Geomagnetic Reversals: a Stochastic Exit Problem

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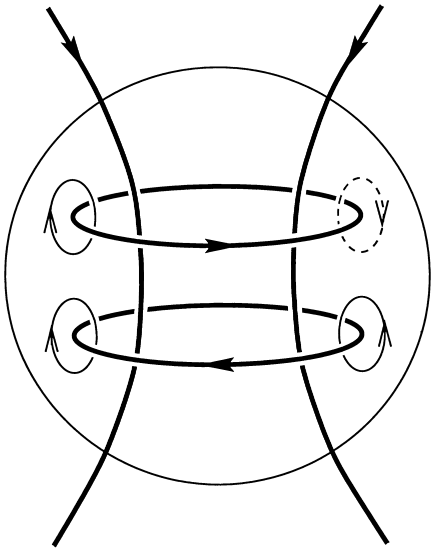
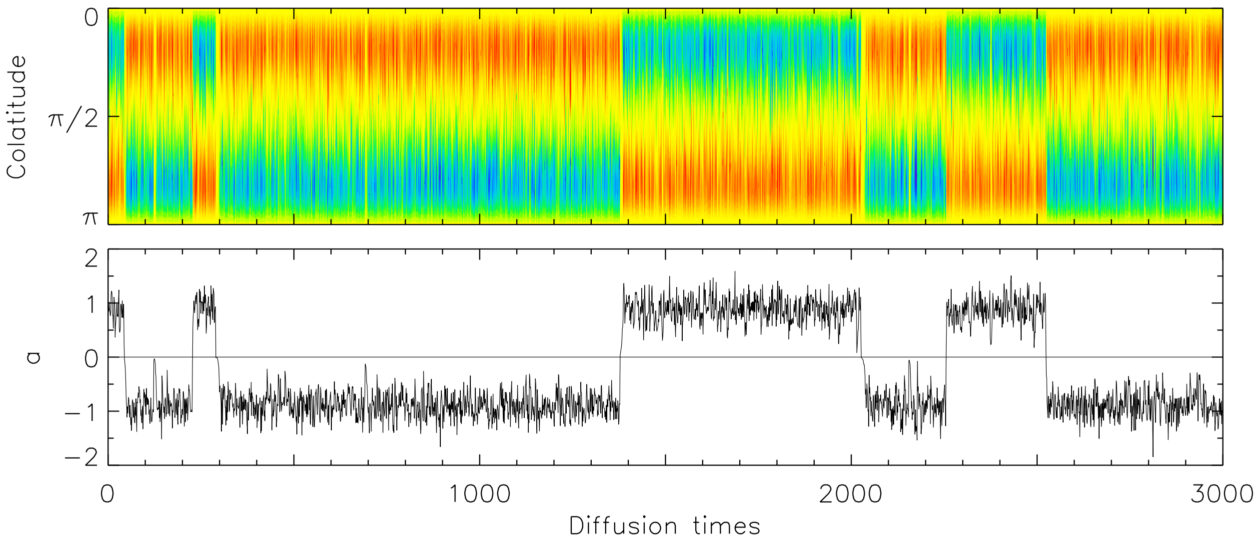


$$\frac{\partial\langle \boldsymbol{B}\rangle}{\partial t} = \boldsymbol{\nabla}\times[\boldsymbol{v}\times +\alpha - +(\beta+\eta)\boldsymbol{\nabla}\times]\,\langle \boldsymbol{B}\rangle$$

- $\alpha\Omega$, 1D, fundamental mode stationary dipole
- $\alpha = \alpha_0(\theta) + \delta\alpha(\theta, t)$
- nonlinearly quenched



- variable dipole and overtones
- sudden reversals
- quantitative agreement with observed properties of geomagnetic dipole
- details: GAFD 94 (2001) 263;
  PEPI 130 (2002) 143

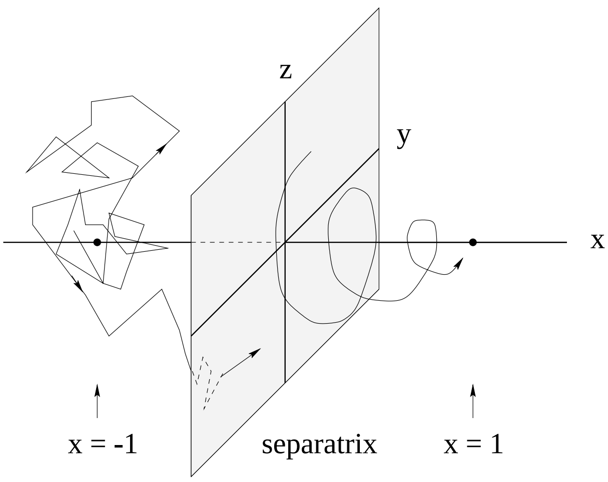
Eigenmode decomposition

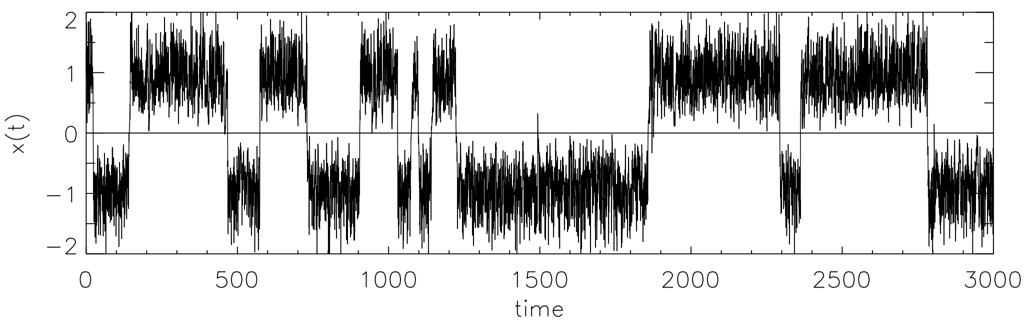
$$\dot{a}_i = \underbrace{\lambda^i a_i}_{\text{linear evol.}} + \underbrace{(1 - a_0^2) E_{ik} a_k}_{\alpha\text{-quenching}} + \underbrace{V_{ik}(\tau) a_k}_{\alpha\text{-fluctuations}}$$

Simpler Model

$$\begin{aligned}
\dot{x} &= (1 - x^2)x + V_{11}x + V_{12}y + V_{13}z \\
\dot{y} &= -ay + cz \ + V_{21}x + V_{22}y + V_{23}z \\
\dot{z} &= -cy - az \ + V_{31}x + V_{32}y + V_{33}z
\end{aligned}$$

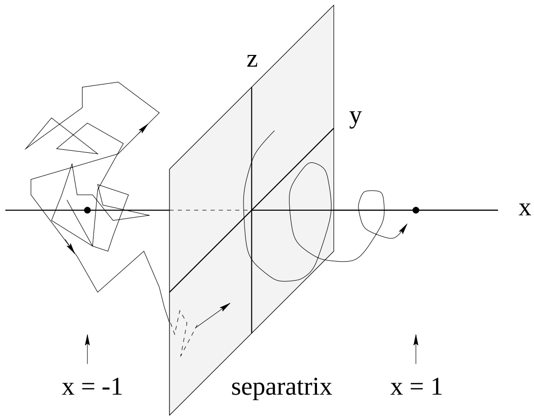
Linearly unstable fundamental mode x, and one stable periodic overtone (y, z; $a > 0$) coupled by multiplicative noise; $V_{ik}(t)$ uncorrelated and equally strong





Thermal noise

$$T_{\mathrm{r}} \propto (\text{chance particle has energy } \Delta U)^{-1}$$
$$\propto \exp(\Delta U/kT) \qquad \text{(Kramers 1940)}$$

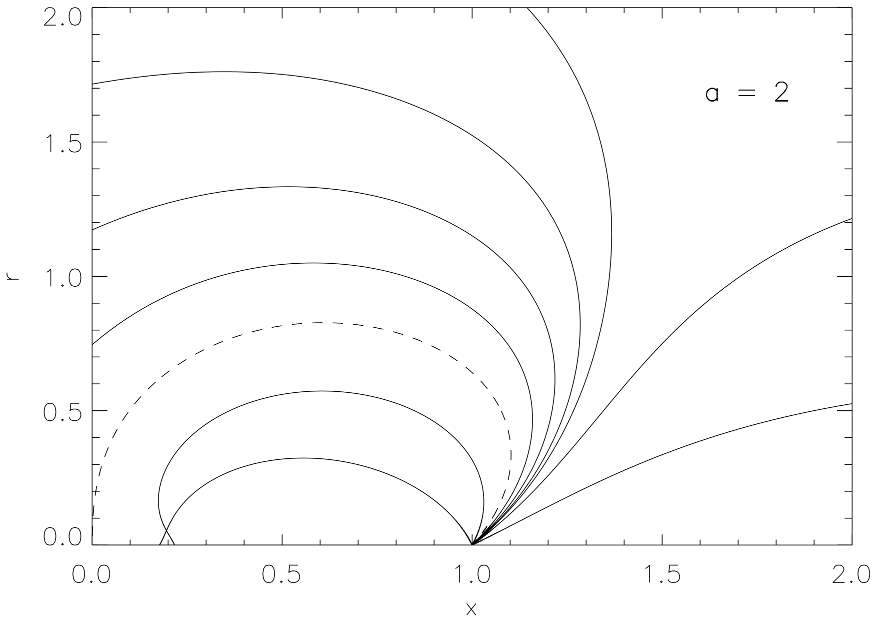


stationary probability distribution $P(x,y,z)$
axisymmetric $\rightarrow P(x,r)$;
$r = (y^2 + z^2)^{1/2}$

Multiplicative noise: no solution known

$P(x,r) =$ probability distribution; $\qquad P(\infty) = 0$
$T(x,r) =$ mean time from x, r to $x = 0$; $\qquad T(0,r) = 0$

- $T(x,r) \simeq T_{\mathrm{i}}$ inside well, and mean time between reversals $T_{\mathrm{r}} = 2T_{\mathrm{i}}$
- $\int_{x\geq 0}(PM^{\dagger}T - TMP)\mathrm{d}V = \pi D \int_0^{\infty} r^3 \mathrm{d}r P(0,r) \left.\frac{\partial T}{\partial x}\right|_{x=0}$
- express $\partial T/\partial x$ in T_{i} by boundary layer expansion of $M^{\dagger}T = -1$ near $x = 0$
- $$T_{\mathrm{r}} = \frac{1}{\sqrt{\pi D(a+1)}} \frac{\int_{x\geq 0} P\mathrm{d}V}{\int_0^{\infty} r^2 \mathrm{d}r P(0,r)}$$
- try $P \propto \exp(-\psi/D)$



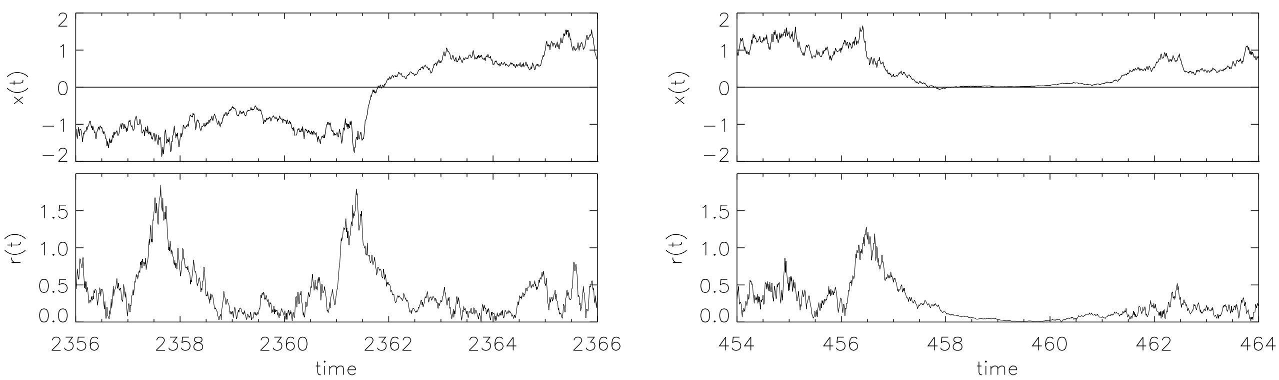
- near (1,0): $\psi \simeq 2(x-1)^2 + ar^2$ from $H = 0$

  on $x = 0$: $\psi \simeq \Delta\psi + \frac{1}{2}(r/r_0)^2$ numerical

$$T_{\mathrm{r}} = \left(\frac{\pi}{D(a+1)}\right)^{1/2} \frac{1}{ar_0^3} \exp(\underbrace{\Delta\psi/D})$$

↑

effective $\Delta U/kT$



Exit strategy

- rays are most likely exit paths in the limit $D \to 0$

$$y = z = 0 \quad \to \quad (\log x)^{\cdot} = 1 - x^2 + V_{11}(t)$$

$$x \simeq 0 \quad \to \quad \dot{x} \simeq V_{12}y + V_{13}z$$

- distribution of exit points $\exp[-\psi(0, r)/D]$ has broad maximum at $r = 0$

Remarks

- T_{r} very sensitive to changes in

  D (i.e. level of turbulence)

  a (i.e. amplitude overtones)

- reversal rate is set by the waiting time for the right jump in r and decrease in x to happen *simultaneously*

- results not contingent on mean field theory: induction equation with $\boldsymbol{u} =$ mean flow $\boldsymbol{u}_0+$turbulence $\delta\boldsymbol{u}$ leads to same equations

- future work: more overtones, correlated $V_{ij}(t)$

Equations

$MP = 0$ and $M^\dagger T = -1$, with

$$M = -\frac{\partial}{\partial x}(1-x^2)x + \frac{a}{r}\frac{\partial}{\partial r}r^2 + \tfrac{1}{2}D\left(\frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right)(x^2+r^2)$$

$$M^\dagger = (1-x^2)x\frac{\partial}{\partial x} - ar\frac{\partial}{\partial r} + \tfrac{1}{2}D(x^2+r^2)\left(\frac{\partial^2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right)$$

$$D \simeq \langle V_{ij}^2\rangle \tau_{\rm c}$$

Try $P \propto \exp(-\psi/D) \rightarrow$

$$(1-x^2)x\frac{\partial\psi}{\partial x} - ar\frac{\partial\psi}{\partial r} + \tfrac{1}{2}(x^2+r^2)\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial r}\right)^2\right] \simeq 0$$

$$H = (1-x^2)xp - arq + \tfrac{1}{2}(x^2+r^2)(p^2+q^2) = 0$$

$$\dot{x} = \frac{\partial H}{\partial p} = (1-x^2)x + (x^2+r^2)p$$

$$\dot{r} = \frac{\partial H}{\partial q} = -ar + (x^2+r^2)q$$

$$\dot{p} = -\frac{\partial H}{\partial x} = -(1-3x^2)p - x(p^2+q^2)$$

$$\dot{q} = -\frac{\partial H}{\partial r} = aq - r(p^2+q^2)$$

$$\dot{\psi} = p\dot{x} + q\dot{r} - H = \tfrac{1}{2}(x^2+r^2)(p^2+q^2)$$