{3}{2}\xi - \frac{3}{4}\xi^2)}{r^2} & r \gg r_d \end{cases}$$

Dynamo effect for $\xi > 1$

Remarks:

- no dynamo for $\xi = 0$: short-correlated flows have positive eddy-diffusivity
- dynamo for $\xi = 2$: analytical solution for $C(r,t) \propto r^{-5/2} e^{5\lambda_1 t/4}$
- For $1 < \xi < 2$ the magnetic field correlation function at scales $r \gg r_d$ is $C(r,t) \propto -e^{-b\,(r/r_d)^{1-\xi/2}}\, e^{\,ct/t_d}$ where $t_d = r_d^2/\kappa$ is the diffusive timescale

Dynamo at all scales. 4

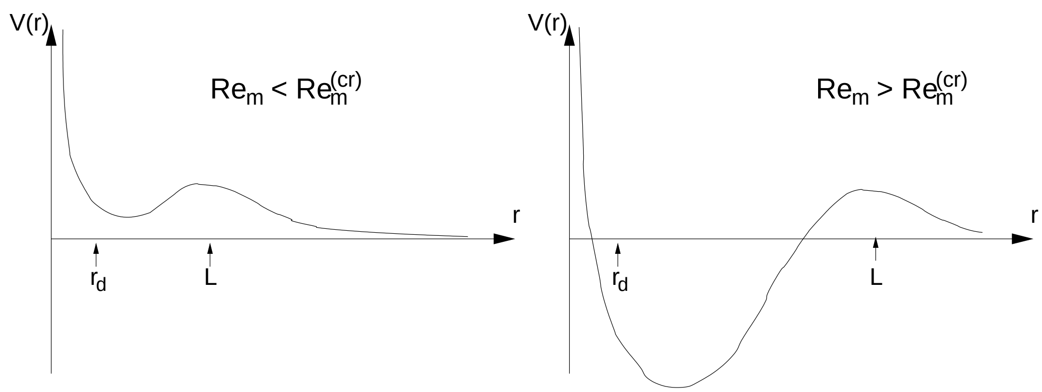
Numerical results. 1

Vincenzi, J. Stat. Phys. **106** (2002)

Find the ground state by variation-iteration method

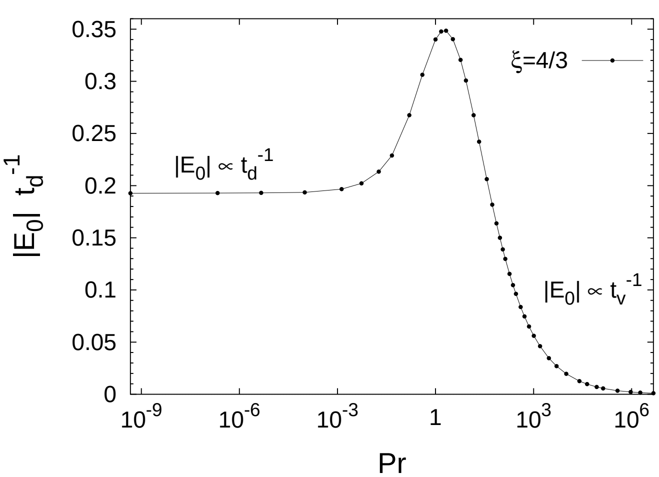
Finite Re_m effects

Existence of a critical $Re_m^{(cr)}$ for the onset of dynamo



Nonzero Pr effects

Maximal growth rate for $Pr \approx 1$



Perspectives

The nonlinear problem: active *vs* passive transport

$$\partial_t \boldsymbol{B} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{v} + \kappa\nabla^2\boldsymbol{B}$$
$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = -\boldsymbol{\nabla} p + (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{B} + \nu\nabla^2\boldsymbol{v}$$

2D MHD: a test case

Celani *et al*, PRL **89** (2002)

$$\partial_t a + (\boldsymbol{v}\cdot\boldsymbol{\nabla})a = \kappa\nabla^2 a + f$$

where a is the magnetic potential $\boldsymbol{B} = \boldsymbol{\nabla}^{\perp} a$