TURBULENCE

IN THE EARTH'S CORE

NOTE CHANGE OF TITLE

THE GEODYNAMO $\left\{ \begin{array}{l} \text{PARADOX} \\ \text{MYSTERY} \end{array} \right.$:

WHY DO NUMERICAL SIMULATIONS DO SO WELL?

GLOSSARY:

LES = LARGE EDDY SIMULATION
= NUMERICALLY RESOLVED SCALES

SGS = SUB-GRID SCALES
= NUMERICALLY UNRESOLVED SCALES

CMB = CORE-MANTLE BOUNDARY

ICB = INNER-CORE BOUNDARY

QUESTION:

WHERE IS LES–SGS TRANSITION TODAY?

ROUGH ANSWER:

DEPTH OF FLUID CORE $\doteqdot 2 \times 10^6$ m

100 HARMONICS $\Rightarrow L_{TRANSITION} = 2 \times 10^4$ m

QUESTION:

WHAT IS Rm AT TRANSITION?

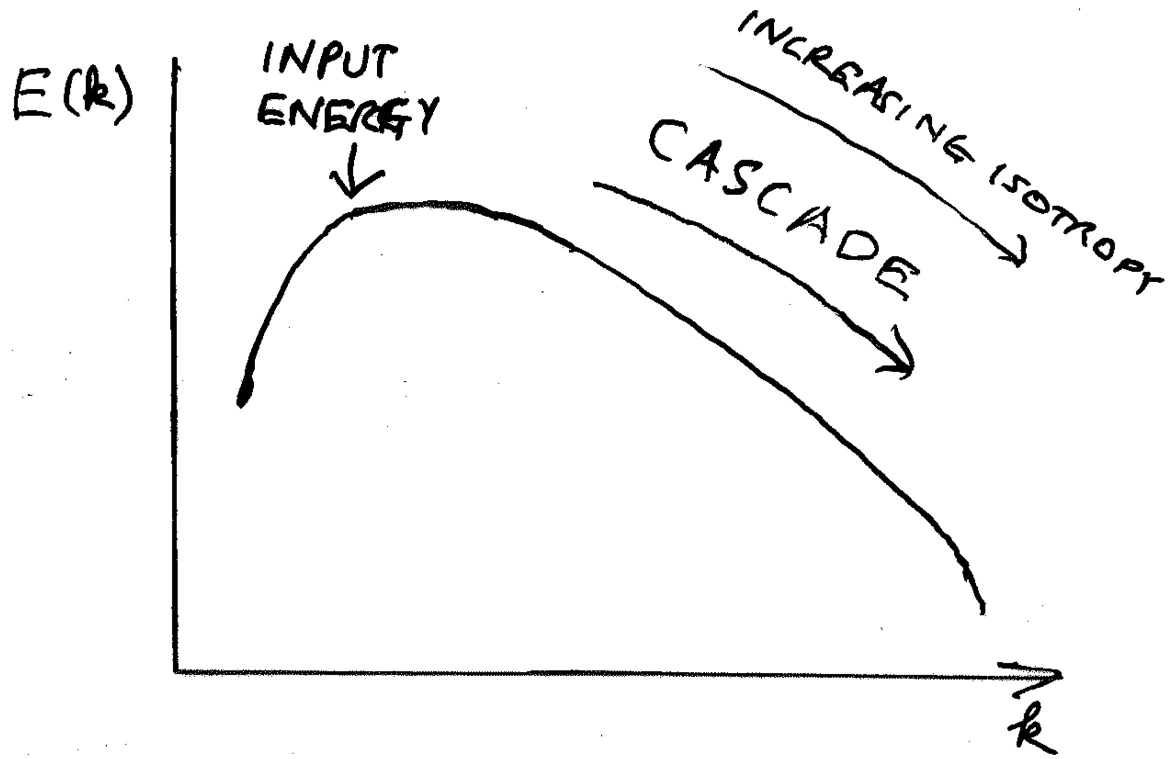
$$\left.\begin{aligned} U &= 10^{-4} \text{ m/s} \\ L &= 2 \times 10^4 \text{ m} \\ \eta &= 2 \text{ m}^2/\text{s} \end{aligned}\right\} \Rightarrow Rm = \frac{UL}{\eta} = 1$$

QUESTION:

DO WE NEED TO ALLOW FOR THE SGS IN THE LES?

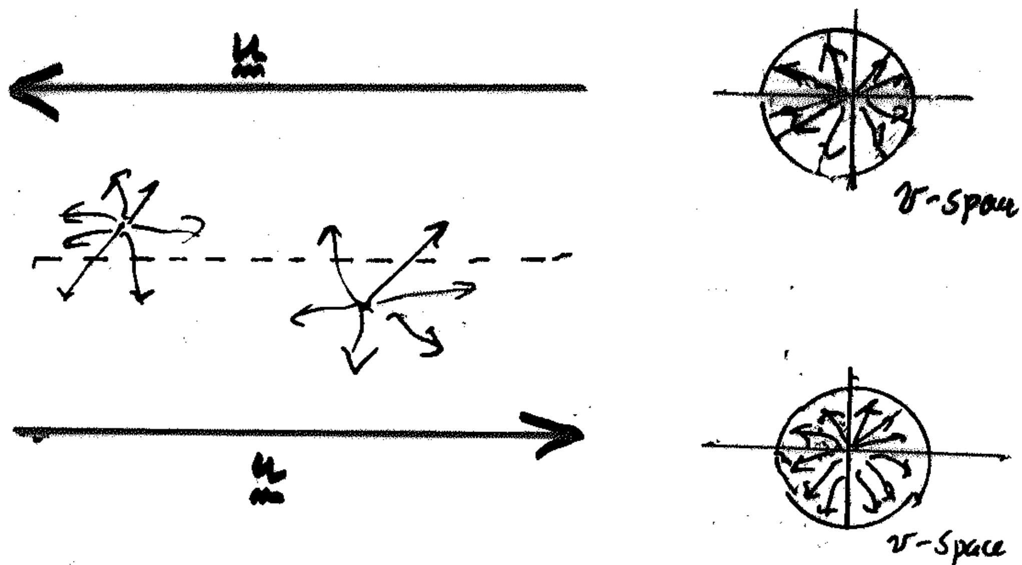
YES! DEFINITELY!

INERTIAL CASCADE



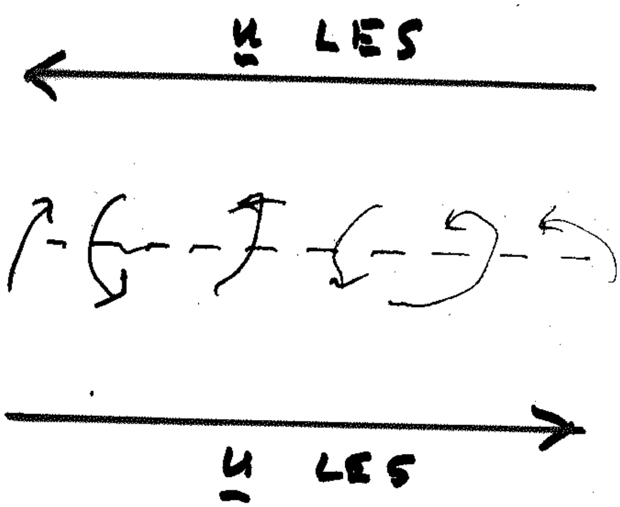
SHEAR FLOW TURBULENCE

VISCOSITY (MOLECULAR LEVEL)



$$\nu_{mol} = \frac{1}{3} v_{rms} \, l_{mfp}$$

VISCOSITY (TURBULENT LEVEL)



THE REYNOLDS
ANSATZ

$$\nu_{turb} = \frac{1}{3} \tilde{u}_{SGS} \, \tau_{SGS}$$

THE REYNOLDS ANSATZ

$$\partial_t \mathbf{u} + \mathbf{u}.\nabla\mathbf{u} = -\nabla\Pi + \nu\nabla^2\mathbf{u}, \qquad \nabla.\mathbf{u} = 0$$

ENSEMBLE AVERAGE

$$\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}}.\nabla\bar{\mathbf{u}} = -\nabla\bar{\Pi} + \nu\nabla^2\bar{\mathbf{u}} - \nabla.\overleftrightarrow{\mathbf{R}},$$

$$\nabla.\bar{\mathbf{u}} = 0$$

$$\overleftrightarrow{\mathbf{R}} = \overline{\mathbf{u}\mathbf{u}} - \bar{\mathbf{u}}\,\bar{\mathbf{u}}$$

= REYNOLDS STRESS TENSOR

REYNOLDS'S IDEA:

$$\overleftrightarrow{\mathbf{R}} = -2\nu_t\,\overleftrightarrow{\mathbf{e}}, \qquad e_{ij} = \frac{1}{2}(\nabla_i\bar{u}_j + \nabla_j\bar{u}_i)$$

$\Rightarrow$

$$\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}}.\nabla\bar{\mathbf{u}} = -\nabla\bar{\Pi} + (\nu + \nu_t)\nabla^2\bar{\mathbf{u}}$$

$$\simeq -\nabla\bar{\Pi} + \nu_t\nabla^2\bar{\mathbf{u}}$$

QUESTIONS AND COMMENTS

1. THE REYNOLDS ANSATZ PROVIDES THE FIRST EXAMPLE OF A

   LOCAL TURBULENCE THEORY

2. IT IS BASED ON A PHYSICAL IDEA
   (MAYBE TOO SIMPLISTIC BUT NEVERTHELESS....)

3. IT DOES RECOGNIZE ENERGY EXPENSES OF SGS

4. IT HELPS NUMERICAL WORK

5. IT CAN BE GENERALIZED BY MAKING ν_t DEPEND ON LOCAL PROPERTIES OF LES FLOW.
   e.g. SMAGORINSKY $\nu_t = \nu_t(\nabla u_{LES})$

6. IT CAN BE GENERALIZED TO OTHER TRANSPORT:

MOLECULAR	TURBULENT
$\nu = 10^{-6}\ m^2/s$	$\nu_t = \frac{1}{3} U_{SGS} L_{SGS} \sim 1\ m^2/s$
$K = 5\times10^{-6}\ m^2/s$	$K_t = \frac{1}{3} U_{SGS} L_{SGS} \sim 1\ m^2/s$
$D = 3\times10^{-10}\ m^2/s$	$D_t = \frac{1}{3} U_{SGS} L_{SGS} \sim 1\ m^2/s$
$\eta = 2\ m^2/s$	$\eta_t = \frac{1}{3} U_{SGS} L_{SGS} \sim 1\ m^2/s$

"ALL TURBULENT DIFFUSIVITIES ARE THE SAME"

SOME OBJECTIONS

1. ν_t CONSTANT IN TIME?! (DNS)

2. ν_t ISOTROPIC?! NOT LIKELY UNLESS CASCADE IS IMPORTANT, BUT BUOYANCY INJECTS ENERGY AT ALL WAVE NUMBERS!

   IF SGS ARE ANISOTROPIC, THEN

   $$\overleftrightarrow{\nu}_t, \ \overleftrightarrow{\chi}_t, \ \overleftrightarrow{D}_t, \ \overleftrightarrow{\eta}_t$$

   ARE TENSORS, e.g. TURBULENT HEAT FLUX

   $$= -\rho c_p \overleftrightarrow{K}_t \cdot \nabla T$$

3. WHAT ABOUT "BACK SCATTER UP THE SPECTRUM"?

   e.g. Λ-EFFECT (aka AKA)

   α - EFFECT (in MFE)

SGS DYNAMICS, PART I.

THE EFFECT OF $\mathbf{B}$ ON $\mathbf{u}'$

WE ASSUMED $R_m \ll 1$ FOR SGS

$$\Rightarrow \quad \mathbf{b} = O(R_m \mathbf{B}) \ll O(\mathbf{B})$$

BUT DYNAMICAL EFFECT OF $\mathbf{b}$ IS LARGE

$$\mathbf{j} = \sigma(-\nabla\phi + \mathbf{u}\times\mathbf{B}) \approx \sigma\,\mathbf{u}\times\mathbf{B}$$

$$\therefore\ \mathbf{J}\times\mathbf{B} \approx \sigma(\mathbf{u}\times\mathbf{B})\times\mathbf{B} = -\rho\,\mathbf{u}_\perp/\tau_d$$

WHERE

$$\tau_d = \frac{\rho}{\sigma B^2} = \left\{\begin{matrix}\text{magnetic}\\ \text{Joule}\end{matrix}\right. \text{damping time}$$

LET

$$\mathbf{V}_A = \frac{\mathbf{B}}{\sqrt{(\mu\rho)}} = \text{Alfvén velocity}$$

THEN

$$\tau_d = \frac{\eta}{V_A^{\,2}}$$

$$B = 100 \text{ GAUSS } (G) \qquad \rho = 10^4\,\text{kg/m}^3$$

$$V_A = 0.1\ \text{m/s}, \qquad \tau_d = 200\,\text{s}$$

SHORTEST TIME IN CORE DYNAMICS!

COMPARE WITH

$$\tau_\nu = \frac{L^2}{\nu} = \text{VISCOUS DECAY TIME}$$

$$L = 2 \times 10^4\,\text{m}, \qquad \tau_\nu = 2 \times 10^{[illegible]}\ \text{s}$$

RATIO IS

$$\frac{\tau_\nu}{\tau_d} = \frac{\sigma B^2 L^2}{\rho \nu} = M^2$$

(M IS HARTMANN #, M^2 IS CHANDRASEKHAR #)

$$M \doteqdot 10^6 \qquad M^2 \doteqdot 10^{12}$$

EXAMPLE: WAVE-LIKE SOLUTIONS

$$u \propto e^{i\underline{k}\cdot\underline{x} + \gamma t} \qquad \underline{k}\cdot\underline{u} = 0$$

$\Rightarrow$

$$\gamma_* = 0$$

WHERE

SMALL

$$\gamma_* = \gamma + \gamma_B + \nu k^2,$$

$$\gamma_B = \frac{(\hat{\underline{k}}\cdot\underline{V}_A)^2}{\eta}$$

LARGE BUT DIRECTIONAL

$$\hat{\underline{k}} = \underline{k}/k$$

SGS DYNAMICS, PART 2

THE EFFECT OF $\mathbf{B}$ AND Ω ON u'

ROSSBY NUMBER: $R_o = \frac{U}{\Omega L} \simeq 10^{-4}$

$U = 10^{-4}$ m/s $\Omega = 7 \times 10^{-5} s^{-1}$, $L = 2 \times 10^4$ m

ROSSBY RADIUS OF DEFORMATION $\simeq 1$ m

EKMAN NUMBER: $E = \frac{\nu}{\Omega L^2} \simeq 3 \times 10^{-11}$

PART 1 SUGGESTS TRANSFORMING

$$\nu \rightarrow \frac{L^2}{\tau_d} = \frac{V_A^2 L^2}{\eta}$$

THEN

$$P_m = \frac{\nu}{\eta} \rightarrow \left(\frac{V_A L}{\eta}\right)^2 = Lu^2 \qquad Lu = 10^3$$

$$E = \frac{\nu}{\Omega L^2} \rightarrow \frac{V_A^2}{2\Omega\eta} = \Lambda \qquad \Lambda = 100$$

Lu = LUNDQUIST #, Λ = ELSASSER #.

EXAMPLE WAVE-LIKE SOLUTIONS

$$\gamma_*^2 + \Omega_*^2 = 0$$

WHERE $\Omega_* = 2\boldsymbol{\Omega}\cdot\hat{k}$

DAMPED INERTIAL WAVES: $\gamma = \pm i\Omega_* - \gamma_B - \nu k^2$

SGS DYNAMICS, PART 3

THE EFFECT OF $\mathbf{B}$, $\mathbf{\Omega}$ AND $\mathbf{g}$ ON u'

GEOPHYSICAL DIGRESSION

ADIABATIC HEAT LOSS $\sim$ 5 TW

JOULE HEATING $\sim$ 1 TW

CONVECTIVE HEAT LOSS $\sim$ 5 TW ??

$$1 \text{ TW} = 10^{12} \text{ W}$$

CONVECTIVE HEAT FLUX $= \rho C_p \overline{T u_r}$

$= 300 \, \Delta T \ \text{W/m}^2$

$\overset{?}{\Rightarrow}$ CONVECTIVE HEAT FLOW ACROSS CMB

$= 4 \times 10^{16} \, \Delta T \ \text{W}$

$= 5 \text{ TW} \Rightarrow \Delta T = 10^{-4} \text{ K}$

IF $\alpha = 10^{-5} \text{ K}^{-1}$ THEN $\dfrac{\Delta\rho}{\rho} = 10^{-9}$

(B & R GOT $\dfrac{\Delta\rho}{\rho} = 10^{-8}$)

MEAN DENSITY GRADIENT:

$$\boldsymbol{\beta} \rho = \boldsymbol{\nabla} \rho \qquad \beta = 5 \times 10^{-15} \text{ m}^{-1}$$

BUOYANCY PARAMETER:

$$N = (g\beta)^{1/2} \qquad g = 5 \text{ m/s}^2$$

$$N = 10^{-7} \text{ s}^{-1}$$

EXAMPLE: WAVE-LIKE SOLUTIONS

$$\gamma_* \mathbf{u} + 2\boldsymbol{\Omega}\times\mathbf{u} = -i\Pi\mathbf{k} + c\,\mathbf{g}$$

$p'/\bar{p}$

$$\Rightarrow \quad \mathbf{u} = -\frac{[\gamma_*\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{g}) + \Omega_*\hat{\mathbf{k}}\times\mathbf{g}]}{\gamma_*^2+\Omega_*^2}\,c$$

THERMAL EQUATION:

$$(\gamma + \kappa k^2)c = -\boldsymbol{\beta}\cdot\mathbf{u}$$

DISPERSION RELATION

RADIUS VECTOR $= \mathbf{r}$

$$\boxed{(\gamma+\kappa k^2)(\gamma_*^2+\Omega_*^2) = \gamma_* N^2(\hat{\mathbf{k}}\times\hat{\mathbf{r}})^2}$$

CUBIC EQUATION, ONE REAL POSITIVE ROOT IF

$$N^2 > \frac{(\gamma_B+\nu k^2)^2+\Omega_*^2}{\gamma_B+\nu k^2}\,\kappa k^2$$

AND $\mathrm{Re}(\gamma) < N$.

LET $$\hat{\mathbf{k}}_* = \frac{\boldsymbol{\Omega}\times\mathbf{B}}{|\boldsymbol{\Omega}\times\mathbf{B}|}$$

THEN FOR $\mathbf{k} \propto \hat{\mathbf{k}}_*$

$$(\gamma+\kappa k^2)(\gamma+\nu k^2) = N^2\sin^2\vartheta = \tilde{N}^2$$

(ϑ = angle between $\hat{\mathbf{k}}_*$ and $\mathbf{r}$).

UNSTABLE FOR $L = \frac{\pi}{k} \gtrsim 5\,\mathrm{m}$??

LOWER LIMIT ON k_1, k_2, k_3

SGS MEANS $(L_1, L_2, L_3) \leq 2 \times 10^4 \, m$

OR $(k_1, k_2, k_3) \geq 10^{-4} \, m^{-1}$

$$\mathbf{k} = k_0 \hat{\mathbf{k}}_* + \mathbf{k}_1 \qquad \mathbf{k}_1 \cdot \hat{\mathbf{k}}_* = 0$$

$\Rightarrow$

$$\gamma_B = \frac{(\mathbf{v}_A \cdot \mathbf{k}_1)^2}{\eta k_0^2} \qquad \Omega_* = \frac{2\boldsymbol{\Omega} \cdot \mathbf{k}_1}{k_0}$$

SUPPOSE

$$\gamma \ll \gamma_B + \nu k_0^2$$

$\Rightarrow$

$$(\gamma + \chi k_0^2)\left[(\gamma_B + \nu k_0^2)^2 + \Omega_*^2\right] = (\gamma_B + \nu k_0^2)\tilde{N}^2$$

MAXIMUM γ FOR MODES WITH

$$\gamma_B + \nu k_0^2 = \Omega_*$$

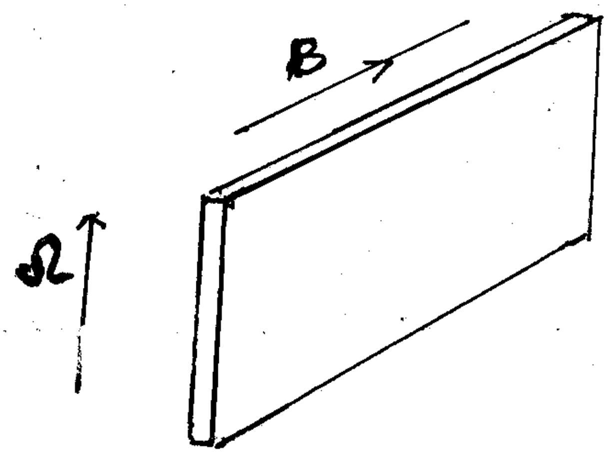
E.G. SUPPOSE $\nu k_0^2 << \gamma_B, \Omega_*$ THEN

$$\gamma_B = \Omega_*$$

E.G.,

$$k_0 / k_1 = \Lambda \qquad \Lambda = 100$$

PLATE-LIKE CELLS.



FOR THESE CELLS

$$\gamma = \frac{N^2}{2\Omega_*} - \kappa k_0^2$$

SELF-CONSISTENCY.

INITIAL SUPPOSITION $\gamma \ll \gamma_B + \nu k_0^2 = \Omega_*$

$\Rightarrow \quad \gamma \ll N \ll \Omega_*$

$\Rightarrow \quad \frac{k_0}{k_1} \ll \frac{2\Omega}{N} \doteqdot 2000$

IF νk_0^2 AND κk_0^2 ARE NEGLIGIBLE

NOTHING FIXES THE <u>SCALE</u> OF THE CELL.

REQUIRED: PHYSICAL PRINCIPLE FOR k_0

B & M SUGGEST EQUIPARTITION

LEADS TO $\quad k_0 = \frac{2\Omega}{V_A}$ $\qquad k_0 = 10^{-3}\ m^{-1}$, $\quad k_1 = 10^{-5}\ m^{-1}$

JOULE DISSIPATION = BUOYANCY SUPPLY

$\Rightarrow \quad u' = g\tau_d \Delta\rho/\rho \doteqdot 10^{-5}$ m/s

$\Rightarrow \quad$ 0.3 TW FOR WHOLE CORE

LET'S STOP AND ASSESS STATUS!

1. PRESENT "THEORY" IS MINOR...
   GENERALIZATION OF B&M (GAFD, 1990)

2. CLEARLY WAVE TREATMENT IS TOO SIMPLISTIC
   BUT BRINGS OUT STRIKING CONSEQUENCES
   OF ANISOTROPY

3. ARGUMENTS CAN BE MADE BASIS FOR
   LOCAL TURBULENCE THEORY.
   B&M ESTIMATE TENSOR DIFFUSIVITIES

$$\kappa_* \doteq \Lambda^{-2} \kappa_\perp \qquad (\Lambda^{-2} \simeq 10^{-4})$$

   (JOHN DONALD'S INTERMEDIATE MODEL)

4. OBVIOUS SHORTCOMINGS INCLUDE
   (a) $\gamma > 0$ BUT FINITE AMPLITUDE EFFECTS
   ARE IGNORED.
   (NO WORSE THAN MIXING LENGTH "THEORY")
   (b) DIFFICULTY OVER SCALE OF PLATES
   (TEND TO BE TOO BIG. SHREDDED?)

MODERN METHODS

ILLUSTRATED FOR NON-MAGNETIC, NON-ROTATING CASES

METHOD #1. SIMILARITY

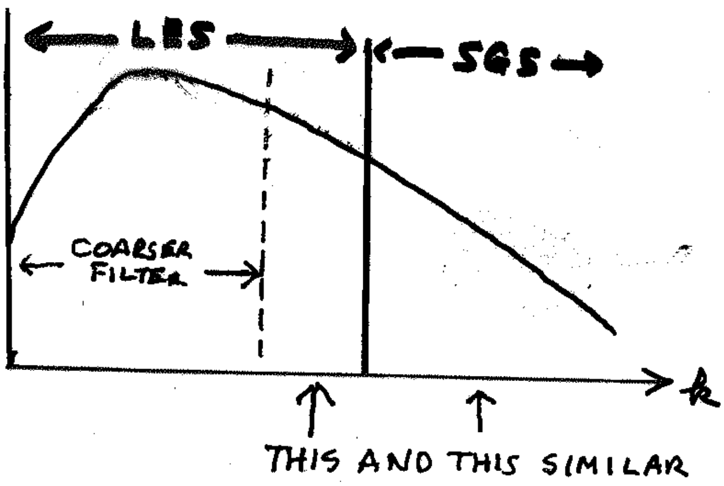
$\bar{u}$ IS REGARDED AS A "FILTERED" u

$\overleftrightarrow{R}$ IS CALLED THE "SGS STRESS TENSOR"

EXAMPLE: GAUSSIAN FILTER

$$\tilde{u}(x,t) = \frac{1}{(2\pi)^{3/2}\Delta^3}\int e^{-(x-x')^2/2\Delta^2}\, u(x',t)\, d^3x'$$

SIMILARITY ASSUMPTION:



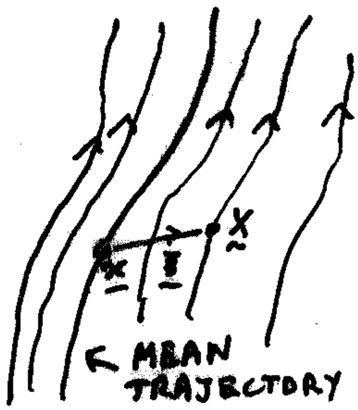
GERMANO'S IDENTITY LEADS TO

$$R_{ij} = C\left(\widetilde{\bar{u}_i \bar{u}_j} - \tilde{\bar{u}}_i\, \tilde{\bar{u}}_j\right)$$

THE ANSATZ

NOTHING TO DO WITH MFE

METHOD #2. CAMASSA-HOLM α-METHOD



STEP 1 THROW AWAY ν

STEP 2 FORMULATE VARIATIONAL PRINCIPLE

e.g. BAROTROPIC FLUID

$$A = \int_{t_1}^{t_2} L\, dt$$

$$L = \int \rho\left(\tfrac{1}{2}u^2 - U\right) d^3x$$

HAMILTON'S PRINCIPLE $\delta A = 0$

IS EQUIVALENT TO EQUATION OF MOTION

STEP 3 CONSTRUCT MEAN LAGRANGIAN $\bar{L}$

IT INVOLVES MOMENTS OF $\underline{\xi}$

$$\overline{\underline{\xi}}, \quad \overline{\xi_i \xi_j}, \quad \overline{\xi_i \partial_k \xi_j}, \quad \overline{\partial_k \xi_i \partial_k \xi_j}, \ldots\ldots$$

STEP 4 CHOOSE ANSATZ!

$\bar{L}$ NOW SPECIFIED

STEP 5 FIND MEAN TRAJECTORY FROM HAMILTON'S PRINCIPLE

e.g. BAROTROPIC FLUID

ANSATZ $\quad (\partial_t + \bar{\underline{u}}\cdot\nabla)\,\underline{\xi} = 0$ ← THE ANSATZ

$$\Rightarrow \quad (\partial_t + \bar{\underline{u}}\cdot\nabla)\,\overline{\xi_i \xi_j} = 0$$

$$\overline{\xi_i \xi_j} = \alpha^2 \delta_{ij}$$ ← THE α-MODEL!

CONSEQUENCES FOR BAROTROPIC FLUID

$$\mathbf{u} = (1 - \alpha^2 \nabla^2)\bar{\mathbf{u}}$$

$$\Rightarrow \quad \bar{\mathbf{u}}(x,t) = \frac{1}{4\pi\alpha^2} \int \frac{e^{-|x-x'|/\alpha}}{|x-x'|} \mathbf{u}(x',t)\, d^3x'$$

FILTERED $\mathbf{u}$

$$\partial_t \mathbf{u} + \bar{\mathbf{u}}\cdot\nabla\mathbf{u} + u_j \nabla \bar{u}_j = -\nabla\Pi$$

KELVIN'S THEOREM

STEP 6 RESTORE VISCOUS TERM

e.g. barotropic fluid

$$\partial_t \mathbf{u} + \bar{\mathbf{u}}\cdot\nabla\mathbf{u} + u_j \nabla \bar{u}_j = -\nabla\Pi + \nu\nabla^2\mathbf{u}$$

$$= \nu\nabla^2\bar{\mathbf{u}} - \nu\alpha^2\nabla^4\bar{\mathbf{u}}$$

SUMMARY: NEEDS FOR FUTURE

(1) CONVINCE THE WORLD
OF NEED FOR BETTER "SGS" PARAMETERIZATION

(2) GENERALIZE MODERN METHODS
(DO THEY LEAD ANYWHERE?)

(3) IMPROVE ANSATZ BY UNDERSTANDING
SGS DYNAMICS BETTER

(a) DNS

(b) LABORATORY STUDIES FOR SMALL Rm

(c) BETTER "THEORY" THAN ABOVE!