REMARKS ON KINETIC HELICITY, α–EFFECT AND DYNAMO ACTION

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MEAN–FIELD DYNAMO THEORY

All dynamo mechanisms
revealed by mean–field dynamo theory
require some deviation
of the small–scale motions
from reflectional symmetry.

Consider
ISOTROPIC NON-MIRRORSYMMETRIC TURBULENCE.

It shows an α-effect, $\boldsymbol{\mathcal{E}} = \alpha \boldsymbol{B}$,
which makes a dynamo possible.

Consider further
high–conductivity limit, i.e. $\tau_{cor} \ll \lambda_{cor}^2/\eta$,
and second–order correlation approximation.
Then

$$\alpha = -\frac{1}{3}\int_0^\infty \langle \boldsymbol{u}'(\boldsymbol{x},t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}'(\boldsymbol{x},t-\tau))\rangle \, \mathrm{d}\tau \,,$$

or

$$\alpha = -\frac{1}{3}\langle \boldsymbol{u}'(\boldsymbol{x},t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}'(\boldsymbol{x},t))\rangle \, \tau_{cor} \qquad (*)$$

Result $(*)$ occurs also in other approximations
(e.g. Vainshtein et al. 1983, Rädler et al. 2003).

It is often overinterpreted in the sense that
the averaged kinetic helicity $\langle \boldsymbol{u}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}')\rangle$
is crucial for any α–effect dynamo
or even any mean–field dynamo.

With
ANISOTROPIC TURBULENCE
the situation is more complex.

If an α–effect at all exists (see below)
α is no longer a scalar but a tensor.

With the assumptions introduced so far
it applies

$$\text{trace}(\alpha) = -\int_0^\infty \langle \boldsymbol{u}'(\boldsymbol{x},t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}'(\boldsymbol{x},t-\tau))\rangle \, \mathrm{d}\tau \,,$$

or

$$\text{trace}(\alpha) = -\langle \boldsymbol{u}'(\boldsymbol{x},t) \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}'(\boldsymbol{x},t))\rangle \, \tau_{cor} \,.$$

BUT the quantity of interest for dynamos
is not trace(α).

Consider, e.g., the $\alpha\omega$–dynamo,

$$\eta \Delta' A + \alpha_{\varphi\varphi} B - \partial_t A = 0$$

$$\eta \Delta' B + (\boldsymbol{\nabla}\omega \times \boldsymbol{\nabla}(r \sin\vartheta A))_\varphi - \partial_t B = 0 \,.$$

Note that

$$\text{trace}(\alpha) = -\int_0^\infty \langle \frac{1}{r} u_r(t) \partial_\vartheta u_\varphi(t-\tau)$$

$$+\frac{\cos\vartheta}{r\sin\vartheta} u_r(t) u_\varphi(t-\tau) - \frac{1}{r\sin\vartheta} u_r(t) \partial_\varphi u_\vartheta(t-\tau)$$

$$+\frac{1}{r}\sin\vartheta) u_\vartheta(t) \partial_\varphi u_r(t-\tau)$$

$$-\frac{1}{\sin\vartheta} u_\vartheta(t) \partial_r u_\varphi(t-\tau) - \frac{1}{r} u_\vartheta(t) u_\varphi(t-\tau)$$

$$+u_\varphi(t) \partial_r u_\vartheta(t-\tau) + \frac{1}{r} u_\varphi(t) u_\vartheta(t-\tau)$$

$$-\frac{1}{r} u_\varphi(t) \partial_\vartheta(t-\tau) \rangle \, \mathrm{d}\tau$$

and

$$\alpha_{\varphi\varphi} = -\int_0^\infty \langle \frac{\cos\vartheta}{r\sin\vartheta} u_r(t) u_\varphi(t-\tau)$$

$$-\frac{1}{r\sin\vartheta} u_r(t) \partial_\varphi u_\vartheta(t-\tau) + \frac{1}{r\sin\vartheta} u_\vartheta(t) \partial_\varphi u_r(t-\tau)$$

$$-\frac{1}{r} u_\vartheta(t) u_\varphi(t-\tau) \rangle \, \mathrm{d}\tau \,.$$

Anisotropic turbulence ...

Examples

$$\alpha_{\varphi\varphi} = f\,\mathrm{trace}(\alpha)$$
$$(= -f\,\langle \boldsymbol{u}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}')\rangle\,\tau_{cor})$$

- Isotropic turbulence $f = \frac{1}{3}$

- Convection in layer ($\boldsymbol{g} \parallel \boldsymbol{\Omega}$)
  - Brandenburg et al. 1990 $f \approx -\frac{1}{2}$
  - Kleeorin & Rogachevskij 2003 $f < 0$ possible

- Inhomogeneous anisotropic turbulence ($\boldsymbol{\nabla}\overline{u'^2}$, $\boldsymbol{\Omega}$)
  - Rüdiger et al. 1993 $f < 0$
  - Rädler et al. 2003 $f = \frac{12}{25}$

Return to
ISOTROPIC NON-MIRRORSYMMETRIC TURBULENCE.

Consider now
low–conductivity limit, i.e. $\tau_{cor} \gg \lambda_{cor}^2/\eta$,
and second–order correlation approximation.
Then

$$\alpha = -\frac{1}{12\pi\eta}\int_\infty \langle \boldsymbol{u}'(\boldsymbol{x},t)\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{\xi},t))\rangle \frac{\mathrm{d}^3\xi}{\xi}$$

$$= -\frac{1}{12\pi\eta}\int_\infty \langle \boldsymbol{u}'(\boldsymbol{x},t)\cdot(\boldsymbol{\xi}\times\boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{\xi},t))\rangle \frac{\mathrm{d}^3\xi}{\xi^3}\,.$$

Introduce the vector potential $\boldsymbol{\psi}$ of $\boldsymbol{u}$
so that $\boldsymbol{u}' = \boldsymbol{\nabla}\times\boldsymbol{\psi} + \boldsymbol{\nabla}\cdots$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\psi} = 0$.
This implies

$$\boldsymbol{\psi}(\boldsymbol{x},t) = \frac{1}{4\pi}\int_\infty (\boldsymbol{\xi}\times\boldsymbol{u}'(\boldsymbol{x}+\boldsymbol{\xi},t))\frac{\mathrm{d}^3\xi}{\xi^3}\,.$$

Then

$$\alpha = -\frac{1}{3\eta}\langle \boldsymbol{u}'(\boldsymbol{x},t)\cdot\boldsymbol{\psi}(\boldsymbol{x},t)\rangle$$

$$= -\frac{1}{3\eta}\langle \boldsymbol{\psi}(\boldsymbol{x},t)\cdot(\boldsymbol{\nabla}\times\boldsymbol{\psi}(\boldsymbol{x},t))\rangle\,.$$

Homogeneous isotropic turbulence,

low–conductivity limit,

$\alpha = -\frac{1}{3\eta}\langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi})\rangle$, ...

The quantity $\langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi})\rangle$,
which is now the relevant quantity
for dynamo action,
is basically different from $\langle \boldsymbol{u}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}')\rangle$.

It can be non–zero
even if $\langle \boldsymbol{u}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}')\rangle$ vanishes.

See also
modified Roberts dynamo
below.

Consider now
HOMOGENEOUS AXISYMMETRIC
TURBULENCE.

Assume, e.g., that it deviates
from a homogeneous isotropic turbulence
only by the action of a Coriolis force
defined by an angular velocity $\boldsymbol{\Omega}$.

In this case there is no α–effect,
and $\langle \boldsymbol{u}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}') \rangle = \langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi}) \rangle = 0$.

Nevertheless a dynamo is possible
due to a contribution to $\boldsymbol{\mathcal{E}}$
proportional to $\boldsymbol{\Omega} \times (\boldsymbol{\nabla} \times \boldsymbol{B})$
(the'$\boldsymbol{\Omega} \times \boldsymbol{j}$–effect')
in combination with a differential rotation.
(Rädler 1969,70,76,86,
Roberts and Stix 1972, Stix 1976)

Note that the turbulence considered here
well deviates from reflectional symmetry.
But this kind of deviation is not indicated
by $\langle \boldsymbol{u}' \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}') \rangle$ or $\langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi}) \rangle$.

Homogeneous axisymmetric turbulence,
contribution to $\boldsymbol{\mathcal{E}}$
proportional to $\boldsymbol{\Omega} \times (\boldsymbol{\nabla} \times \boldsymbol{B})$, ...

There are several related dynamo mechanisms
with other contributions to $\boldsymbol{\mathcal{E}}$
containing derivatives of $\overline{\boldsymbol{B}}$
and differential rotation.
(Rädler 1986)

It remains to be investigated
to which extend such mechanisms
play a role in the geodynamo.

"LAMINAR" DYNAMO THEORY

In many dynamo models
the kinetic helicity $\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u})$
is unequal to zero.

However,
this is not a necessary condition
for dynamo action.
There are quite a few examples
of working dynamos
in which $\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u})$ vanishes everywhere.

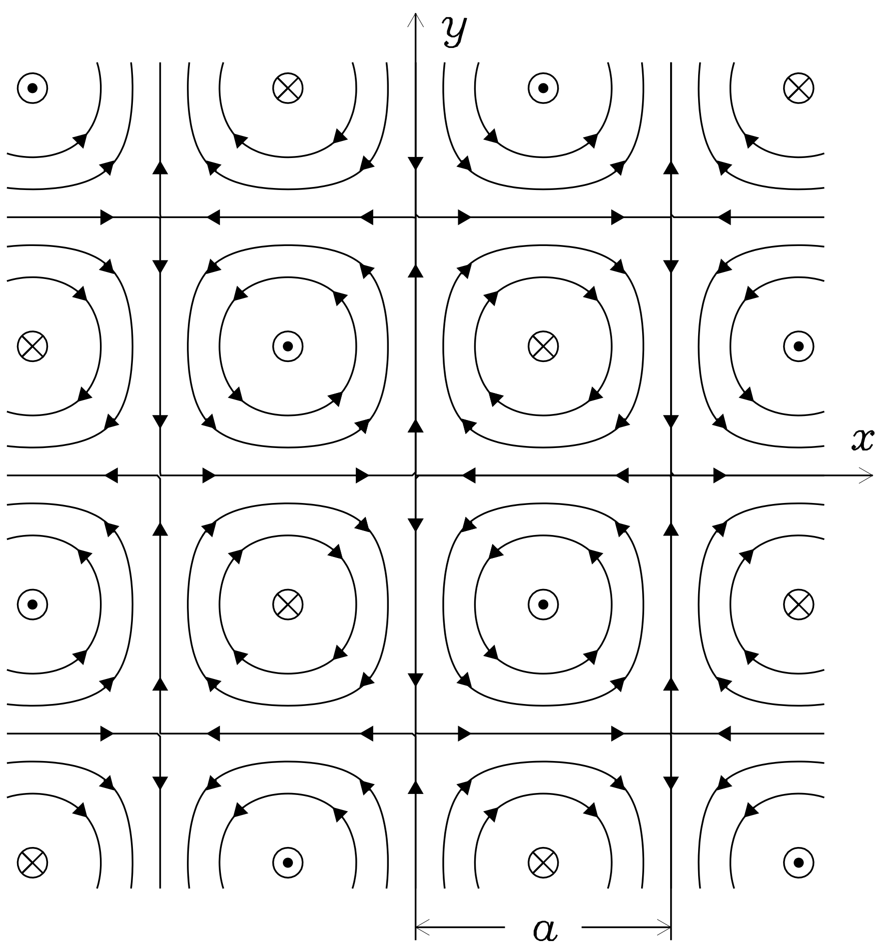
- Gailitis 1970,93,95

- Zheligovsky and Galloway 1998

Consider now
the ROBERTS DYNAMO
(Roberts 1972).

In a slightly modified form it was realized
in the KARLSRUHE DYNAMO EXPERIMENT.
(Müller and Stieglitz 2000,
Stieglitz and Müller 2001,02)

- The original Roberts dynamo



Start from

$$\eta\boldsymbol{\nabla}^2\boldsymbol{B} + \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) - \partial_t\boldsymbol{B} = \boldsymbol{0}\,,\ \boldsymbol{\nabla}\cdot\boldsymbol{B} = 0\,,$$

with

$$u_x = -u_\perp\frac{\pi}{2}\sin(\frac{\pi}{2}x)\ \cos(\frac{\pi}{2}y)$$

$$u_y = +u_\perp\frac{\pi}{2}\cos(\frac{\pi}{2}x)\ \sin(\frac{\pi}{2}y)$$

$$u_z = -u_\parallel(\frac{\pi}{2})^2\ \sin(\frac{\pi}{2}x)\ \sin(\frac{\pi}{2}y)\,.$$

Note that $\boldsymbol{u}\cdot(\boldsymbol{\nabla}\times\boldsymbol{u}) \geq 0$.

Define

$$R_{m\perp} = \frac{u_\perp a}{2\eta} \quad \text{and} \quad R_{m\parallel} = \frac{u_\parallel a}{\eta}.$$

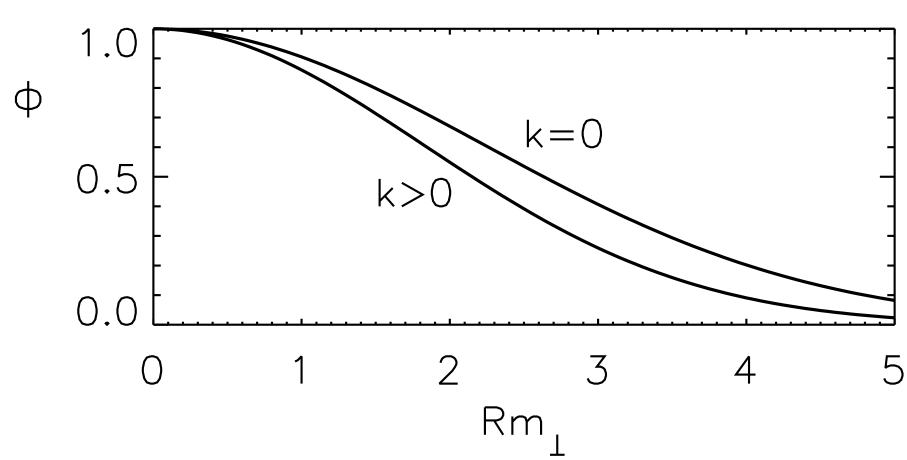
Look for solutions of the form

$$\boldsymbol{B} = \Re(\hat{\boldsymbol{B}}(x,y)\exp(\mathrm{i}kz + pt))\,.$$

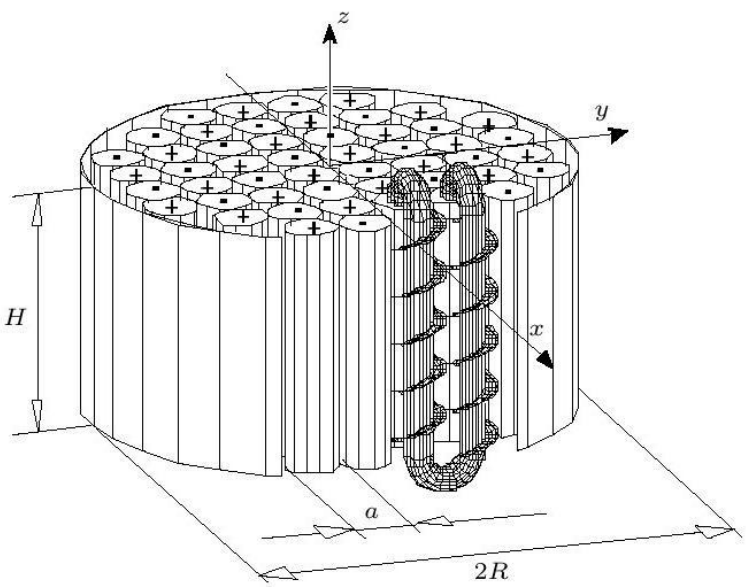
Non–decaying solutions, i.e. dynamo, if

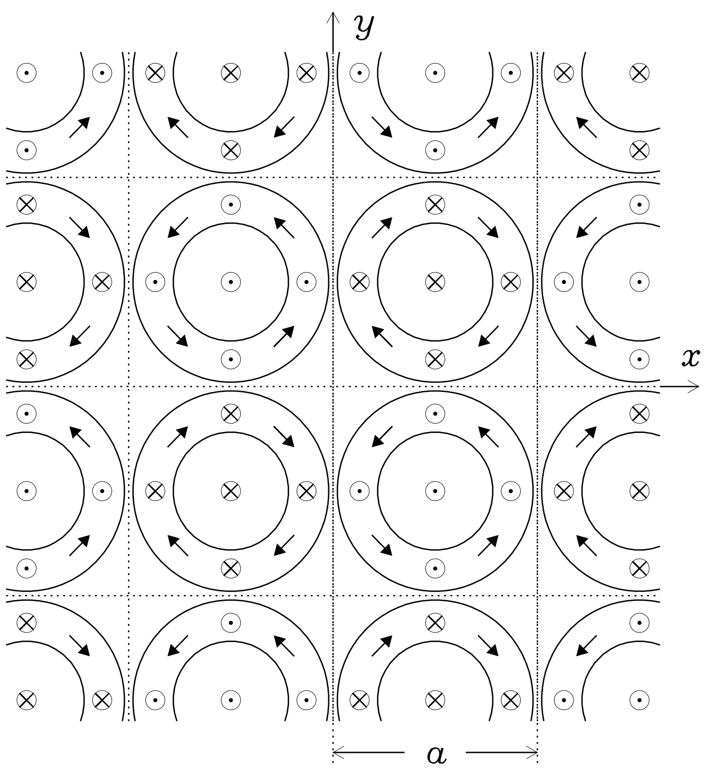
$$R_{m\perp}R_{m\parallel}\phi(R_{m\perp}) \geq \frac{32}{\pi}\frac{a}{l}\,,$$

where $l = \frac{2\pi}{k}$ period length of $\boldsymbol{B}$ in z–direction.



◦ The Karlsruhe realization
of the Roberts dynamo





Again $\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \geq 0$

Excitation condition
similar to that of the original Roberts dynamo.

∘ Mean–field approach to dynamos of Roberts type with a more general flow patterns

Assume that $\boldsymbol{u}$ periodic in x and y with period length $2a$.

Define mean fields $\overline{F}$ by averaging over x and y,

$$\overline{F} = \langle F \rangle \,,$$

$$\langle F(x, y, z) \rangle = \frac{1}{4a^2} \int_{-a}^{a} \int_{-a}^{a} F(x + \xi, y + \eta, z) \, \mathrm{d}\xi \, \mathrm{d}\eta \,.$$

Then

$$\overline{\boldsymbol{u}} = \boldsymbol{0}$$

and

$$\eta \boldsymbol{\nabla}^2 \overline{\boldsymbol{B}} + \boldsymbol{\nabla} \times \boldsymbol{\mathcal{E}} - \partial_t \overline{\boldsymbol{B}} = \boldsymbol{0} \,, \; \boldsymbol{\nabla} \cdot \overline{\boldsymbol{B}} = 0 \,,$$

with

$$\boldsymbol{\mathcal{E}} = \overline{\boldsymbol{u} \times \boldsymbol{B}} \,.$$

Assume that $\boldsymbol{\mathcal{E}}$
is a linear and homogeneous functional of $\overline{\boldsymbol{B}}$.

Assume further that $\overline{\boldsymbol{B}}$ and $\boldsymbol{\mathcal{E}}$
are independent of x and y.

Represent quantities depending on z
in the form

$$F(z) = \int_{-\infty}^{\infty} \hat{F}(k) \exp(\mathrm{i}kz) \, \mathrm{d}k \, .$$

Then

$$-\eta k^2 \hat{\overline{\boldsymbol{B}}} + \mathrm{i}k\boldsymbol{e} \times \hat{\boldsymbol{\mathcal{E}}} - \partial_t \hat{\overline{\boldsymbol{B}}} = \boldsymbol{0} \, , \quad \boldsymbol{e} \cdot \hat{\overline{\boldsymbol{B}}} = 0 \, ,$$

where $\boldsymbol{e}$ is the unit vector in z–direction.

The above assumption on $\boldsymbol{\mathcal{E}}$ implies

$$\hat{\mathcal{E}}_i = \hat{a}_{ij} \hat{\overline{B}}_j \, .$$

Assume now that $\boldsymbol{u}$ is independent of z,
steady,
changes its sign
if shifted along x or y–axis by a length a
and under 90° rotation about z–axis.

Then $\hat{a}_{ij}$ must have the structure

$$\hat{a}_{ij} = -\hat{\alpha}_\perp(k)(\delta_{ij} - e_i e_j) + \mathrm{i}\hat{\beta}(k) k \epsilon_{ijk} e_k \,,$$

with two functions $\hat{\alpha}_\perp$ and $\hat{\beta}$ of k.

Thus

$$\hat{\boldsymbol{\mathcal{E}}} = -\hat{\alpha}_\perp(\hat{\overline{\boldsymbol{B}}} - (\boldsymbol{e}\cdot\hat{\overline{\boldsymbol{B}}})\boldsymbol{e}) - \mathrm{i}\hat{\beta}\, k\, \boldsymbol{e} \times \hat{\overline{\boldsymbol{B}}} \,.$$

This is equivalent to

$$\begin{aligned}\boldsymbol{\mathcal{E}}(z,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha_\perp(\zeta) \\ (\overline{\boldsymbol{B}}(z+\zeta,t) - (\boldsymbol{e}\cdot\overline{\boldsymbol{B}}(z+\zeta,t))\,\boldsymbol{e})\mathrm{d}\zeta \\ -\frac{1}{2\pi}\boldsymbol{e}\times\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\beta(\zeta)\,\overline{\boldsymbol{B}}(z+\zeta,t)\,\mathrm{d}\zeta\end{aligned}$$

with

$$\alpha_\perp(\zeta) = \int_{-\infty}^{\infty} \hat{\alpha}_\perp(k)\exp(\mathrm{i}k\zeta)\mathrm{d}k$$

$$\beta(\zeta) = \int_{-\infty}^{\infty} \hat{\beta}(k)\exp(\mathrm{i}k\zeta)\mathrm{d}k \,.$$

The equations for $\overline{\boldsymbol{B}}$ allow solutions of the form

$$\overline{\boldsymbol{B}} = C(\cos(kz), \pm\sin(kz), 0)\,\exp(pt)$$

$$p = -(\eta + \widehat{\beta}(k))k^2 \pm \widehat{\alpha}_\perp(k)k.$$

That is,

a dynamo occurs if $\dfrac{|\widehat{\alpha}_\perp(k)|}{(\eta+\widehat{\beta}(k))k} \geq 1.$

Assume (as will be confirmed later) that $\widehat{\alpha}_\perp$ takes a finite non–zero value and $\widehat{\beta}$ remains also finite as $k \to 0$.

Then a dynamo is always possible if only k is sufficiently small.

Consider the limit $k \to 0$.

Then

$$\boldsymbol{\mathcal{E}} = -\alpha_\perp(\overline{\boldsymbol{B}} - (\boldsymbol{e} \cdot \overline{\boldsymbol{B}})\,\boldsymbol{e}) \text{ with } \alpha_\perp = \hat{\alpha}_\perp(0).$$

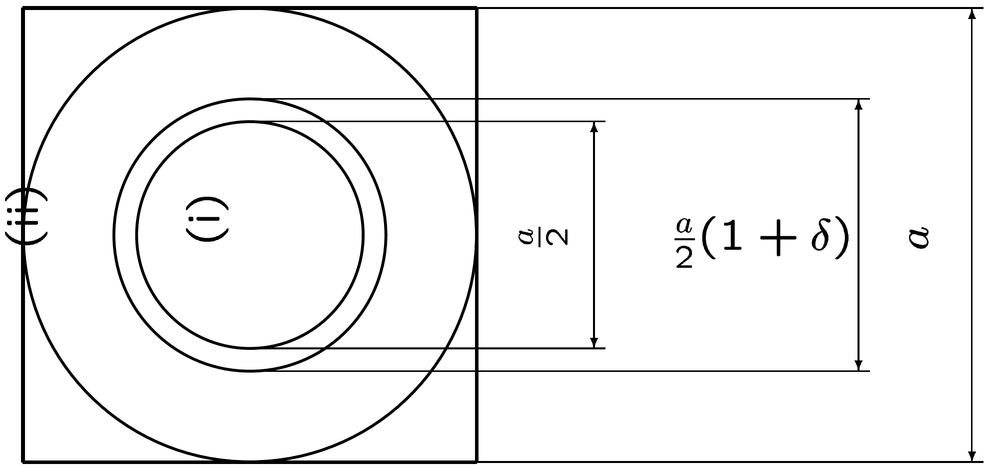
Assuming that $\alpha_\perp$ is small
(but large enough for dynamo action)
we may calculate it along the lines
of the second–order correlation approximation.

This yields

$$\alpha_\perp = \frac{1}{2\eta}\langle \boldsymbol{u}' \cdot \boldsymbol{\psi} \rangle = \frac{1}{2\eta}\langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi}) \rangle\,,$$

with $\boldsymbol{\psi}$ such that $\boldsymbol{u} = \boldsymbol{\nabla} \times \boldsymbol{\psi}$.

It is now easy
to find examples with $\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0$,
in which $\langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi}) \rangle \neq 0$
and therefore a dynamo is possible.



Regions with moving fluid

(i) $r \leq \frac{a}{4}$

$u_r = u_\varphi = 0\,, \quad u_z = -u = \text{const}$

V_1 rate of volumetric flow through cross-section $r \leq \frac{a}{4}$

$\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) = 0$

(ii) $\frac{a}{4}(1+\delta) \leq r \leq \frac{a}{2}$

$u_r = 0\,, \quad u_\varphi = -\omega r\,, \quad u_z = -\varepsilon\omega\frac{a}{2}$

V_2 rate of volumetric flow through $\frac{a}{4}(1+\delta) \leq r \leq \frac{a}{2}$, $0 \leq z \leq a$

$\boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \sim \varepsilon$

Averages

$$\langle \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) \rangle = \frac{16\pi\varepsilon}{a^5(1-(1+\delta)^2/4)^2}\, V_2^2$$

$$\langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi}) \rangle = \frac{2}{a^3}(V_1 + \frac{\pi}{2}\varepsilon V_2)\, V_2$$

(independent of δ)

Modified Roberts dynamo

Numerical results
(independent of the mean–field approach
and the approximations discussed so far)

Marginal dynamo states
with $ak = 0.9$

δ	ε	$V_1/a\eta$	$V_2/a\eta$	
0	0.288	2	0.736	Karlsruhe
0	0	2	0.805	no helicity
0.2	0	2	0.965	no helicity

Helicity is unnecessary
for Roberts–type dynamos — but it helps !

Cf. Gilbert et al. 1988
"Helicity is unnecessary for α–effect dynamos,
but it helps"

The results presented here might suggest
that a necessary condition
for steady dynamo action
is that $\boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi})$
does not vanish everywhere.

But this is not correct.
See, e.g.,
Gailitis dynamo
or
Herzenberg dynamo.

In these cases $\langle \boldsymbol{\psi} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\psi}) \rangle = 0$.