Rotating MHD Boundary Layers and Waves with application to Self-gravitating Rapidly Rotating Spherical Shells

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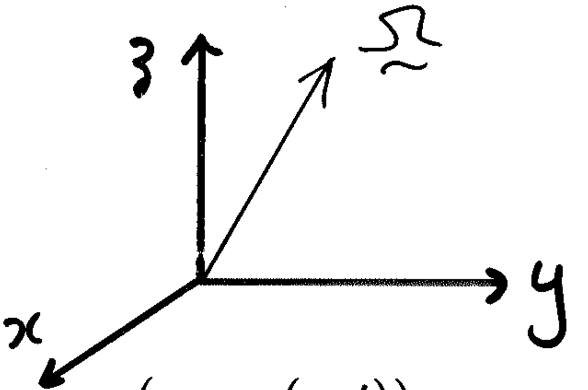
Inertial Waves

Consider the incompressible ($\nabla \cdot \mathbf{u} = 0$), coplanar motion



$$\mathbf{u}(z,t) = (u,\ v,\ 0)$$

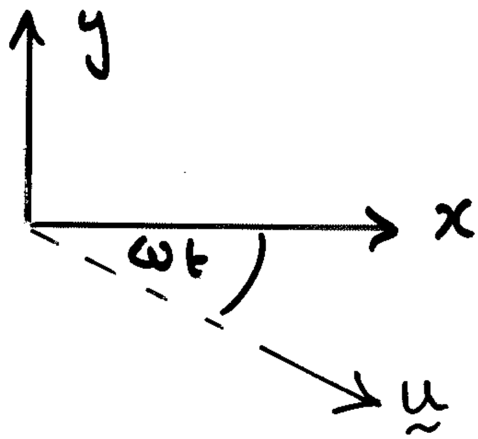
satisfying

$$\frac{\partial \mathbf{u}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{u} = -\nabla p \qquad (p = p(z,t)).$$

The x and y components satisfy

$$\frac{\partial u}{\partial t} - 2\Omega_z v = 0\,,$$

$$\frac{\partial v}{\partial t} + 2\Omega_z u = 0\,,$$



where $\mathbf{\Omega} = (\Omega_x,\ \Omega_y,\ \Omega_z)$. Thus

$$\frac{\partial}{\partial t}(u + \mathrm{i}v) + 2\mathrm{i}\Omega_z(u + \mathrm{i}v) = 0$$

with solution

$$u + \mathrm{i}v = (u_0 + \mathrm{i}v_0)\exp(-\mathrm{i}\omega t)\,,$$

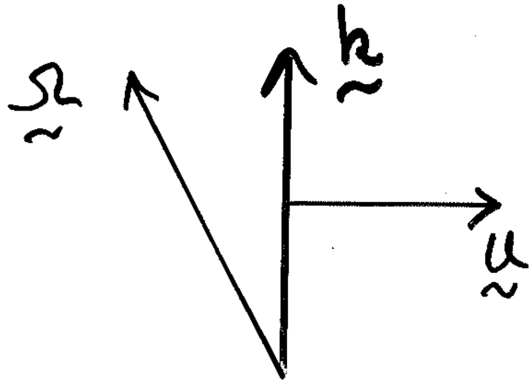
where $\omega = 2\Omega_z$.

Inertial waves

$$\mathbf{u} = \mathrm{Re}\left\{\hat{\mathbf{u}}\, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right\}$$

are circularly polarised

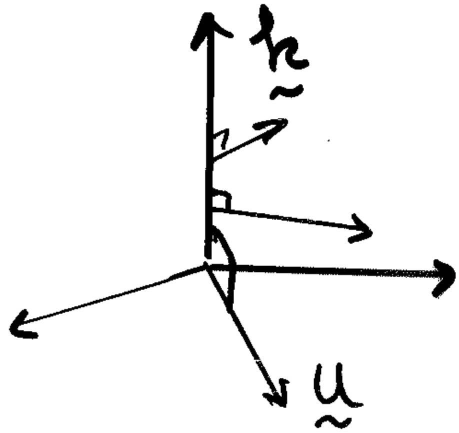
$$|\mathbf{u}| = |\hat{\mathbf{u}}| \quad \text{and} \quad \mathbf{k}\cdot\mathbf{u} = 0\,.$$



Their frequency is

$$\omega = \pm\omega_C \;=\; \pm 2\boldsymbol{\Omega}\cdot\mathbf{k}/|\mathbf{k}|$$

and they are highly dispersive.



Taylor-Proudman Theorem

The solution of

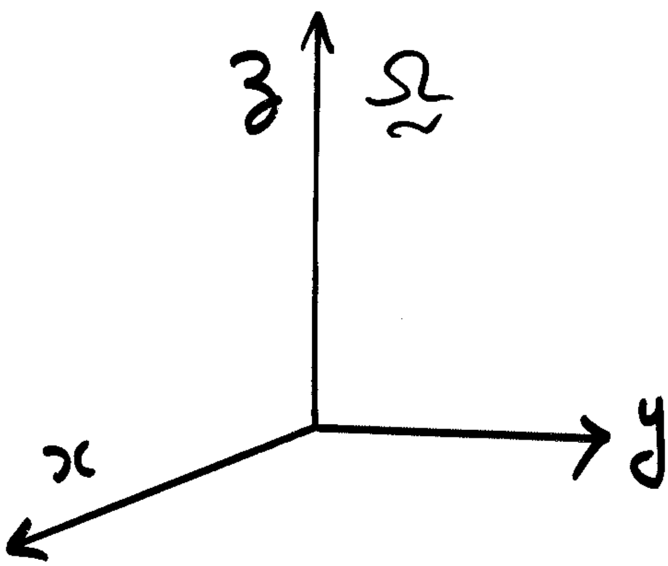
$$2\boldsymbol{\Omega}\times\mathbf{u}_G = -\boldsymbol{\nabla}p$$

with $\boldsymbol{\Omega} = (0, 0, \Omega)$ is

$$\mathbf{u}_G = \left(\frac{\partial\psi}{\partial y}\,, -\frac{\partial\psi}{\partial x}\,, w\right),$$

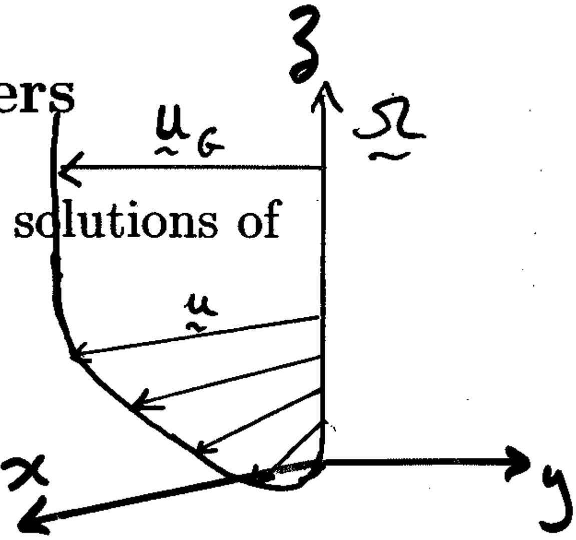
where

$$\psi = \psi(x, y, t)\,, \qquad w = w(x, y, t)\,.$$



This is the zero frequency $(\omega = 0)$ inertial wave.

Ekman Layers



Reinstate viscosity and consider solutions of

$$-2\Omega(v - v_G) = \nu \frac{\partial^2 u}{\partial z^2}\,,$$

$$2\Omega(u - u_G) = \nu \frac{\partial^2 v}{\partial z^2}\,,$$

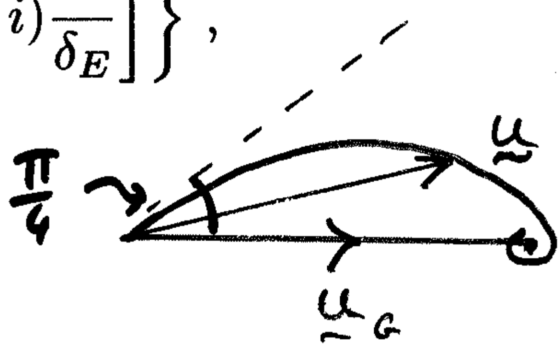
subject to $\mathbf{u} = 0$ on $z = 0$ and $\mathbf{u} \to \mathbf{u}_G$ as $z \uparrow \infty$. The components of $\mathbf{u} = (u, v)$ are determined by the real and imaginary parts of

$$Z \;=\; Z_G \left\{ 1 \;-\; \exp\left[-(1+i)\frac{z}{\delta_E} \right] \right\}\,,$$

where $Z \equiv u + \mathrm{i}v$,
$Z_G \equiv u_G + \mathrm{i}v_G$ and

$$\delta_E = \sqrt{\nu/\Omega}$$

is the **Ekman layer thickness.**



EKMAN SPIRAL

Integration gives

$$\int_0^\infty (Z - Z_G)\,\mathrm{d}z \;=\; -\frac{1-\mathrm{i}}{2} Z_G\,,$$

which determines the **mass flux deficit**

$$\int_0^\infty (\mathbf{u} - \mathbf{u}_G)\,\mathrm{d}z \;=\; \tfrac{1}{2}\delta_E\,(-u_G - v_G\,,\, u_G - v_G)$$

in the boundary layer.

Mass continuity determines

$$\frac{\partial w}{\partial z} = -\frac{\partial}{\partial x}(u - u_G) - \frac{\partial}{\partial y}(v - v_G)$$

and so the vertical velocity $w \to w_E$ as $z/\delta_E \uparrow \infty$ is

$$w_E = -\frac{\partial}{\partial x}\left[\int_0^\infty (u - u_G)\mathrm{d}z\right] - \frac{\partial}{\partial y}\left[\int_0^\infty (v - v_G)\mathrm{d}z\right]$$

giving

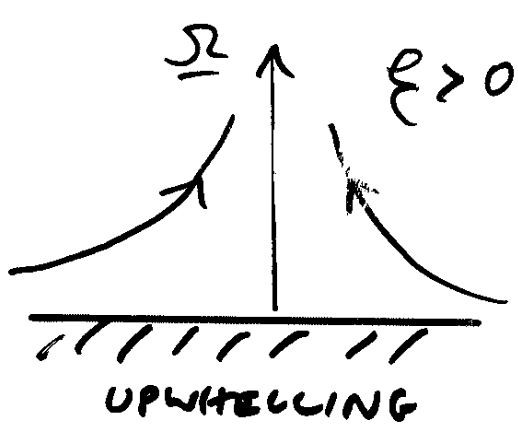
$$w_E = \frac{\delta_E}{2}\left[\left(\frac{\partial u_G}{\partial x} + \frac{\partial v_G}{\partial y}\right) + \left(\frac{\partial v_G}{\partial x} - \frac{\partial u_G}{\partial y}\right)\right]$$

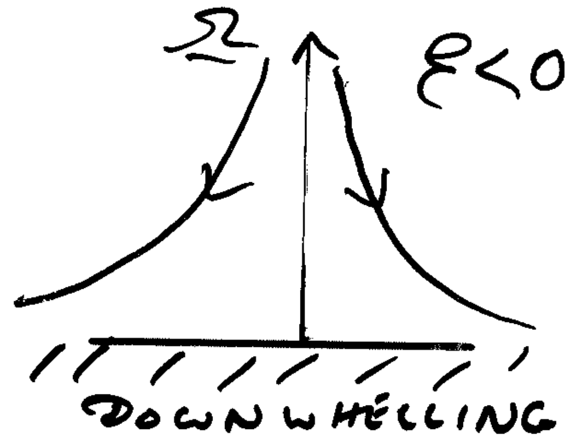
or equivalently, since $\nabla \cdot \mathbf{u}_G = 0$,
the **Ekman pumping** velocity is

$$w_E = \tfrac{1}{2}\delta_E \zeta \, ,$$

where $$\zeta = \hat{\mathbf{z}} \cdot \nabla \times \mathbf{u}_G = -\nabla^2 \psi$$

is the **axial vorticity**.





Spatially Damped Steady Waves

Consider the wave-like solutions

$$\mathbf{u} = \mathrm{Re}\left\{\hat{\mathbf{u}}\,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\right\},$$

with complex wave vector $\mathbf{k}$, of

$$2\boldsymbol{\Omega}\times\mathbf{u} = -\boldsymbol{\nabla}p + \nu\nabla^2\mathbf{u}\,.$$

The dispersion relation is

$$[\mathrm{i}\nu\mathbf{k}\cdot\mathbf{k}]^2 \;=\; [2\boldsymbol{\Omega}\cdot\mathbf{k}]^2/\mathbf{k}\cdot\mathbf{k}\,.$$

Ekman layers with unit normal $\hat{\mathbf{n}}$:

At small Ekman number, we have $\mathbf{k} \approx k\hat{\mathbf{n}}$ giving $k^2 \approx \mp 2\mathrm{i}\hat{\mathbf{n}}\cdot\boldsymbol{\Omega}/\nu$, i.e.

$$\mathrm{i}k \;\approx\; \pm\,\frac{1+i}{\delta_E} \qquad\qquad \delta_{E\theta} = \sqrt{\frac{\nu}{\hat{\mathbf{n}}\cdot\boldsymbol{\Omega}}}\,.$$

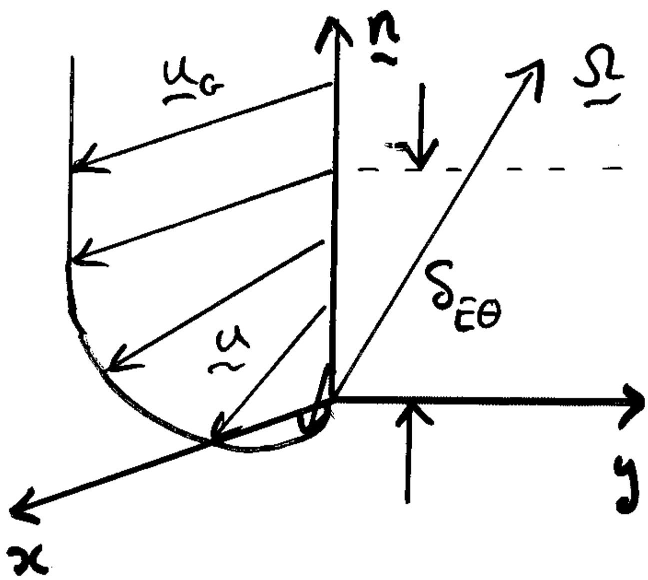
Alternatively we write

$$\delta_{E\theta} = \left(\frac{E}{\cos\theta}\right)^{1/2} H\,,$$

where

$$E \;=\; \frac{\nu}{H^2\Omega}$$

is the **Ekman number.**



Side Wall $E^{1/3}$-layers

With $\boldsymbol{\Omega} = (0, 0, \Omega)$ set $\mathbf{k} = (k_\perp\,,\,0\,,\,k_\parallel)$, where $k_\parallel$ is real and $k_\perp$ is complex. Assume that $|k_\perp| \gg |k_\parallel|$ and so obtain the approximate result

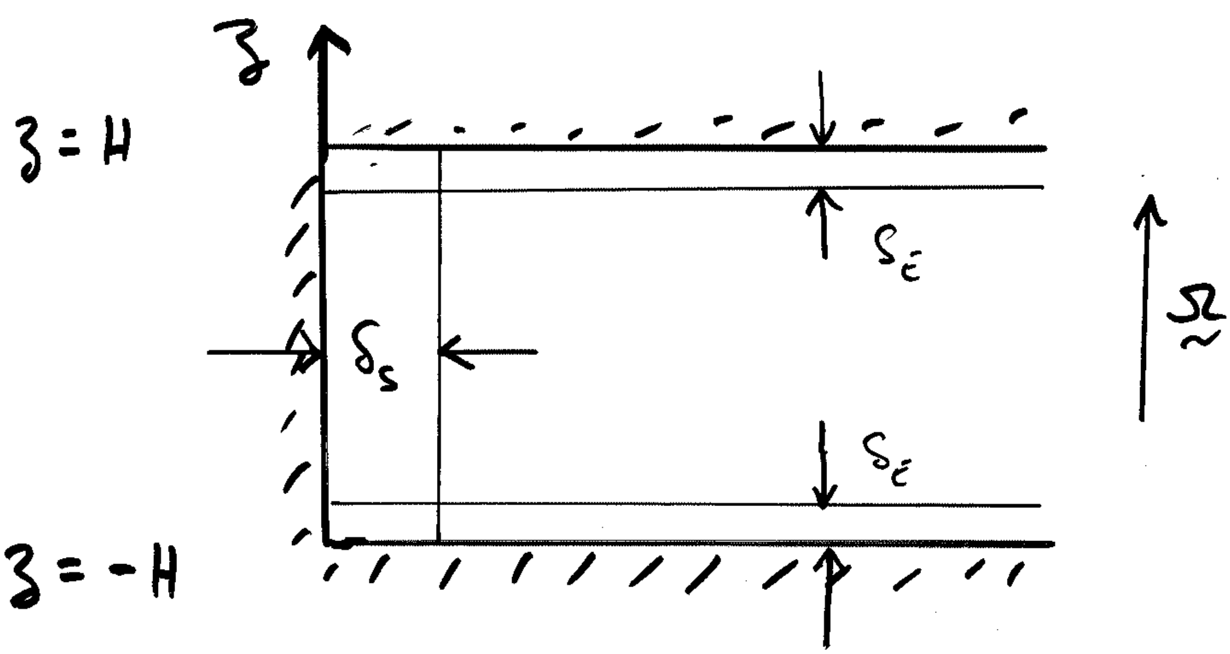
$$\mathrm{i}\,k_\perp \approx \left(\pm 2\frac{k_\parallel \Omega}{\nu}\right)^{1/3}.$$

Here we are envisaging disturbances of the type

$$\mathbf{u} = \mathrm{Re}\left\{\hat{\mathbf{u}}\,\cos(k_\parallel z)\mathrm{e}^{\mathrm{i}\,k_\perp x}\right\}\,,$$

where $k_\parallel = \pi/H$. This gives us the possibility of constructing boundary layers on the length

$$\delta_S \equiv \left(\frac{\nu H}{\Omega}\right)^{1/3} = \left(\delta_E^2 H\right)^{1/3} = E^{1/3} H\,.$$



Boundary Singularity

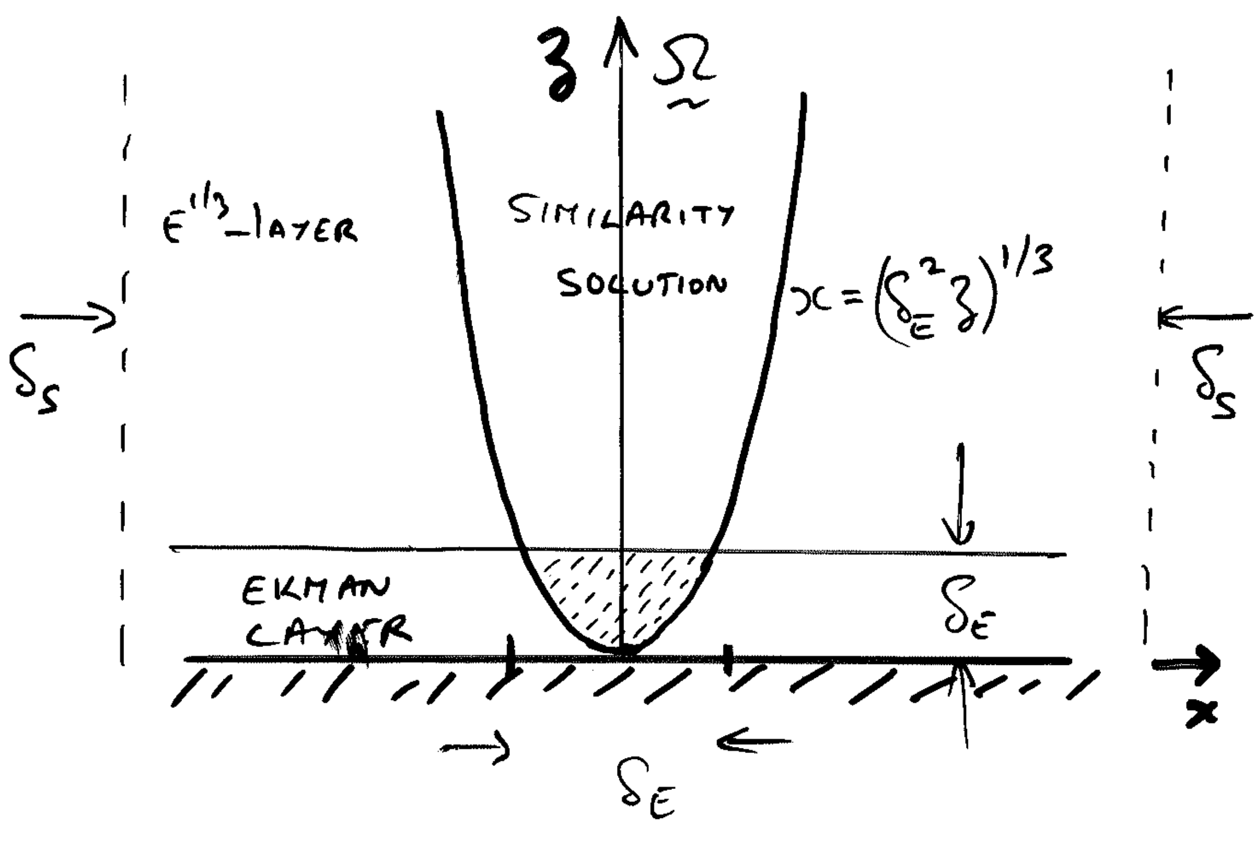
In the neighbourhood of a boundary singularity at $(0,0,0)$, we may make the similarity assumption that $x \sim \delta_S(H)$, in which $H \sim z$. Thus solutions depend on the similarity variable

$$\xi = \frac{x}{(\delta_E^2 z)^{1/3}},$$

when $\xi = O(1)$ and z is sufficiently large.

Plane boundary

The similarity solution intersects the Ekman layer at $z = \delta_E$, when $x = (\delta_E^2 z)^{1/3} = \delta_E$. Thus both solutions fail when $x \sim z = O(E^{1/2})$.



Example: The boundaries $z = \pm H$ move with velocity $(0, \pm(\text{sgn}x)V, 0)$, so that the geostrophic velocity of the mainstream flow vanishes; $\mathbf{u}_G = \mathbf{0}$.

The x–component of the Ekman flux carried by the Ekman layer on the lower boundary $z = -H$ is

$$Q = -(\text{sgn}x)\tfrac{1}{2}\delta_E V.$$

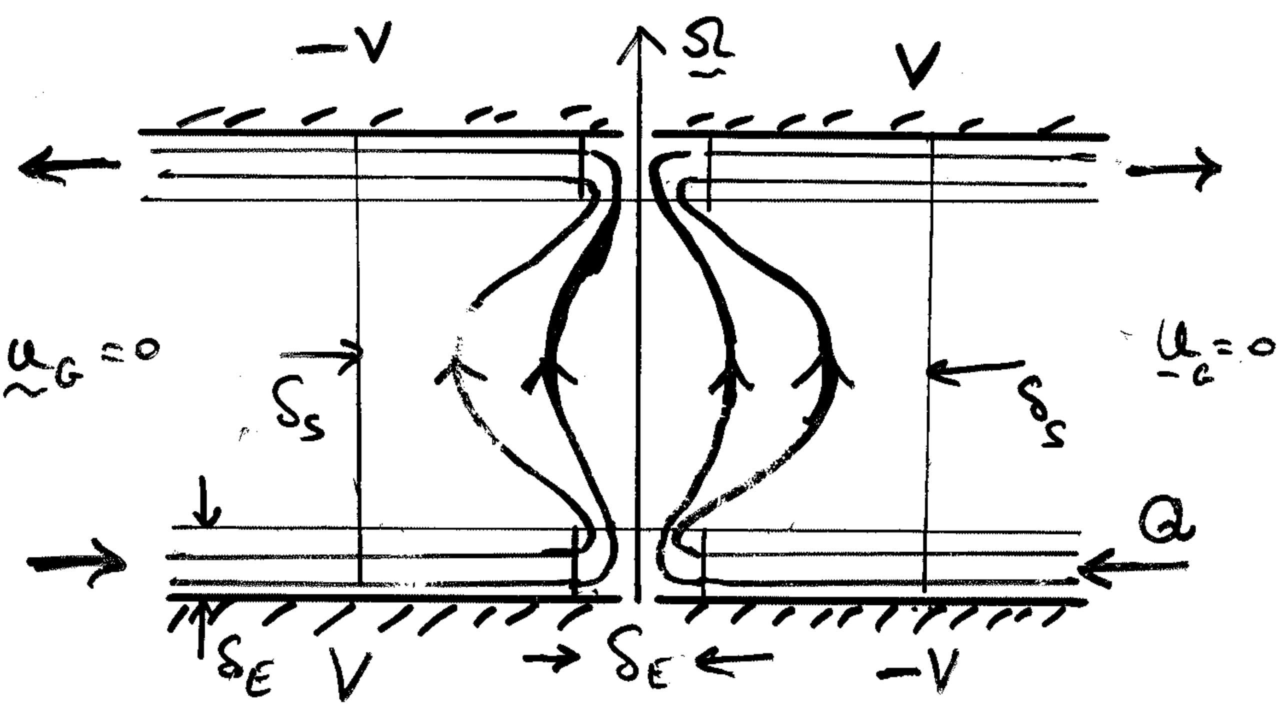
This determines the Ekman pumping velocity

$$w_E = \delta_E V\,\delta(x)$$

out of the lower Ekman layer. The small $E^{1/2} \times E^{1/2}$ regions in the neighbourhood of $(0, 0, \pm H)$ provide a sink–source combination.

In the similarity regions the stream-surfaces are

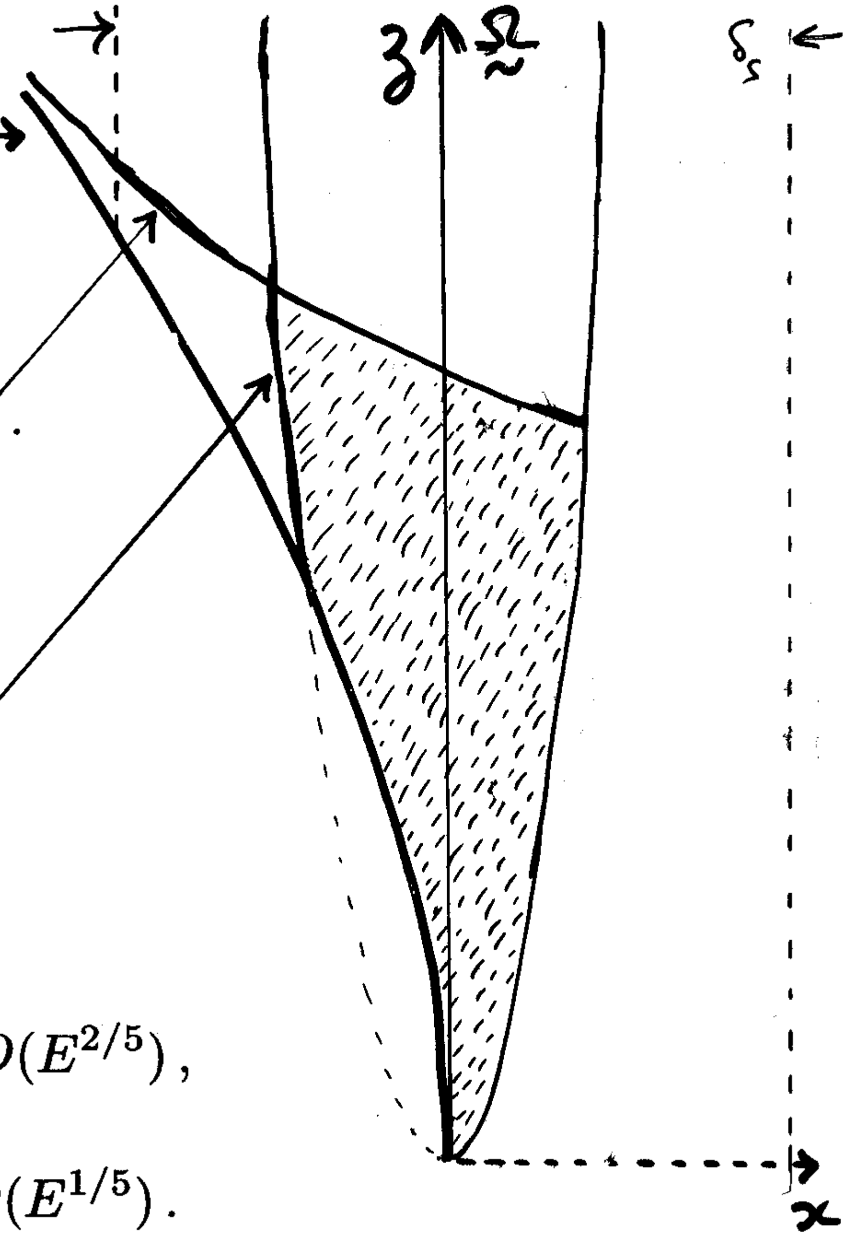
$$\xi = x/[\delta_E^2(z \mp H)^{1/3}] = \text{constant}.$$



Tangent cylinder

Consider an inner sphere radius r_i. Correct to lowest order its boundary $x = x_B(z)$ is located at

$$x_B = -\frac{z^2}{2r_i}.$$



Ekman layer width is

$$\delta_{E\theta} = \delta_E \left(\frac{r_i}{z}\right)^{1/2}.$$

Similarity solution width is

$$\delta_S = (\delta_E^2 z)^{1/3}.$$

All widths are the same when

$$|x| \sim \left(\delta_E^4 r_i\right)^{1/5} = O(E^{2/5}),$$

$$z \sim \left(\delta_E^2 r_i^3\right)^{1/5} = O(E^{1/5}).$$

Here the Ekman layer and similarity solutions fail.

Quasi-geostrophic Flow Between Two Parallel Planes

Relative to boundaries $z = \pm H$ with $\mathbf{\Omega} = (0, 0, \Omega)$,
we write $\mathbf{u} = (u, v, w)$ and consider
the axial vorticity equation

$$\Big[2\mathbf{\Omega}\cdot\mathbf{\nabla}\times(-v, u, 0) \;=\Big] \; - 2\Omega\frac{\partial w}{\partial z} \;=\; \nu\nabla^2\zeta\,.$$

Integrating between the boundaries
(EXCLUDING the **Ekman layers**) and use of
the **Ekman jump condition** determines

$$\frac{\delta\Omega}{H}\zeta \;=\; \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\zeta\,.$$

Thus the geostrophic flow $\mathbf{u}_G = (u_G, v_G)$:

$$u_G = \frac{\partial\psi}{\partial y}\,, \qquad v_G \;=\; -\frac{\partial\psi}{\partial x}\,, \qquad \psi \;\equiv\; -\,\frac{p_G}{2\Omega}\,.$$

is obtained from

$$\zeta \;=\; \frac{\partial v_G}{\partial x} - \frac{\partial u_G}{\partial y} \;=\; -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi\,.$$

Side Wall $E^{1/4}$-layers

Example: When $\mathbf{u}_G = (0, v_G(x))$, the boundary layer solutions of

$$\frac{\delta\Omega}{H} v_G = \nu \frac{\partial^2 v_G}{\partial x^2}$$

have the form

$$v_G \propto \exp[\pm(x/\Delta)],$$

where

$$\Delta = \sqrt{\frac{\nu H}{\Omega\delta}} = \sqrt{\delta H} = E^{1/4} H \, .$$

Example: The boundaries $z = \pm H$ move with velocity (0, $-(\mathrm{sgn} x)V$, 0), so that the quasi-geostrophic velocity in the $E^{1/4}$–layer is

$$v_G = -(\mathrm{sgn} x) V \left[1 - \exp\left(-|x|/\Delta\right)\right] .$$

This determines the Ekman pumping velocity

$$w_E = \delta_E V \left[-\frac{1}{2\Delta} \exp\left(-\frac{|x|}{\Delta} \right) + \delta(x) \right]$$

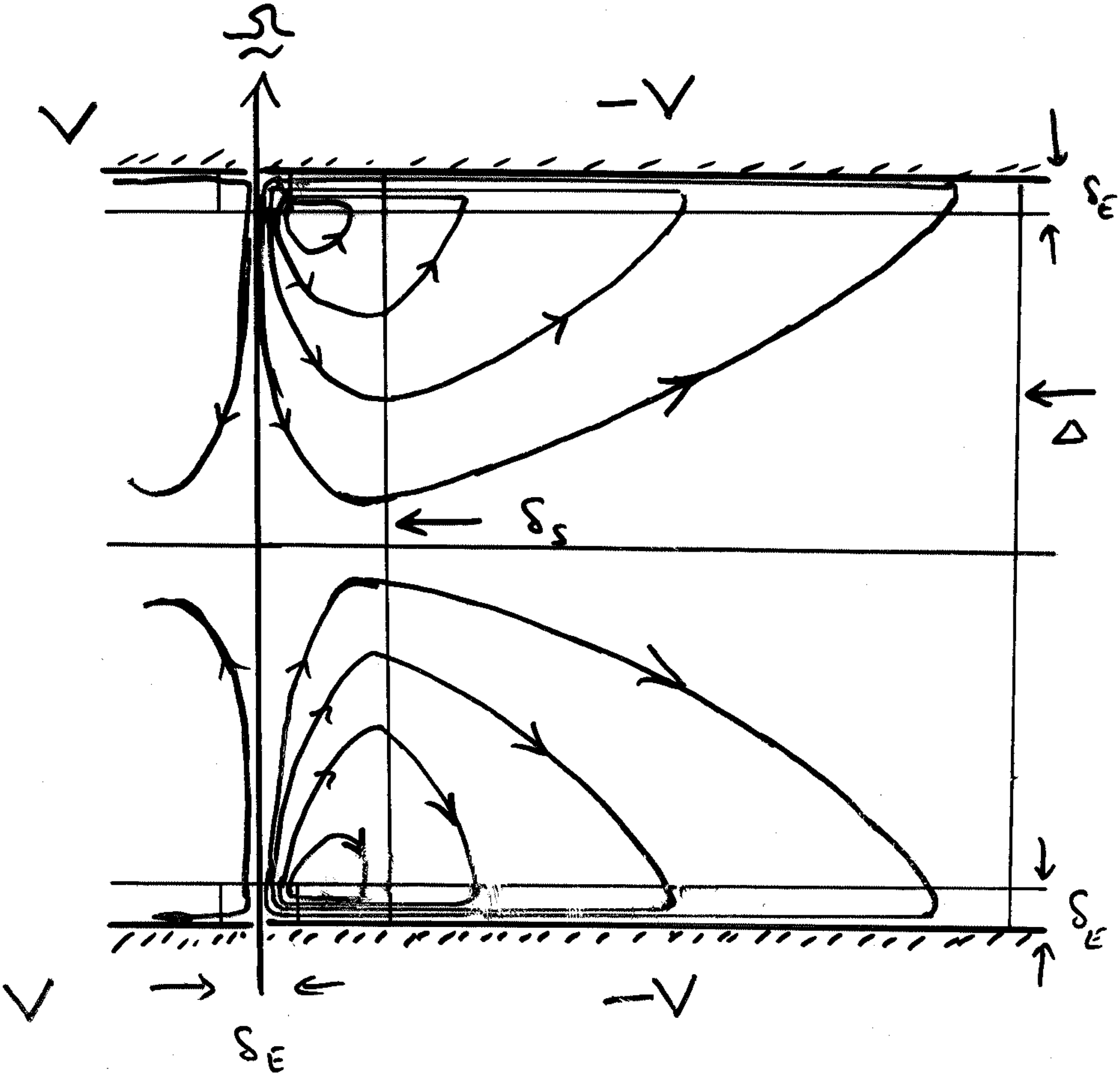
out of the lower Ekman layer.

Outside the $E^{1/3}$–layer, where $|x| \gg HE^{1/3}$, the axial velocity is

$$w = \frac{\delta_E}{2\Delta} V \frac{z}{H} \exp\left(-\frac{|x|}{\Delta}\right) = O(E^{1/4}) .$$

For this velocity, the stream-surfaces are

$$w = \text{constant}.$$



The Governing Equations

The ϕ-component of **equation of motion** is

$$-2\frac{\partial}{\partial z}\left(\frac{\psi}{s}\right) = E\Delta(s\Omega);$$

the ϕ-component of **the vorticity equation** is

$$-2\frac{\partial}{\partial z}(s\Omega) = E\Delta^2\left(\frac{\psi}{s}\right),$$

where $$\Delta \equiv \nabla^2 - s^{-2}.$$

The Ekman Boundary conditions

$H = z_T - z_B,$

$\cos\theta_T = \sqrt{1-(s/\alpha)^2},$

$\cos\theta_B = \sqrt{1-s^2}.$

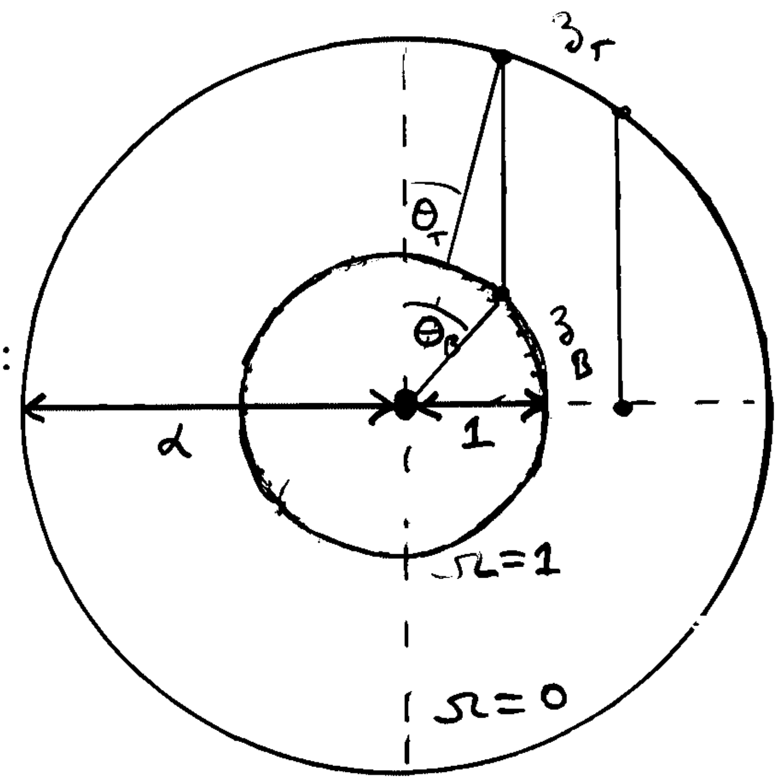
Ekman layer fluxes are:

top boundary

$$2\psi_T = \frac{E^{1/2}s^2}{\sqrt{\cos\theta_T}}\Omega;$$

bottom boundary

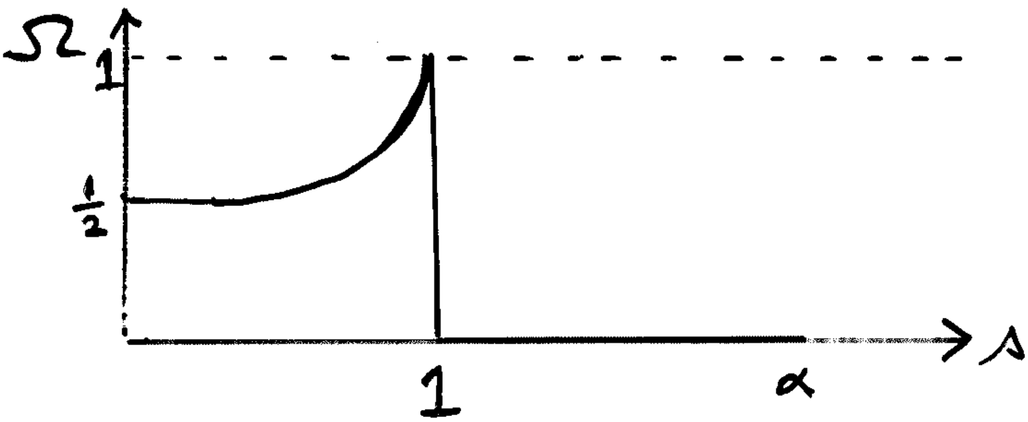
$$2\psi_B = \begin{cases} -\dfrac{E^{1/2}s^2}{\sqrt{\cos\theta_B}}(1-\Omega) & (s<1); \\ 0 & (s<1). \end{cases}$$



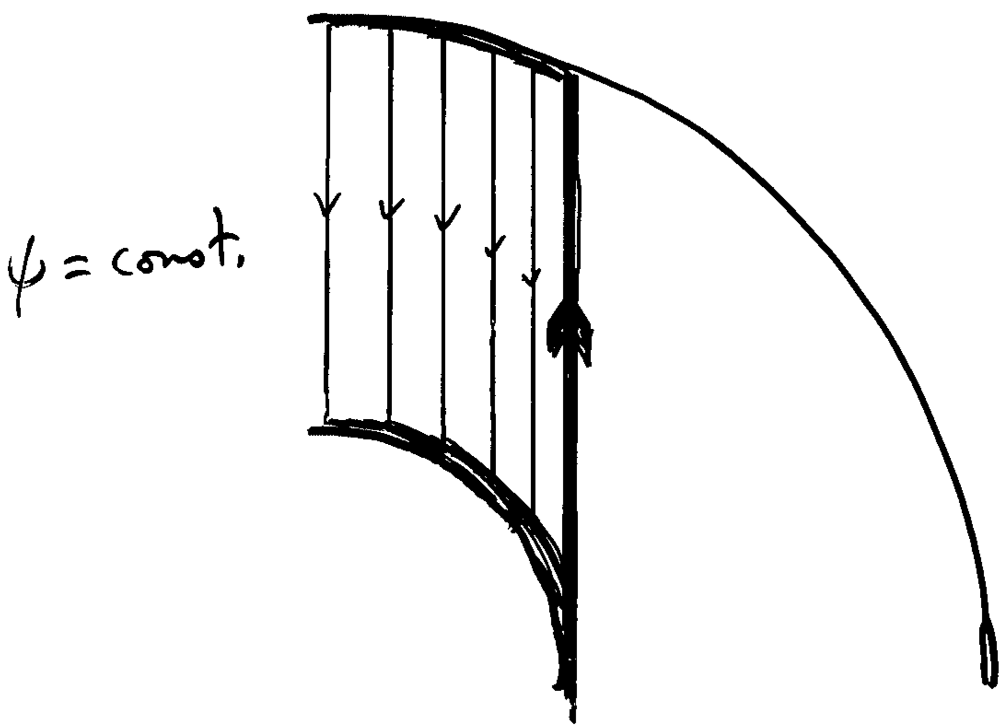
The Proudman Solution (1956)

Set $\psi = \psi_T = \psi_B$ to obtain

$$1 - \Omega = \begin{cases} \dfrac{(1-s^2)^{1/4}}{(1-s^2)^{1/4} + (1-(s/\alpha)^2)^{1/4}} & (s < 1); \\ 1 & (s > 1). \end{cases}$$



$$2\psi = \begin{cases} -\dfrac{E^{1/2}s^2}{(1-s^2)^{1/4} + (1-(s/\alpha)^2)^{1/4}} & (s < 1); \\ 0 & (s > 1). \end{cases}$$



with solution

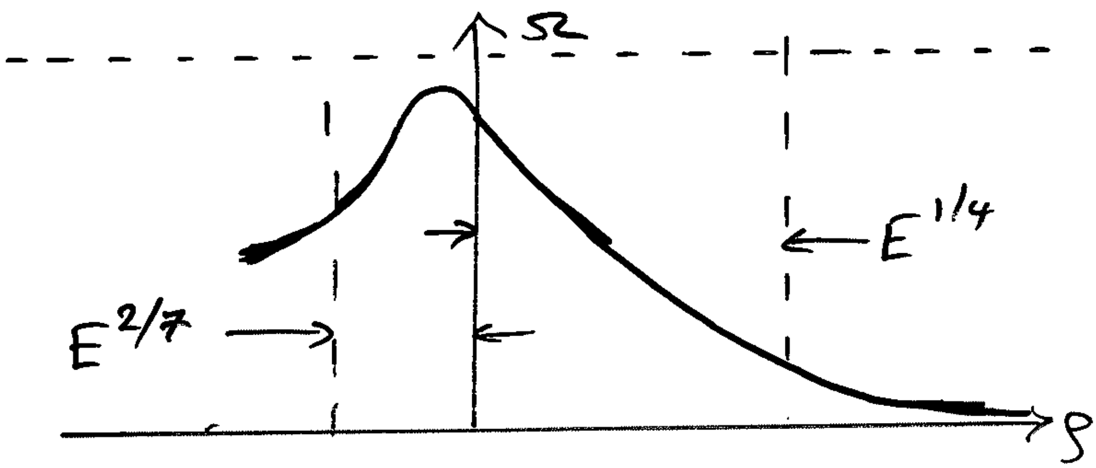
$$1 - \Omega = B\mathcal{F}\left(\frac{\bullet \rho}{E^{2/7}}\right) + E^{1/14}\mathcal{G}\left(\frac{\bullet \rho}{E^{2/7}}\right),$$

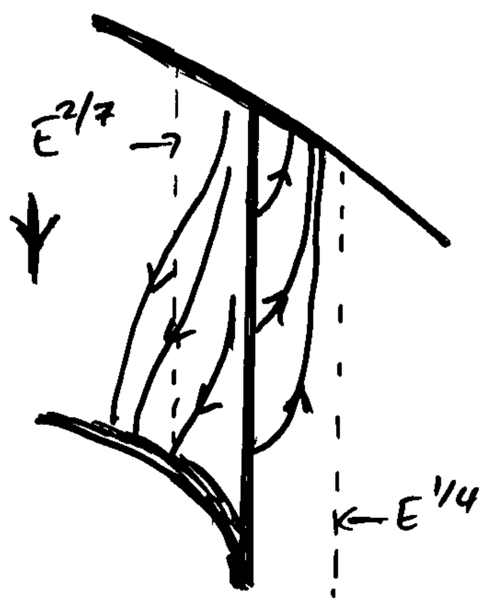
where $\mathcal{F}$ ($\mathcal{F}(0) = 1$) and $\mathcal{G}$ are appropriate solutions. Continuity of Ω and $\mathrm{d}\Omega/\mathrm{d}\rho$ at $\rho = 0$ give

$$A \approx B \qquad (1 - A) \bullet E^{-1/4} \approx B \bullet E^{-2/7}$$

with solution

$$A \approx \bullet E^{-\frac{1}{4}+\frac{2}{7}} = \mathcal{O}(E^{1/28}).$$



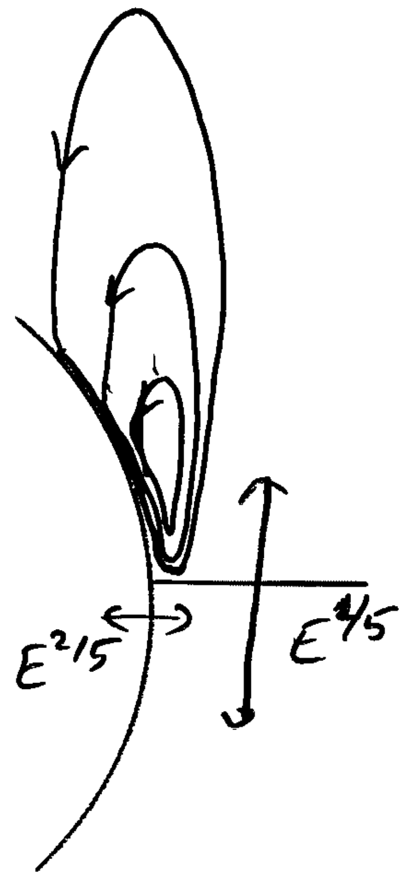


Similarity Solution; $E^{1/5} \ll z \ll 1$

For small z, Stewartson's integral reduces to

$$\widetilde{\psi} = \frac{E^{\frac{19}{42}} A \bullet}{2\pi} \int_{-\infty}^{\infty} \frac{[\exp(-\frac{1}{2}|k|^3 z) + \mathrm{i}k\rho/E^{1/3}]\,\mathrm{d}k}{(-ik)^{3/4}}\,.$$

which leads to the similarity forms



$$\widetilde{\psi} = A\,E^{\frac{2}{5}} \left(\frac{E^{\frac{1}{5}}}{z}\right)^{\frac{1}{12}} F(\xi)\,,$$

$$E^{-\frac{1}{3}}\widetilde{\Omega} = A \left(\frac{E^{\frac{1}{5}}}{z}\right)^{\frac{5}{12}} G(\xi)\,,$$

where F and G are functions

of $\qquad \xi = \dfrac{\rho}{(Ez)^{1/3}}\,.$

The eddy recirculates through the

Equatorial Ekman Layer,

where $z = \mathcal{O}(E^{1/5})$ and $\rho = \mathcal{O}(E^{2/5})$. There the volume flux is maximised with

$$\widetilde{\psi} = \mathcal{O}(AE^{\frac{2}{5}}) = \mathcal{O}(E^{\frac{61}{140}}).$$

Steady Rotating MHD

The slow steady motion of an electrically conducting fluid permeated by a uniform magnetic field $\mathbf{B}$ is governed by

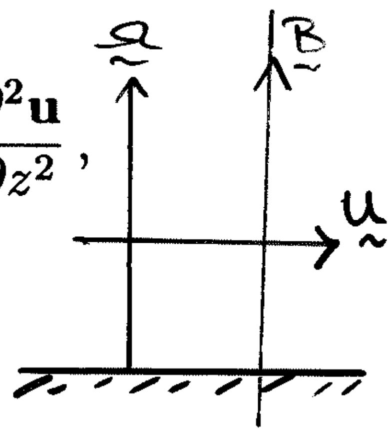
$$2\mathbf{\Omega}\times\mathbf{u} = -\nabla p + \frac{1}{\mu\rho}\mathbf{B}\cdot\nabla\mathbf{b} + \nu\nabla^2\mathbf{u}\,,$$

$$\mathbf{0} = \mathbf{B}\cdot\nabla\mathbf{u} + \eta\nabla^2\mathbf{b}\,,$$

where $\mathbf{b}$ is the perturbation magnetic field.
For the particular case $\mathbf{B} = (0,0,B)$, $\mathbf{\Omega} = (0,0,\Omega)$ and

$$\mathbf{u} = (u(z),\, v(z),\, 0)\,, \qquad \mathbf{b} = (b_x(z),\, b_y(z),\, 0)\,,$$

the equations reduce to



$$2\mathbf{\Omega}\times\mathbf{u} = -\nabla p + \frac{B}{\mu\rho}\frac{\partial\mathbf{b}}{\partial z} + \nu\frac{\partial^2\mathbf{u}}{\partial z^2}\,,$$

$$\mathbf{0} = B\frac{\partial\mathbf{u}}{\partial z} + \eta\frac{\partial^2\mathbf{b}}{\partial z^2}\,.$$

These combine to yield

$$2\mathbf{\Omega}\times(\mathbf{u}-\mathbf{u}_{MG}) = -\frac{B^2}{\eta\mu\rho}(\mathbf{u}-\mathbf{u}_{MG}) + \nu\frac{\partial^2\mathbf{u}}{\partial z^2}\,,$$

where $\mathbf{u}_{MG} = (u_{MG}, v_{MG}, 0)$ is some constant **magneto-geostrophic** flow.

The solution subject to $\mathbf{u} = 0$ on $z = 0$ and $\mathbf{u} \to \mathbf{u}_{MG}$ as $z \uparrow \infty$ is determined by the real and imaginary parts of

$$Z \;=\; Z_{MG}\,[1 \;-\; \exp(-nz)]\;,$$

where $Z_{MG} \equiv u_{MG} + \mathrm{i}v_{MG}$ and n is the root of

$$n^2 \;=\; 2\mathrm{i}\,\frac{\Omega}{\nu} \;+\; \frac{B^2}{\eta\mu\rho\nu}$$

with positive real part.

The **Ekman layer** is recovered in the limit $B = 0$.

The **Hartmann layer** with unidirectional flow is recovered in the limit $\Omega = 0$. It is a boundary layer of width

$$\delta_H \;=\; \sqrt{\eta\mu\rho\nu}/B \;=\; L/M\,,$$

where

$$M^2 \;\equiv\; \frac{L^2B^2}{\eta\mu\rho\nu}\,.$$

In the case of the **Ekman–Hartmann layer**, the relative importance of the magnetic field to the rotation is measured by the **Elsasser number**

$$\Lambda \;\equiv\; \left(\frac{\delta_E}{\delta_H}\right)^2 \;=\; \frac{B^2}{\eta\mu\rho\Omega}\,.$$

Alfvén Waves

Linearised Alfvén waves satisfy

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{1}{\mu\rho}\mathbf{B}\cdot\nabla\mathbf{b}\,, \qquad \frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}\cdot\nabla\mathbf{u}\,.$$

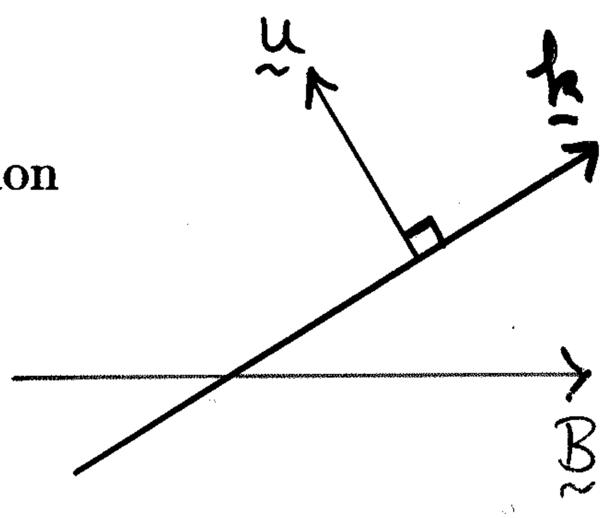
They combine to give the wave equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{1}{\mu\rho}(\mathbf{B}\cdot\nabla)^2\mathbf{u}\,,$$

in which $\mathbf{u}$ is solenoidal: $\nabla\cdot\mathbf{u} = 0$.

Transverse Alfvén waves $\mathbf{k}\cdot\mathbf{u} = 0$
of the form

$$\mathbf{u} = \mathrm{Re}\left\{\hat{\mathbf{u}}\,\mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right\}$$



satisfy the dispersion relation

$$\omega = \pm\mathbf{c}_M\cdot\mathbf{k}\,,$$

where

$$\mathbf{c}_M = \mathbf{B}/\sqrt{\mu\rho}$$

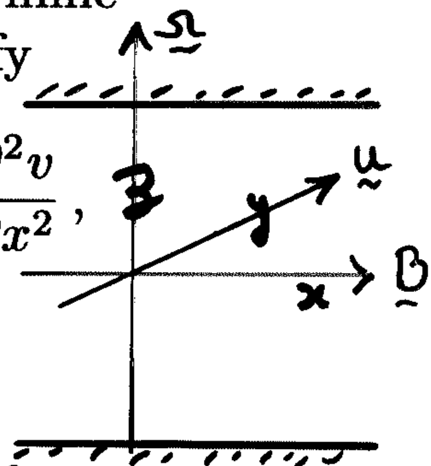
is the **Alfvén velocity**.

Torsional Oscillations

We consider damped torsional waves between two plane boundaries $z = \pm H$. We assume that $\mathbf{B} = (B, 0, 0)$, $\boldsymbol{\Omega} = (0, 0, \Omega)$ and

$$\mathbf{u} = (0,\, v(x,t),\, 0)\,, \qquad \mathbf{b} = (0,\, b(x,t),\, 0)\,.$$

We also assume that Ekman pumping occurs on the boundaries, which limits the size of acceptable $\boldsymbol{\Lambda}$. Then in the presence of viscous and Ohmic dissipation, torsional oscillations satisfy



$$\frac{\partial v}{\partial t} + \frac{\Omega \delta_E}{H} v = \frac{B}{\mu\rho}\frac{\partial b}{\partial x} + \nu \frac{\partial^2 v}{\partial x^2}\,,$$

$$\frac{\partial b}{\partial t} = B\frac{\partial v}{\partial x} + \eta \frac{\partial^2 b}{\partial x^2}$$

The dispersion relation for the damped waves proportional to $\exp[\mathrm{i}(kx - \omega t)]$ is

$$\left(-\mathrm{i}\omega + \frac{\Omega\delta_E}{H} + \nu k^2\right)\left(-\mathrm{i}\omega + \eta k^2\right) = -c_M^2 k^2\,.$$

Which of **lateral friction** or **bottom friction** dominates, depends on the size of the horizontal length scale $1/k$ relative to $E^{1/4}H$.

Rotating MHD Waves

They satisfy

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega}\times\mathbf{u} = \frac{1}{\mu\rho}\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{b}\,, \qquad \frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u}$$

and combine to give

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{1}{\mu\rho}(\mathbf{B}\cdot\boldsymbol{\nabla})^2\mathbf{u} + 2\boldsymbol{\Omega}\times\frac{\partial \mathbf{u}}{\partial t} = 0$$

together with $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$.

Following our previous manipulations, this leads to the dispersion relation

$$\omega - \frac{(\mathbf{c}_M\cdot\mathbf{k})^2}{\omega} = \pm 2\frac{\boldsymbol{\Omega}\cdot\mathbf{k}}{k}\,.$$

Alternatively we may write

$$\omega^2 \pm \omega_C\omega - \omega_M^2 = 0\,,$$

where

$$\omega_C = 2\frac{\boldsymbol{\Omega}\cdot\mathbf{k}}{k}\,, \qquad \omega_M = \mathbf{c}_M\cdot\mathbf{k}$$

are the **inertial** and **Alfvén frequencies** respectively.

MC–Waves

Generally we have

$$\omega_C \gg \omega_M \,.$$

Then the waves split into **fast inertial waves** with frequency

$$\omega \approx \pm\omega_C$$

and **slow MC-waves** with frequency

$$\omega \approx \pm\omega_{MC}\,, \qquad \omega_{MC} = \frac{\omega_M^2}{\omega_C}\,.$$

Another interesting possibility is the **magneto-geostrophic flow** with

$$\omega = 0$$

which occurs when $\mathbf{k} \parallel \mathbf{\Omega} \times \mathbf{B}$. Such motions may play an important role in small scale turbulence.

Stratified Rotating MHD Waves

Consider a Boussinesq fluid density $\rho_0(z)$ relative to co-ordinates such that $\mathbf{g} = -g\hat{\mathbf{z}} = (0, 0, -g)$. The governing equations are

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega}\times\mathbf{u} = -\boldsymbol{\nabla} p + \vartheta\mathbf{g} + \frac{1}{\mu\rho}\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{b} + \nu\nabla^2\mathbf{u}\,,$$

$$\frac{\partial \mathbf{b}}{\partial t} = \mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u} + \eta\nabla^2\mathbf{b}\,,$$

$$\frac{\partial \vartheta}{\partial t} = -\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\hat{\mathbf{z}}\cdot\mathbf{u} + \kappa\nabla^2\vartheta\,.$$

The wave frequency satisfies

$$\sigma - \frac{\omega_C^2}{\sigma} - \frac{\omega_A^2}{\omega + \mathrm{i}\kappa k^2} = 0\,,$$

in which

$$\sigma = \omega + \mathrm{i}\nu k^2 - \frac{\omega_M^2}{\omega + \mathrm{i}\eta k^2}\,,$$

where

$$\omega_C = 2\frac{\boldsymbol{\Omega}\cdot\mathbf{k}}{k}\,, \qquad \omega_M = \mathbf{c}_A\cdot\mathbf{k}\,,$$

$$\omega_A = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\frac{|\mathbf{k}\times\hat{\mathbf{z}}|^2}{k^2}\,.$$

MAC–Waves
The Neglect of Inertia and Viscosity

If we ignore inertia and viscosity, we may make the approximation

$$\sigma = -\frac{\omega_M^2}{\omega + \mathrm{i}\eta k^2}$$

and simplifications follow.

In the case of $\kappa = \eta = 0$ we obtain **MAC–waves** of frequency

$$\omega \approx \pm\omega_{MAC}\,,$$

where

$$\omega_{MAC} = \omega_{MC}\left(1 + \frac{\omega_A^2}{\omega_M^2}\right)^{1/2}\,.$$

Instability is only possible when $\omega_A < 0$ and

$$-\omega_A > \omega_M\,.$$

For a proper study of the instability, dissipation must be included.