INTRODUCTION TO FAST DYNAMOS

[AND SLOW DYNAMOS.]

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14:00 – 15:00 MONDAY.

OBJECTIVES:

- UNDERSTAND BASIC DYNAMO MECHANISMS IN FLUID FLOWS (CLASSIFICATION),
- OBTAIN ASYMPTOTIC / APPROXIMATE GROWTH RATES,
- UNDERSTAND BASIC SATURATION MECHANISM.

STARTING POINT:

- INDUCTION EQUATION, NON-DIMENSIONALISED USING SCALE $\mathcal{L}$ AND MAGNITUDE $\mathcal{U}$ OF THE FLUID FLOW:

$$\partial_t \underline{B} = \nabla \times (\underline{u} \times \underline{B}) + \varepsilon \nabla^2 \underline{B}$$

$$\nabla . \underline{B} = 0$$

- $\varepsilon \equiv R_m^{-1}$, $R_m = \dfrac{\mathcal{U}\mathcal{L}}{\eta}$ MAGNETIC REYNOLDS NUMBER
- KINEMATIC THEORY: $\underline{u}$ GIVEN FLUID FLOW, (NOT RANDOM OR TURBULENT).

$$\partial_t \underline{B} = \nabla \times (\underline{u} \times \underline{B}) + \varepsilon \nabla^2 \underline{B}$$

OR

$$\underbrace{\partial_t \underline{B} + \underline{u}.\nabla \underline{B}}_{\text{ADVECTION}} = \underbrace{\underline{B}.\nabla \underline{u}}_{\text{STRETCHING}} + \underbrace{\varepsilon \nabla^2 \underline{B}}_{\text{DIFFUSION}}.$$

WITH $\nabla.\underline{B} = 0, \quad \nabla.\underline{u} = 0.$

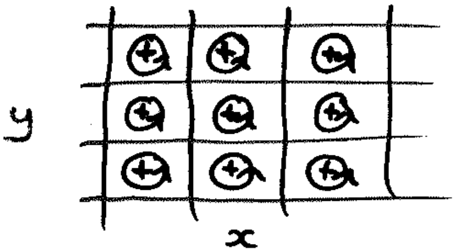
FOCUS ON THE LIMIT $R_m \to \infty$ OR $\varepsilon \to 0$, i.e. OF WEAK DIFFUSIVITY (WITH FIXED CHOICE OF FLOW FIELD $\underline{u}$).

WHY?

- ASTROPHYSICALLY IMPORTANT LIMIT (STARS, GALAXIES).
- ALLOWS ASYMPTOTIC APPROXIMATION GIVING USEFUL FORMULAE & INFORMATION.
- ALLOWS CLASSIFICATION OF DYNAMO MECHANISMS.

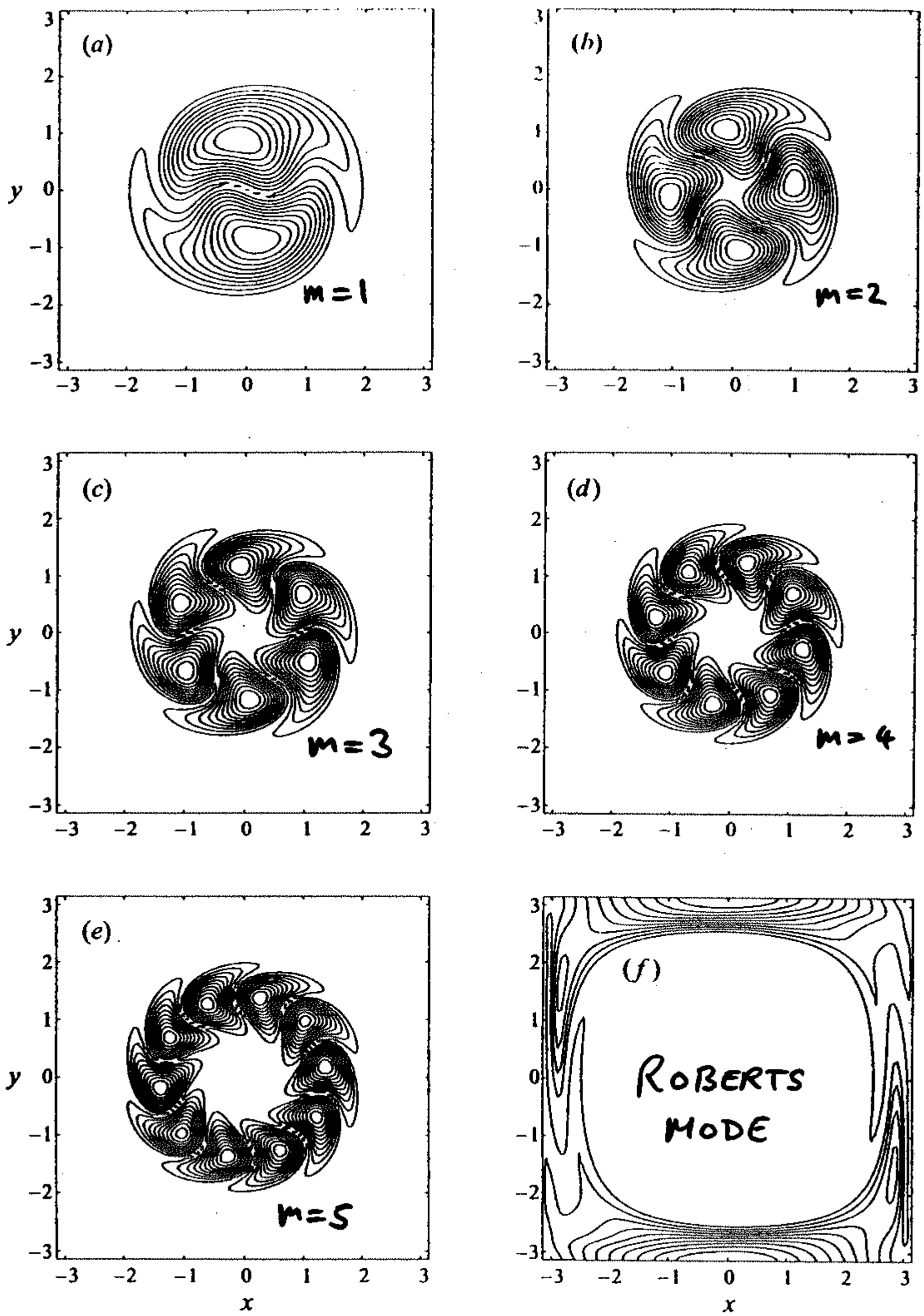
E.G. PLUNIAN, MARTY, ALEMANY JFM 382, 1999.

$$\underline{u} = (\psi_y, -\psi_x, K\psi), \qquad \psi = a(1+\cos x)(1+\cos y).$$



"FAST BREEDER REACTOR"

MAGNETIC FIELD IN 1 CELL OF FLOW:

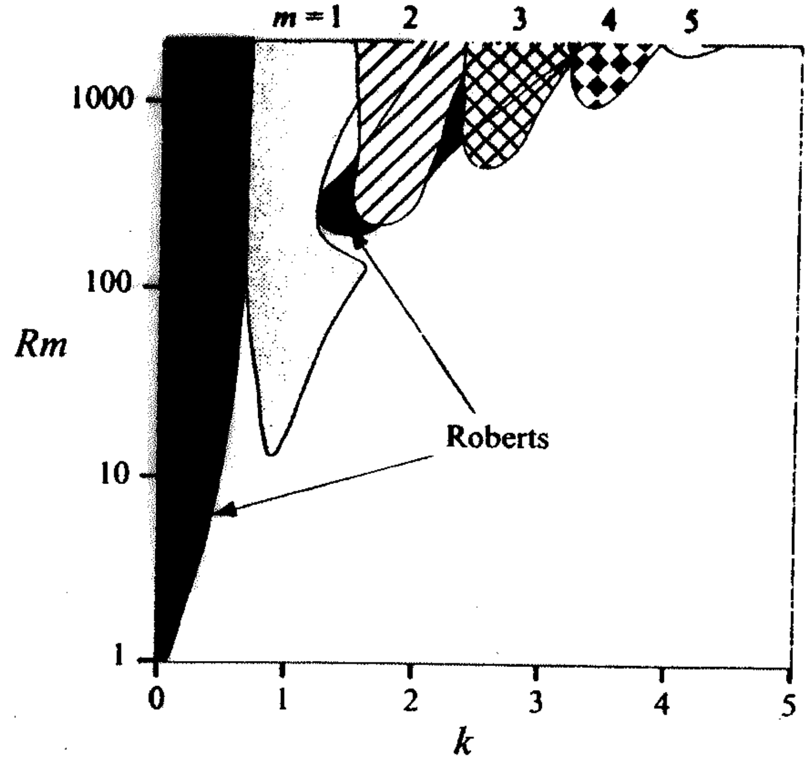


PONOMAREN
MODES

MERICAL RESULTS:

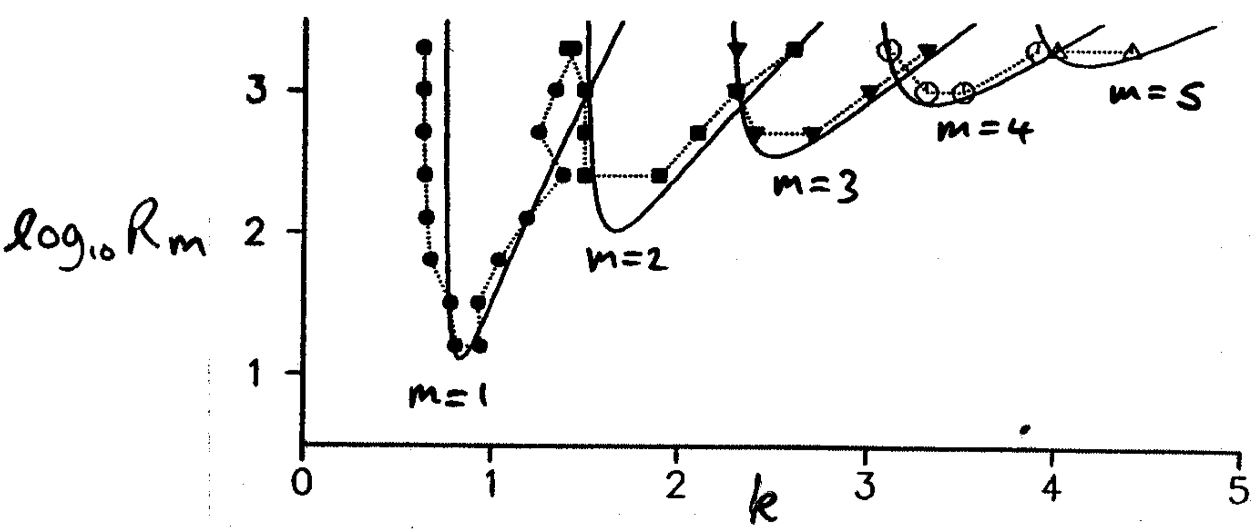
$\underline{B} \propto e^{ikz+pt}$ $\qquad \gamma = \mathrm{Re}\, p$

CRITICAL VALUES OF R_m (FOR ZERO GROWTH RATE, $\gamma = 0$).



ASYMPTOTIC (LARGE-R_m) RESULTS ——— GILBERT, PONTY, GAFD $\underline{93}$, 2000.

NUMERICAL RESULTS ● ■ ▼ ○ △



EXCELLENT AGREEMENT BETWEEN $\underset{\sim}{\text{HIGH}}$-$R_m$ THEORY & $\underline{\text{CRITICAL}}$ R_m VALUES.

BASIC MECHANISMS :

PONOMARENKO DYNAMO (SMOOTH):

Ponomarenko 1973
Gailitis, Freiberg

$$\underline{u} = r\Omega(r)\,\underline{e}_\theta + U(r)\,\underline{e}_z$$

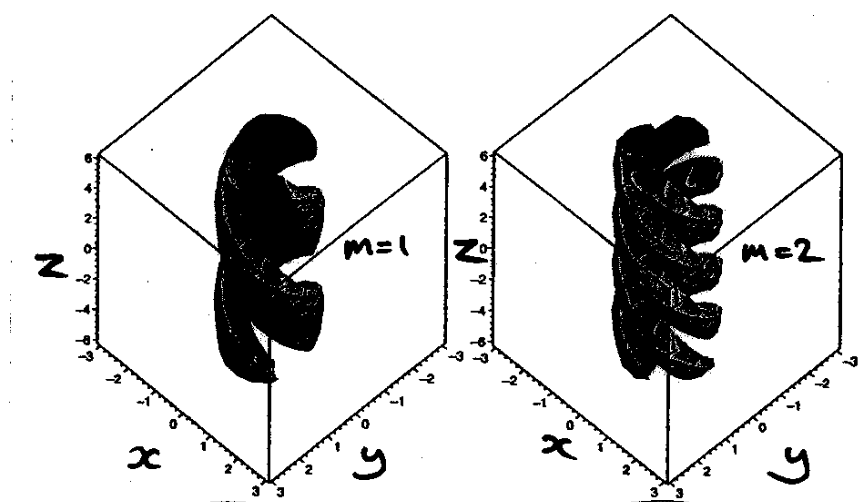
+ z-motion

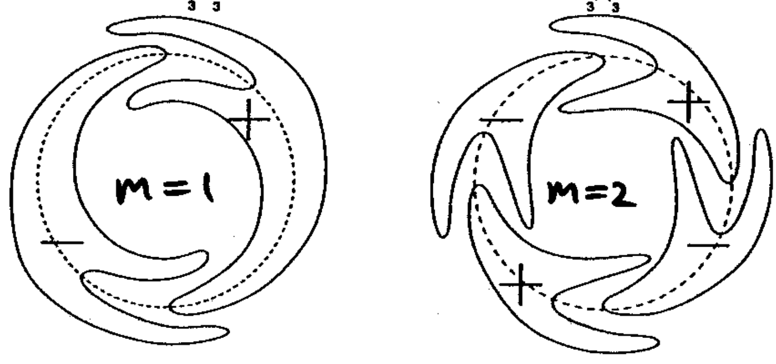
$$\underline{B} = e^{im\theta + ikz + pt}\,\underline{b}(r)$$

m, k ARE WAVENUMBERS WE CAN CHOOSE.

AS $\varepsilon \to 0$ FIELD LOCALISES AT A RADIUS a WHERE

$$m\Omega'(a) + kU'(a) = 0$$





$$\gamma = \mathrm{Re}\, p = \sqrt{\varepsilon |m\Omega'(a)|/a} - \tfrac{1}{2}\sqrt{\varepsilon |m\Omega''(a) + kU''(a)|}$$

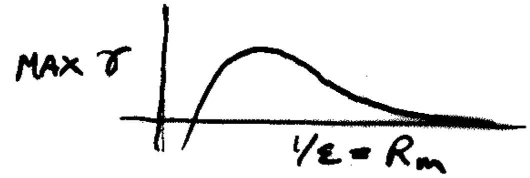
$$- \varepsilon\left(m^2/a^2 + k^2\right).$$

Ruzmaikin, Sokoloff, Shukurov, Gilbert 1988.

ASTEST GROWING MODES HAVE:

$$m, k = O(\varepsilon^{-1/3})$$

$$\gamma = O(\varepsilon^{1/3})$$



"SLOW" DYNAMO: MAXIMUM $\gamma \to 0$ AS $\varepsilon \to 0$

- DYNAMO ACTION (AT LARGE R_m) ONLY POSSIBLE AT A GIVEN RADIUS IF:

$$r \left| \frac{\Omega''(r)}{\Omega'(r)} - \frac{U''(r)}{U'(r)} \right| < 4$$

(GEOMETRICAL CONDITION).

- MECHANISM:

$$\begin{tikzcd} B_r \arrow[r, bend left, "\text{STRETCHING}"', "\Omega"] & B_\theta,\ B_z \arrow[l, bend left, "\text{DIFFUSION}", "\alpha"'] \end{tikzcd}$$

DYNAMO IS DIFFUSION-LIMITED.

- CAN GENERALISE TO FLOWS WITH CLOSED STREAM SURFACES $\underline{u}(x,y)$.

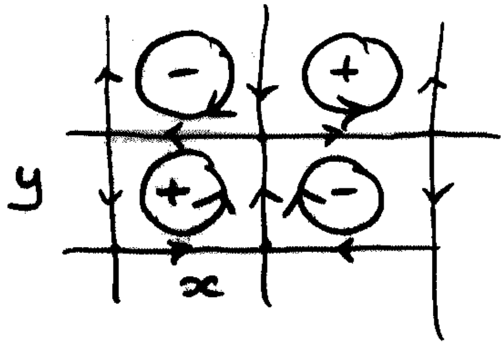
SIMILAR MECHANISM, GROWTH RATES, GEOMETRICAL CONDITION.

Soward 1990,
Gilbert, Ponty 2000.

CELLULAR STEADY FLOWS:

G.O. Roberts 1970.

$\underline{u} = (\sin x \cos y, -\cos x \sin y, K \sin x \sin y)$ $K = \sqrt{2}$



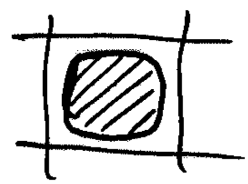
BELTRAMI FLOW
$\nabla \times \underline{u} = K \underline{u}$
ABC FLOW
A=B, C=0, RESCALE

LOW R_m: CLASSICAL α^2 DYNAMO.

HIGH R_m: PONOMARENKO MODES

BUT ONLY WHERE GEOMETRICAL

CONDITION IS MET ////

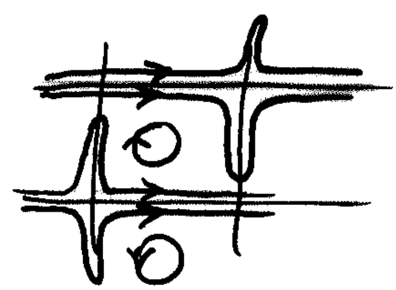


Soward 1990

BUT NOW HAVE "ROBERTS" MODES

FIELD LOCALISED ON SEPARATRICES

& X-POINTS. "SFS" MECHANISM:



PUT $\underline{B} \propto e^{ikz+pt}$

$$\gamma = \alpha k - \varepsilon k^2, \qquad \alpha = -\tfrac{1}{2} K \varepsilon^{1/2} G$$

$$G \simeq 1.0655$$

Childress 1979
Soward 1986

VALID FOR $k = O(1)$.

FASTEST GROWING MODES HAVE:

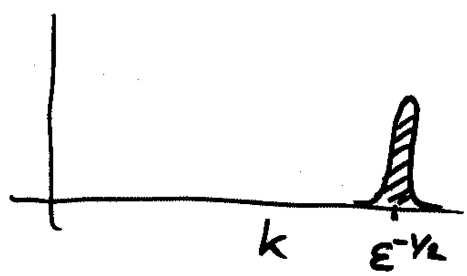
$$\gamma = O\left(\frac{\log\log \varepsilon^{-1}}{\log \varepsilon^{-1}}\right), \qquad k = O\left(\frac{\varepsilon^{-1/2}}{\sqrt{\log \varepsilon^{-1}}}\right)$$

GROWTH RATE GOES TO ZERO AS $\varepsilon \to 0$

(BUT ONLY LOGARITHMICALLY) — SLOW DYNAMO.

FIELD HAS TO ADOPT SMALL SCALES AS

SPECTRUM $E_M(k)$

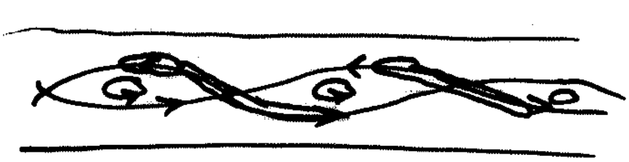


DIFFUSION STILL LIMITING GROWTH.

NO LARGE-SCALE FIELD COMPONENT.

IMPORTANT SLOW DYNAMO MECHANISM.

- ALSO CATS' EYE DYNAMOS. PONOMARENKO & SEPARATRIX MODES



PLANE LAYER

Ponty, Gilbert, Soward.

DEFINITIONS:

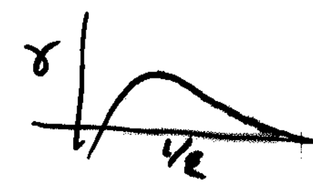
SOLVE THE INDUCTION EQUATION FOR A GIVEN FLOW $\underline{u}$ AND LET $\gamma(\varepsilon)$ BE THE MAXIMUM GROWTH RATE, MAXIMISED OVER ALL MODES / INITIAL CONDITIONS.

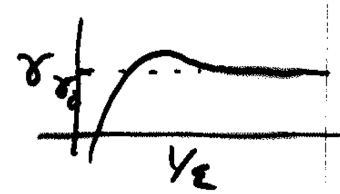
PUT:

$$\gamma_0 = \lim_{\varepsilon \to 0} \gamma(\varepsilon)$$

$\underline{u}$ IS A SLOW DYNAMO IF $\gamma_0 \leq 0$

A FAST DYNAMO IF $\gamma_0 > 0$





Vainshtein + Zeldovich 1972.

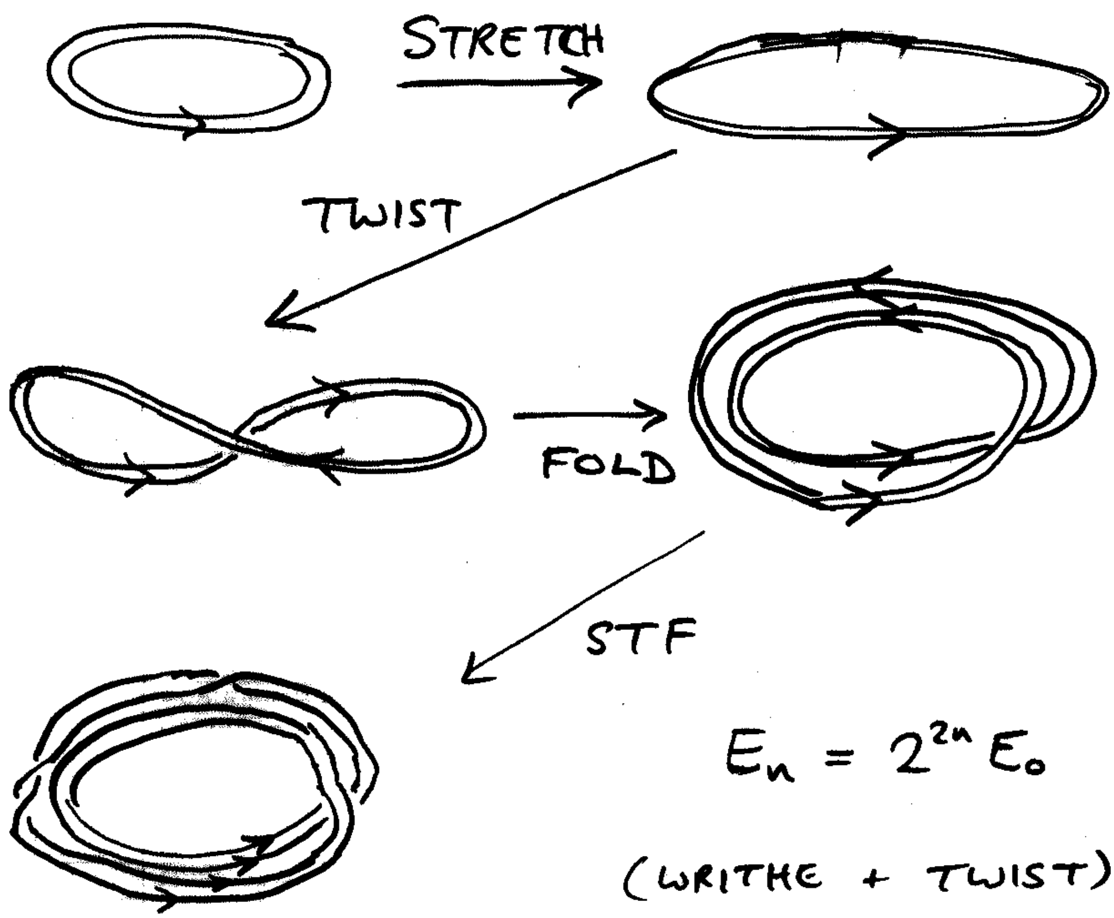
FAST DYNAMO:

NEED TO AVOID <u>RELYING</u> ON DIFFUSION TO AMPLIFY A COHERENT FIELD.

STRETCH - TWIST - FOLD (STF)

MECHANISM / PICTURE

Vainshtein, Zeldovich 197
Alfvén.



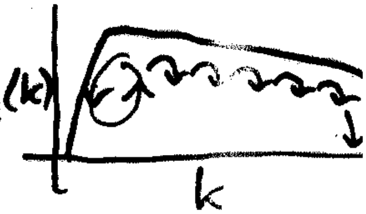
$$E_n = 2^{2n} E_0$$

(WRITHE + TWIST)

"PICTURE" OF FAST DYNAMO: $\gamma_0 = \log 2$

- CHAOTIC (LAGRANGIAN SENSE) FLOW
- AMPLIFIES LARGE-SCALE COHERENT FIELD

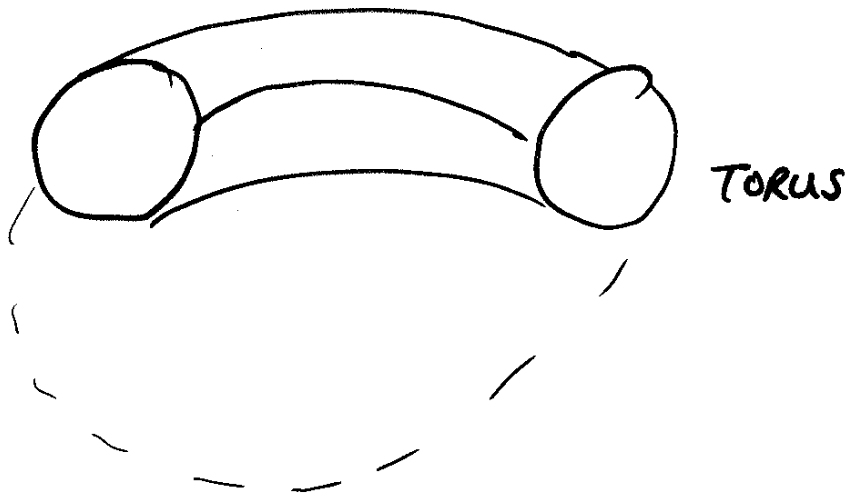
  + CASCADE OF FLUCTUATIONS $E_M(k)$



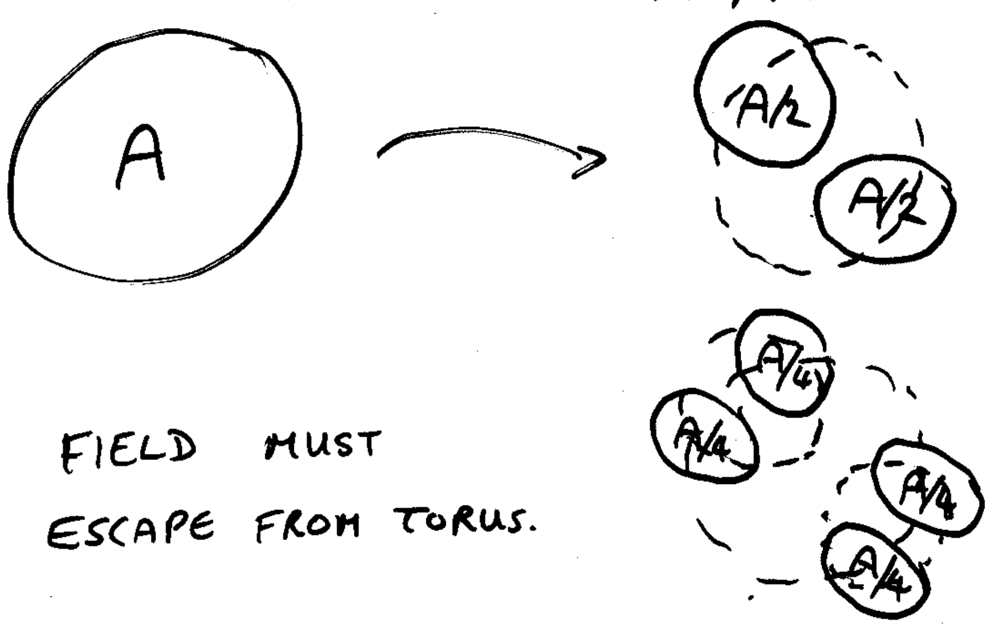
- **PICTURE HARD TO REALISE**

Moffatt, Proctor, Vainshtein, Rosner, So[illegible]aw, Fi
Ott.

PROBLEM WITH S.T.F.

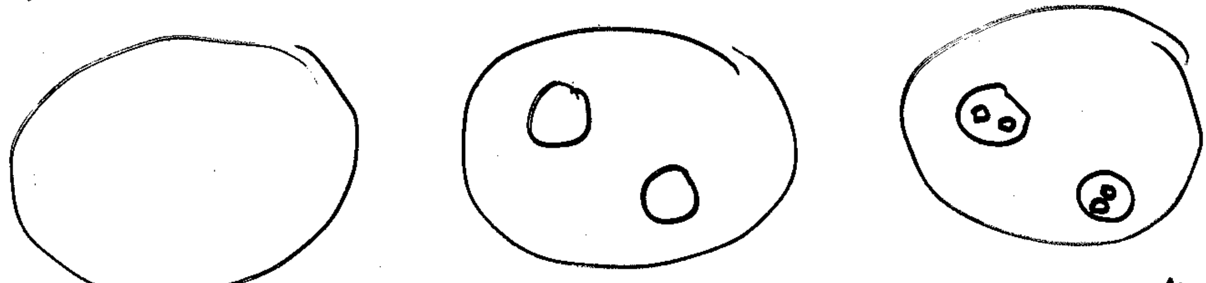


VOLUME-PRESERVING MAP/FLOW



FIELD MUST ESCAPE FROM TORUS.

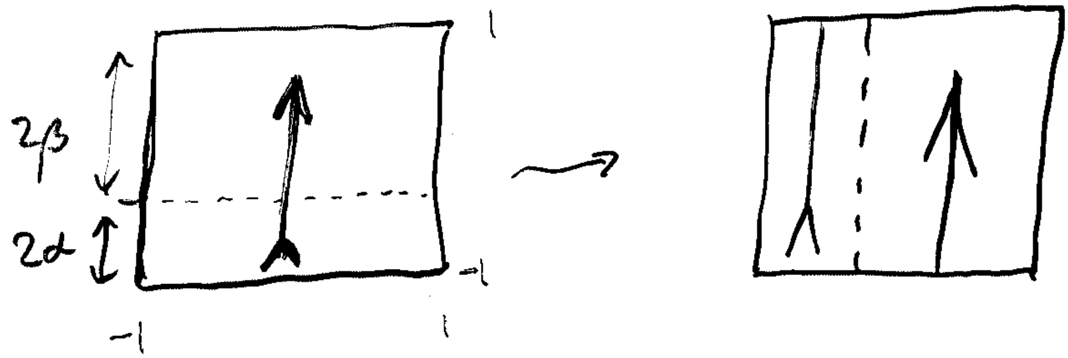
FOR A VOLUME-CONTRACTING FLOW: $\operatorname{div} \underline{u} < 0$:



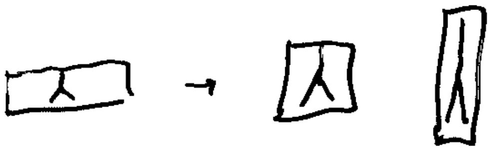
"SOLENOID MAP".

BAKER'S MAP MODEL OF STF

Finn / Ott.



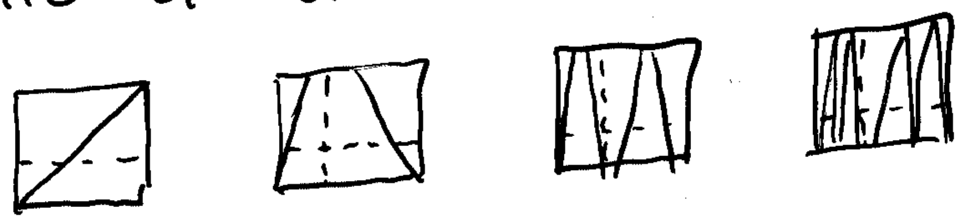
$\alpha+\beta=1$



stretch by factor α^{-1} or β^{-1}

LIAPUNOV EXPONENT $\lambda = \alpha \log \alpha^{-1} + \beta \log \beta^{-1}$

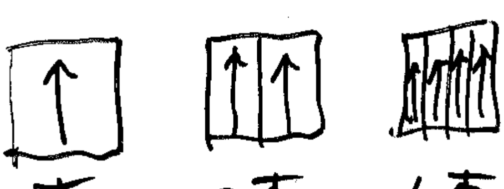
RATE OF GROWTH OF LINE LENGTH:



TOPOLOGICAL ENTROPY = LINE STRETCHING EXPONENT (HERE) = $h = \log 2 \geq \lambda$.

[DIFFERENT AVERAGING : LINES ACCUMULATE IN REGIONS OF STRONG STRETCHING.]

MAGNETIC FLUX (NO DIFFUSION)



Φ 2Φ 4Φ

$\gamma = \log 2 = h$

(NO FOLDING).

• RIGOROUS RESULT:

NEED LAGRANGIAN CHAOS (POSITIVE TOPOLOGICAL ENTROPY) FOR FAST DYNAMO ACTION IN A SMOOTH FLOW: $\gamma_0 \leq h$.

Klapper, Young. 1995
(Also Finn, Ott)

• STEADY, 3-D ABC FLOWS

Galloway, Frisch 1986

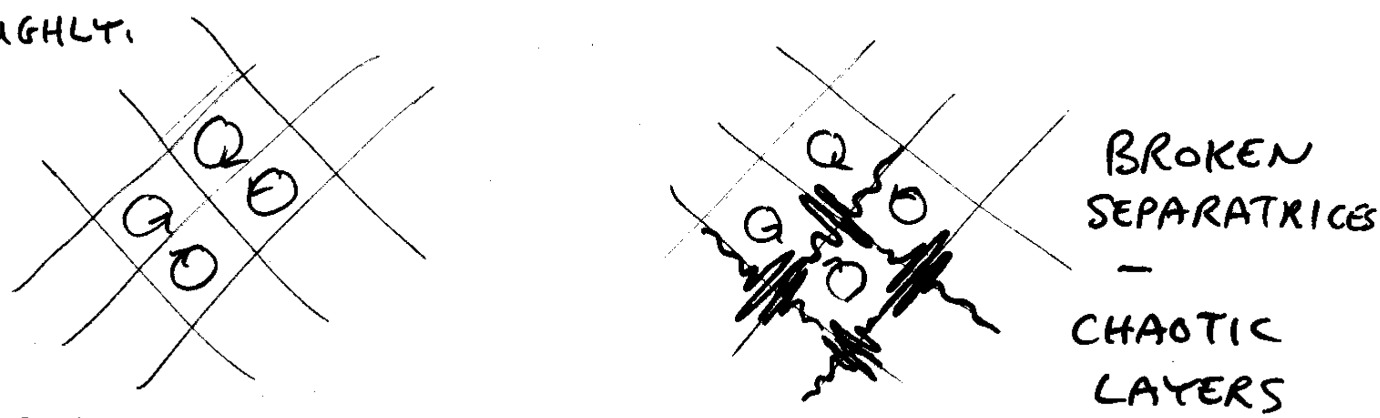
COMPUTATIONALLY INTENSIVE — PICTURE STILL UNCLEAR.

• UNSTEADY, 2-D FLOWS

Galloway, Proctor 1992
Otani 1993

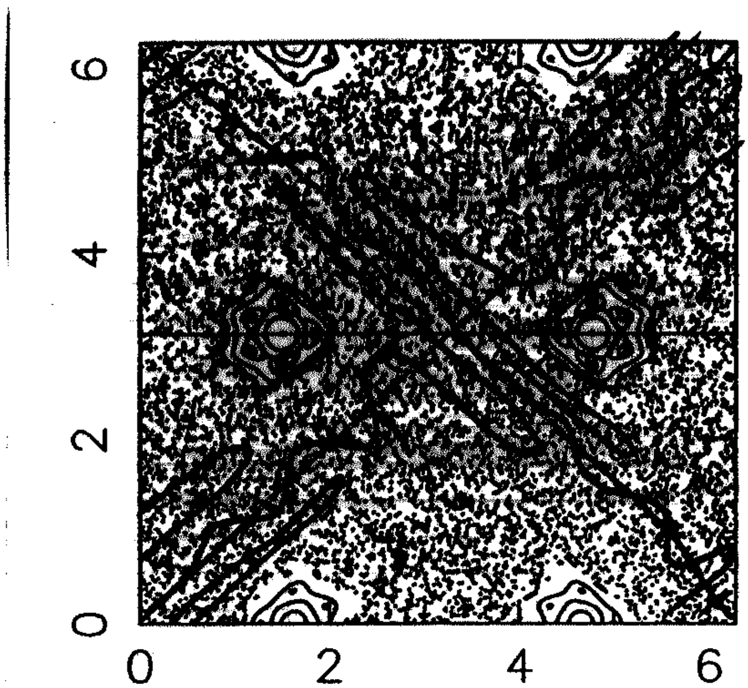
$$\underline{u} = 2\cos^2 t\,(0, \sin x, \cos x) + 2\sin^2 t\,(\sin y, 0, -\cos y)$$

ROUGHLY:



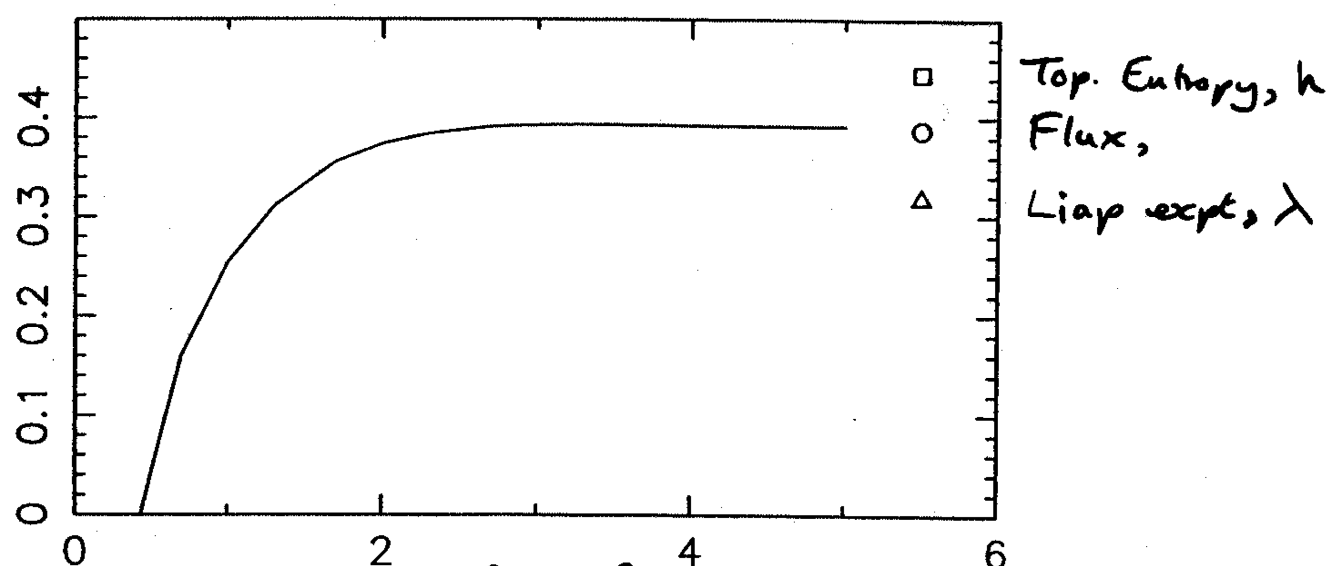
ROBERTS FLOW + TIME DEPENDENCE.

POINCARÉ (STROBOSCOPIC) MAP, PERIOD 2π IN t.



FAST DYNAMO $\gamma_0 \simeq 0.39$,

γ



$\log_{10} R_m = \log_{10} \frac{1}{\varepsilon}$

$\underline{B} \propto e^{ikz+pt}$, $k = 0.8$ LARGE-SCALE FIELD COMPONENT

$\boxed{R_m = \frac{1}{\varepsilon}}$

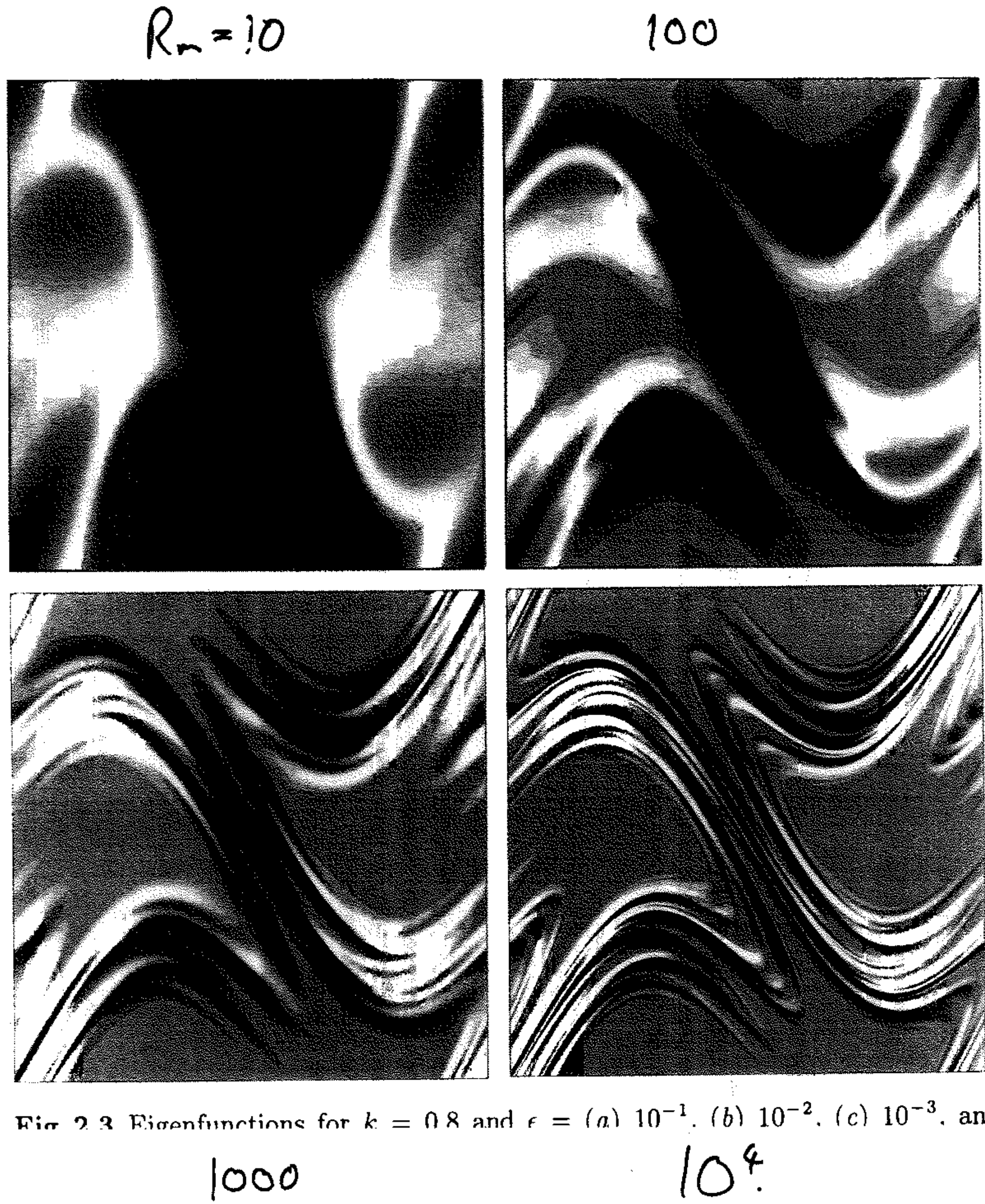


Fig 2.3 Eigenfunctions for $k = 0.8$ and $\epsilon =$ (a) 10^{-1}, (b) 10^{-2}, (c) 10^{-3}, and

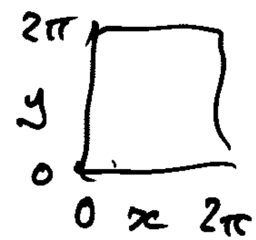
Otani's flow eigenfunctions

$k = 0.8$

$\varepsilon =$

10^{-1} 10^{-2}

10^{-3} 10^{-4}



$z = 0.$

B_x

$R_m = 10^4$

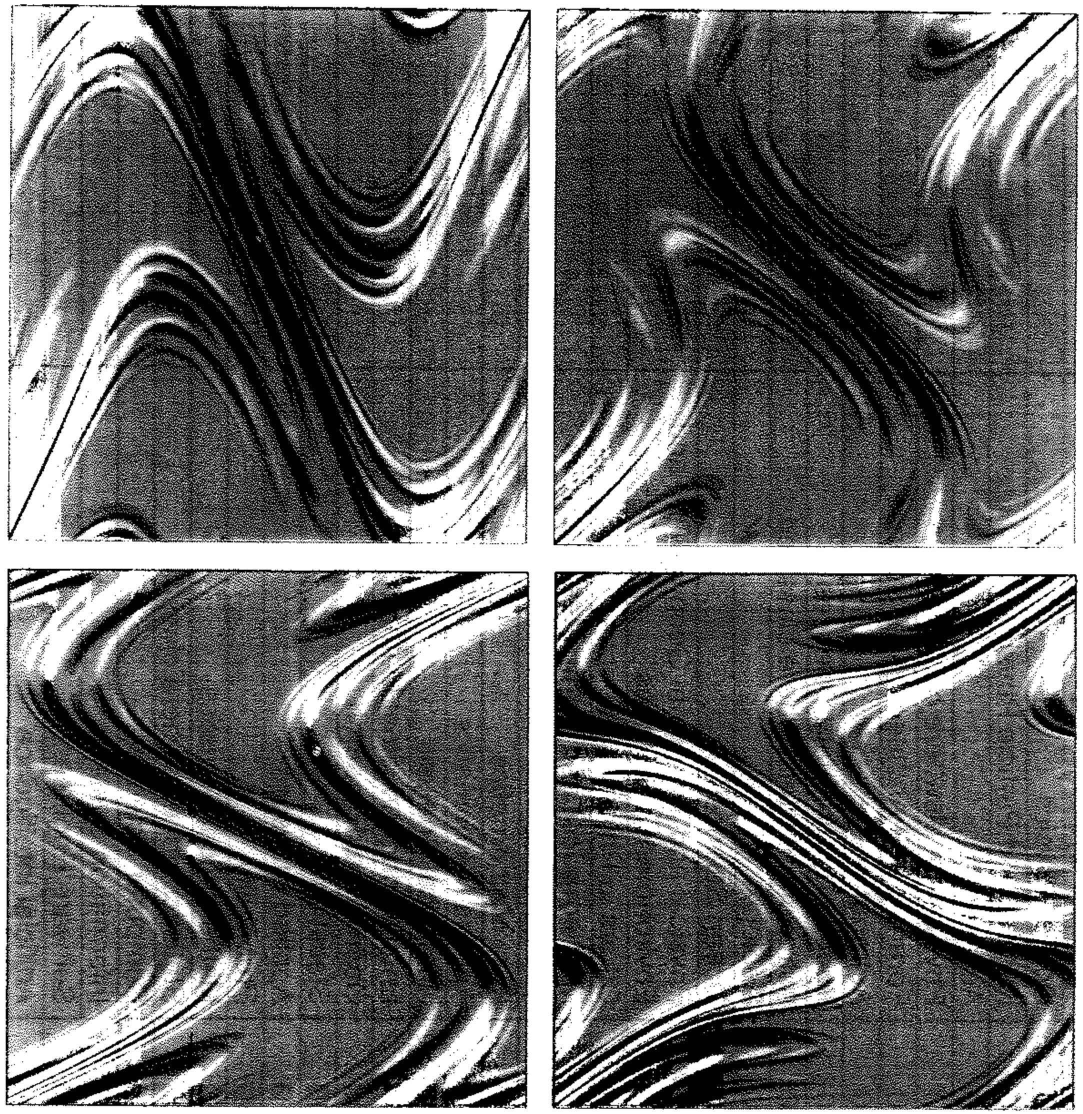
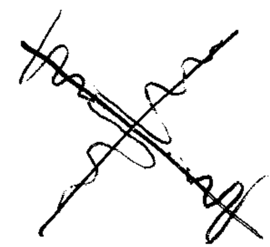


Fig. 2.5 Field evolution with $k = 0.8$ and $\epsilon = 10^{-4}$ for $t - 40\pi = (a)\ \pi/5$, $(b)\ 2\pi/5$,

Otani's flow

$\varepsilon = 10^{-4}$

time-evolution over 1 cycle



mechanism is piling up of sheets of field because of broken separatrix (cf. Melnikov)

+ shear in Z-direction

$\underline{u} = 2\cos^2 t(0, \sin x, \cos x) + 2\sin^2 t(\sin y, 0, -\cos y)$.

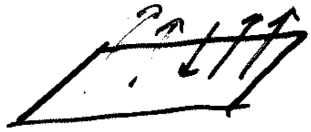
SFS

"STRETCH - FOLD - SHEAR" MECHANISM

Bayly, Childress, Soward, Roberts, ...

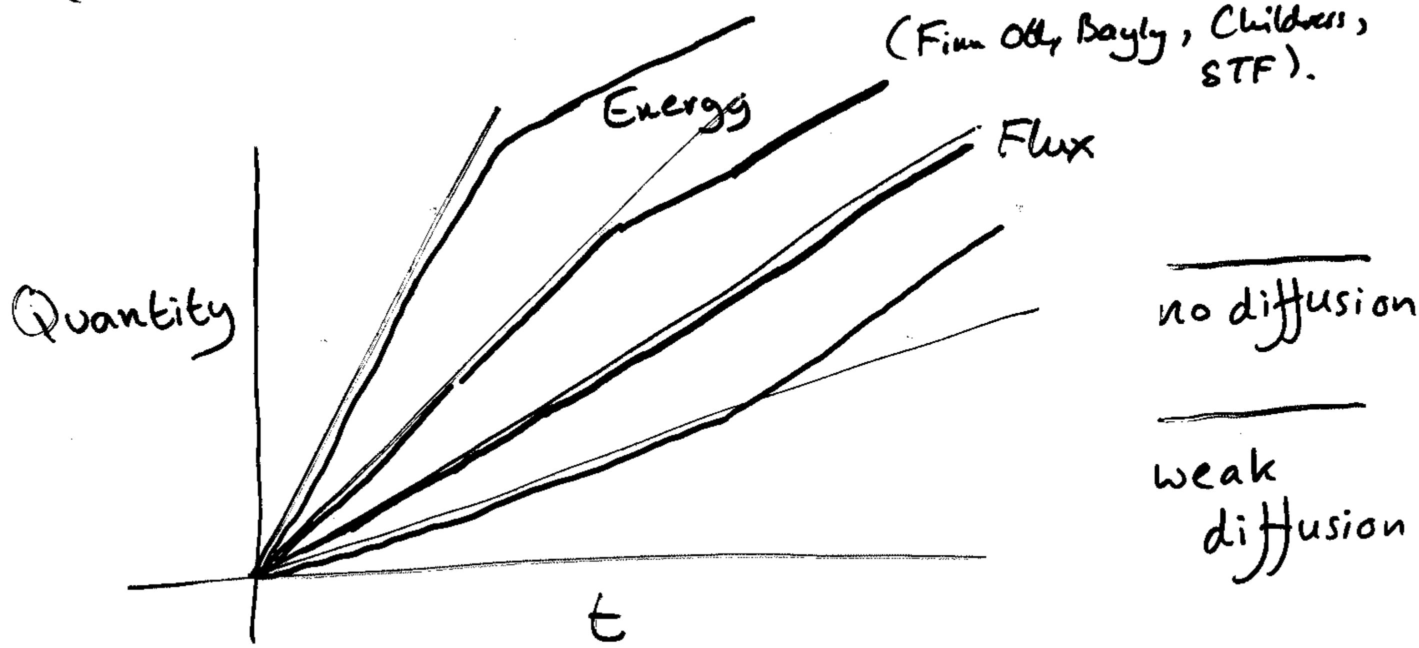
EASURE FLUX GROWTH:

$$\Phi_S = \int_S \underline{B}\cdot d\underline{S} \propto e^{\Gamma_S t}$$



(NO DIFFUSION). CONJECTURE: $\Gamma_S = \gamma_0$

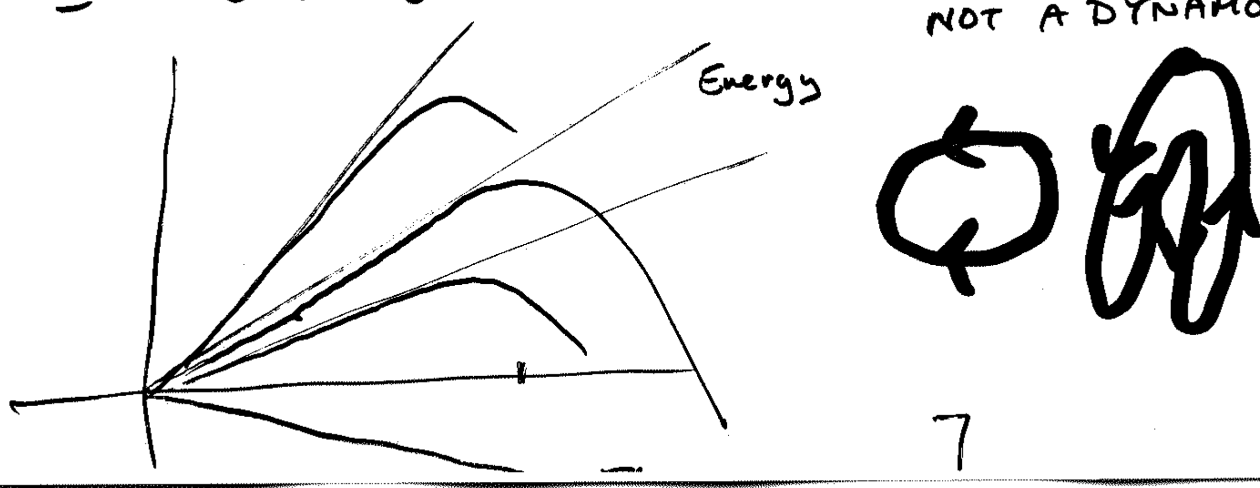
(Finn Ott, Bayly, Childress, STF).



I.E. FLUX IS ONLY ROBUST LINK BETWEEN ZERO DIFFUSION AND WEAK DIFFUSION

[N.B. $\underline{u} = (u_1(x,y,t), u_2(x,y,t), 0)$

NOT A DYNAMO.



ANALYSIS: STUDY DYNAMOS IN MAPS.

APPLY MAP M : $b \longrightarrow Tb$

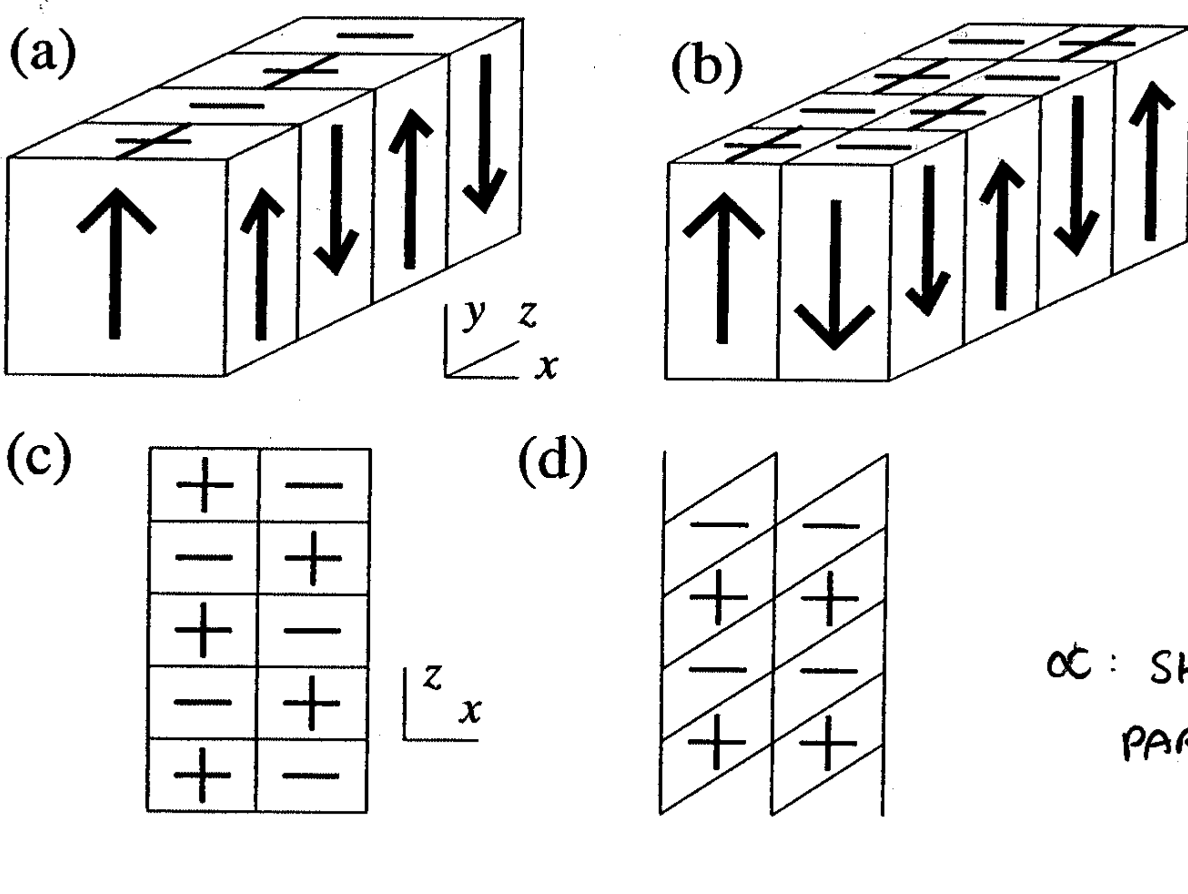
APPLY WEAK DIFFUSION: $b \longrightarrow H_\varepsilon b$

"DYNAMO OPERATOR": $T_\varepsilon \equiv H_\varepsilon T$

STUDY EIGENVALUES & EIGENFUNCTIONS OF T_ε AS $\varepsilon \to 0$. DYNAMO IS FAST IF T_ε HAS EIGENVALUE(S) WITH $|\lambda|$ BOUNDED ABOVE 1 AS $\varepsilon \to 0$

STRETCH-FOLD-SHEAR MECHANISM / MAP

Bayly, Childress



α : SHEAR PARAMETER.

START WITH:

$$\underline{B} = e^{ikz}\, b(x)\, \underline{e}_y + \text{complex conjugate.}$$

TAKE $k=1$ (WLOG).

DYNAMO OPERATOR (NO DIFFUSION) IS:

$$T b(x) = \begin{cases} 2 e^{-i\alpha x}\, b(1+2x) & -1 \leq x < 0. \\ -2 e^{-i\alpha x}\, b(1-2x) & 1 \geq x \geq 0 \end{cases}$$

FOLD GIVES A MINUS SIGN

STRETCHING BY FACTOR 2

CONTRACTION OF SCALE IN x- DIRECTION.

SHEAR GIVES A PHASE SHIFT

DIFFUSION: $\partial_t b = \varepsilon \nabla^2 b$

WITH INSULATING: $b(1) = b(-1) = 0$

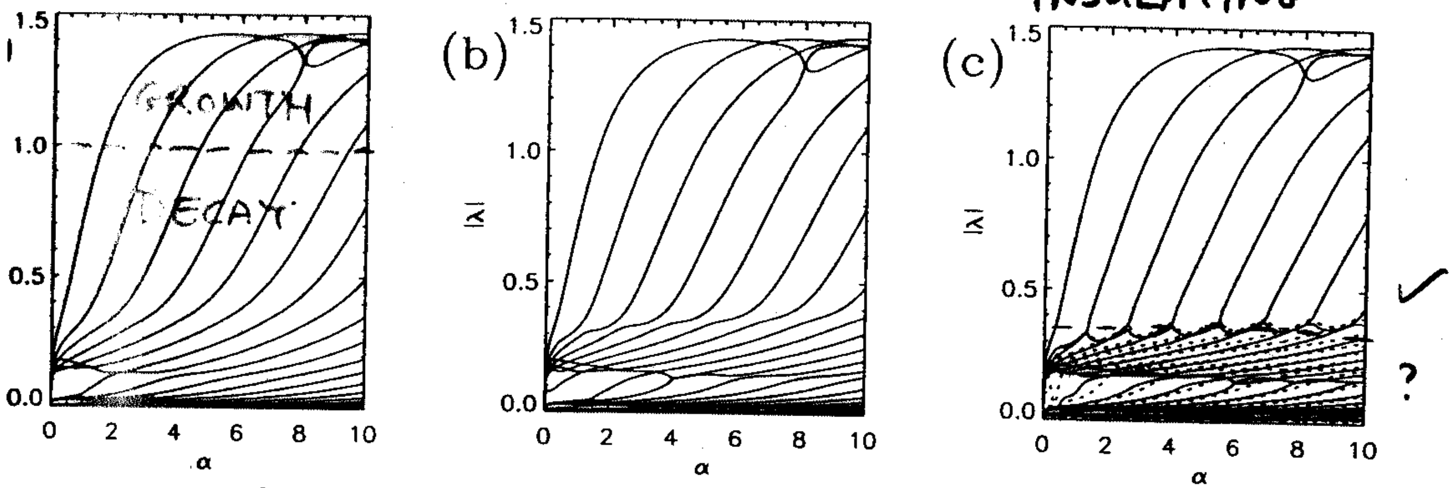
PERFECTLY CONDUCTING: $\partial_x b(1) = \partial_x b(-1) = 0$

OR PERIODIC: $b(x)$ PERIODIC.

$$H_\varepsilon b(x) = \int_{-1}^{1} H_\varepsilon(x,y)\, b(y)\, dy$$

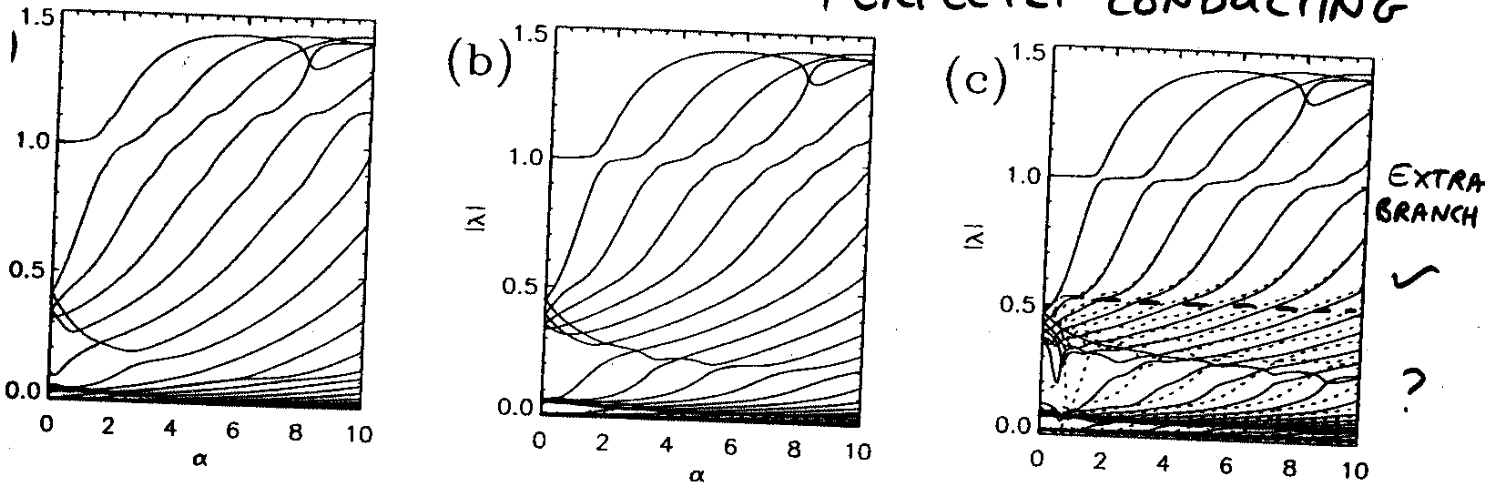
EIGENVALUES $|\lambda|$ AGAINST SHEAR PARAMETER α

INSULATING



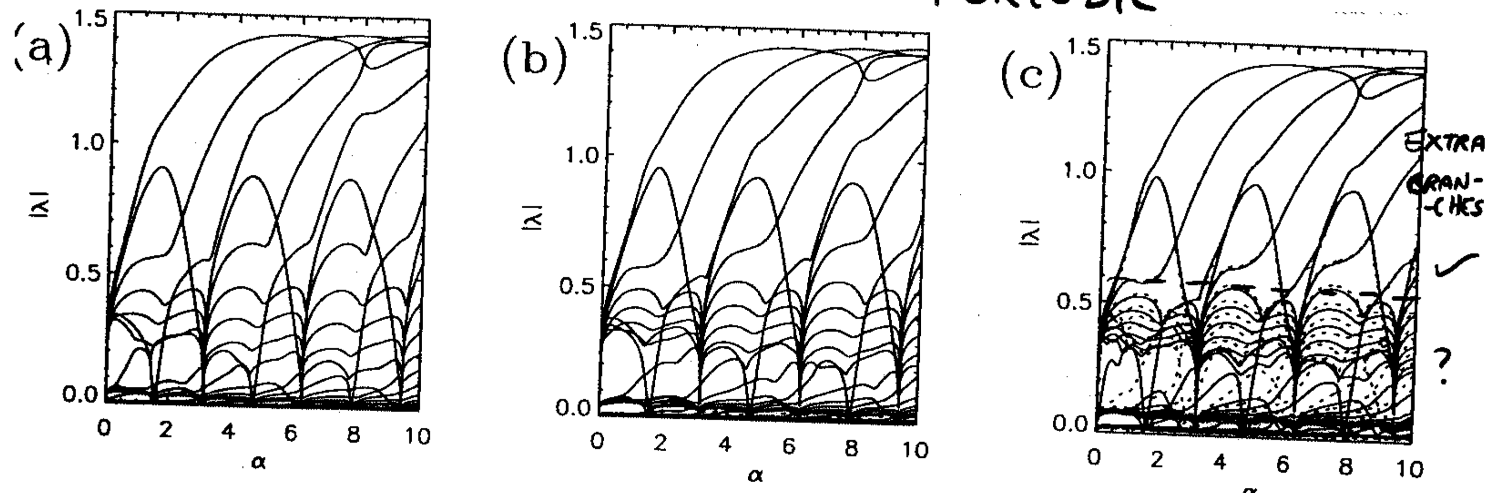
$\varepsilon = 10^{-3}$ $\quad 10^{-4}$ $\quad 10^{-5}$

PERFECTLY CONDUCTING



$\varepsilon = 10^{-3}$ $\quad 10^{-4}$ $\quad 10^{-5}$

PERIODIC



$\varepsilon = 10^{-3}$ $\quad 10^{-4}$ $\quad 10^{-5}$

- CLEAR EVIDENCE OF FAST DYNAMO ACTION FOR $\alpha > \pi/2$ WITH ALL BOUNDARY CONDITIONS: PRETTY ROBUST

- SOME DEPENDENCE ON BOUNDARY CONDITIONS: EXTRA BRANCHES, CONVERGENCE, ... ESP. FOR $|\lambda| \leq 1$.

HOW DO WE MAKE SENSE OF THESE RESULTS ? AND DEVELOP TOOLS TO ANALYSE THIS & RELATED (E.G. PASSIVE SCALAR) PROBLEMS ?

STUDY ADJOINT OPERATOR (IN L^2):

$$T^*_\varepsilon = T^* H_\varepsilon$$

$$T^* c(x) = e^{-i\alpha\frac{1}{2}(x-1)} c\left(\tfrac{1}{2}(x-1)\right) - e^{-i\alpha\frac{1}{2}(1-x)} c\left(\tfrac{1}{2}(1-x)\right)$$

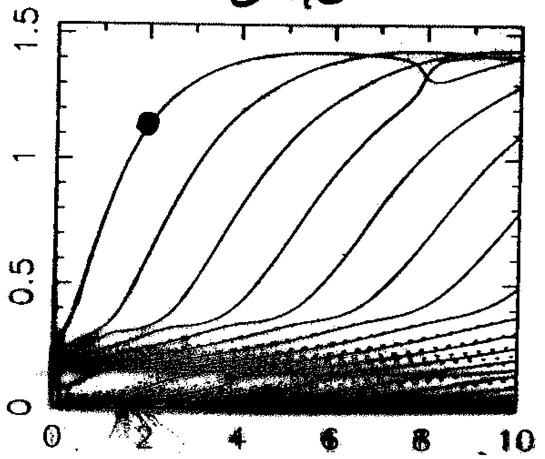
Bayly, Childress

THIS STRETCHES OUT STRUCTURE & HAS SMOOTH EIGENFUNCTIONS !

INSULATING $\alpha = 2$

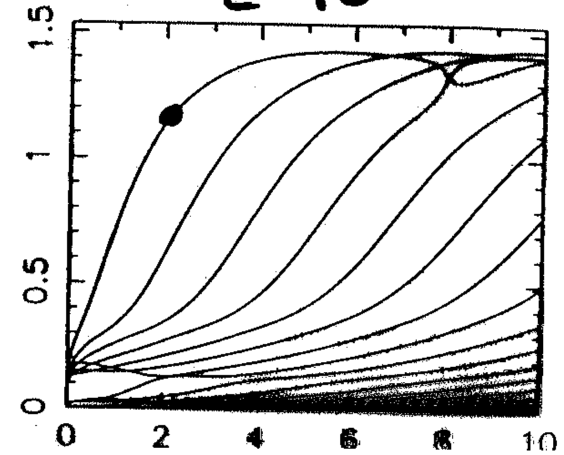
$\varepsilon = 10^{-5}$

$|\lambda|$



$\varepsilon = 10^{-4}$

$|\lambda|$



EIGENFUNCTIONS OF

— Re b

...... Im b

T_ε^*

EIGENFUNCTIONS OF

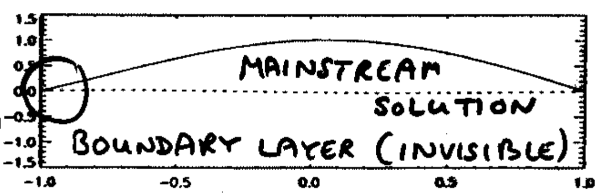
— Re b

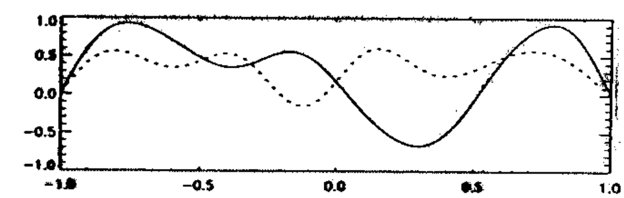
.... Im b

T_ε

I

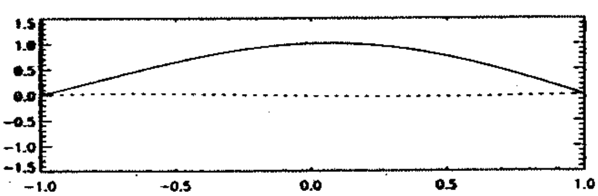
$\varepsilon = 0.01$

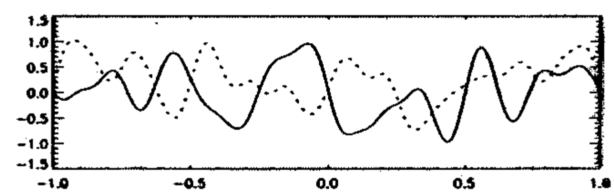




I

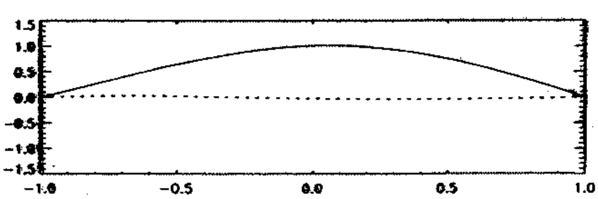
$\varepsilon = 0.001$

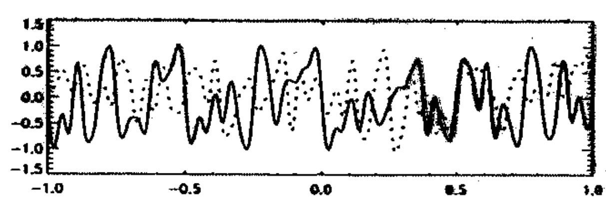




I

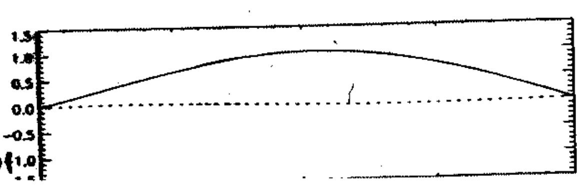
$\varepsilon = 0.0001$

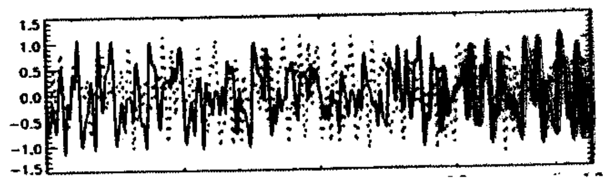




I

$\varepsilon = 0.00001$



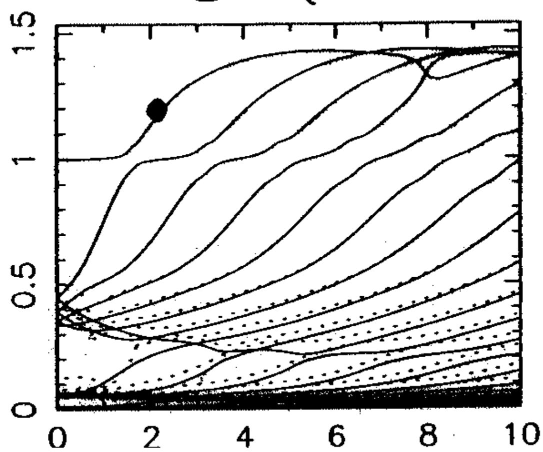


PERFECTLY CONDUCTING

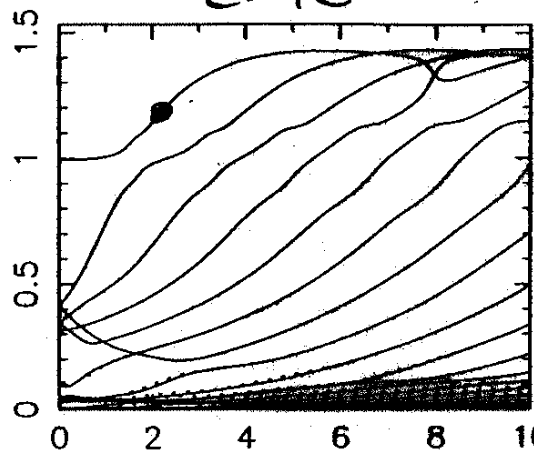
$\alpha = 2$

$\varepsilon = 10^{-5}$ $\varepsilon = 10^{-4}$

$|\lambda|$

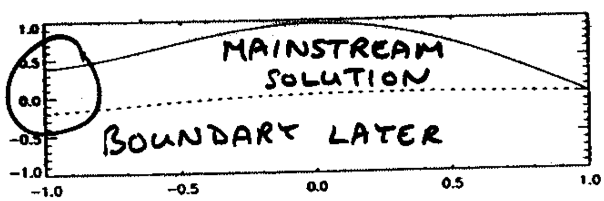


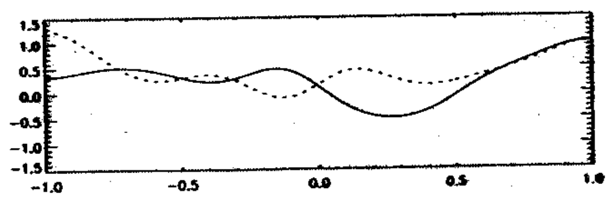
$|\lambda|$



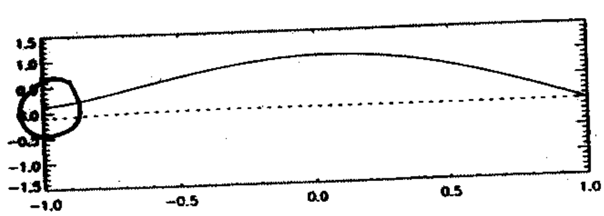
EIGENFUNCTIONS OF T_ε^* EIGENFUNCTIONS OF T_ε

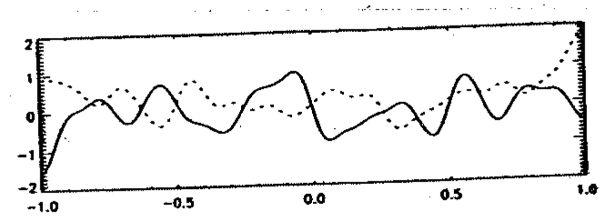
(P) $\varepsilon = 0{\cdot}01$



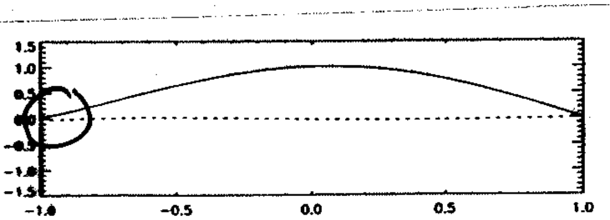


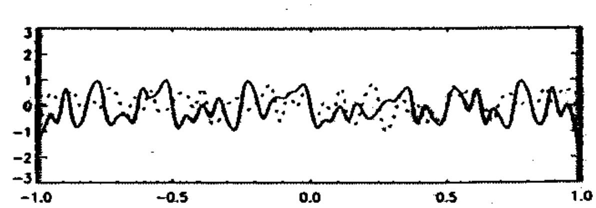
(P) $\varepsilon = 10^{-3}$



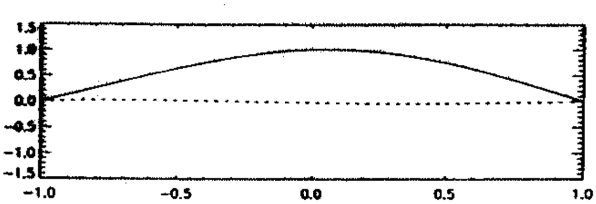


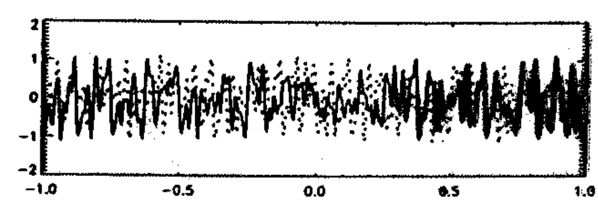
(P) $\varepsilon = 10^{-4}$



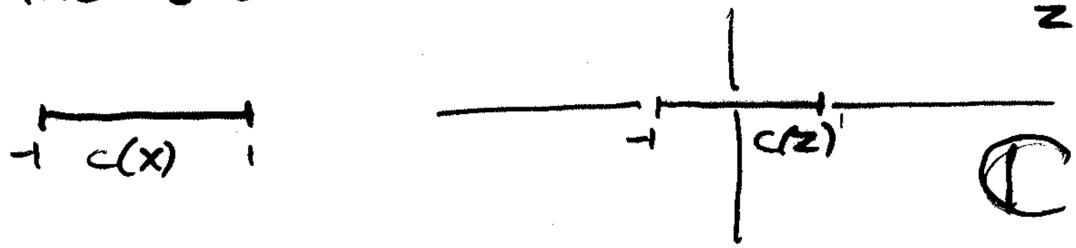


(P) $\varepsilon = 10^{-5}$





IN FACT T^* (NO DIFFUSION) HAS EIGENFUNCTIONS THAT ARE ENTIRE (ANALYTIC EVERYWHERE IN $\mathbb{C}$)



AND T (NO DIFFUSION) HAS EIGENFUNCTIONS THAT ARE ANALYTIC EVERYWHERE IN $\mathbb{C}$ EXCEPT ON $[-1,1]$.

"HYPERFUNCTIONS" INCLUDE DISTRIBUTIONS.

E.G. HORIZONTAL BRANCH:

$$T^*: \quad c(z) = e^{i\alpha(z-1)} - e^{i\alpha(1-z)}, \qquad \lambda = e^{i\alpha}$$

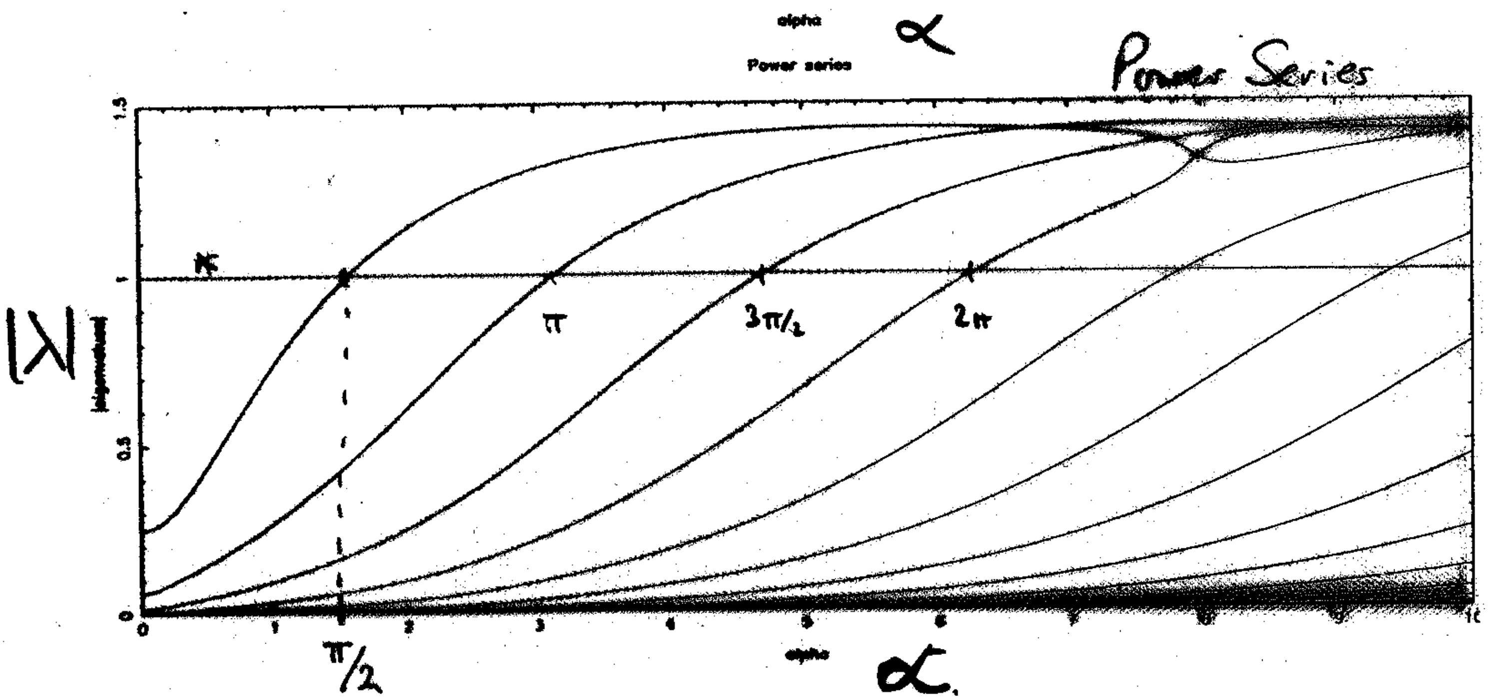
$$T: \quad b(z) = \frac{e^{-i\alpha}}{z-1} - \frac{e^{i\alpha}}{z+1}, \qquad \lambda = e^{i\alpha}$$

CAN CALCULATE EIGENVALUES IN THIS COMPLEX SETTING USING POWER SERIES EXPANSIONS:

$$c = \sum c_n z^n \qquad T^*c = \lambda c$$

$$b = \sum b_n z^{-n-1} \qquad Tb = \lambda b$$

EIGENVALUES:



ZERO DIFFUSION,

POWER SERIES EXPANSIONS

ENTIRE EIGENFUNCTIONS OF T^*

DISTRIBUTION EIGENFUNCTIONS OF T.

CLOSE RELATION OF EIGENVALUES (& EIGENFUNCTIONS OF T^*) WITH ZERO DIFFUSION, AND THOSE OF T_ε, T_ε^* ....

WRITE:

$$\lambda(\varepsilon) = \lambda(0) + \underbrace{\lambda^{ms} + \lambda^{bl}}_{\text{LEADING DIFFUSIVE CORRECTIONS}} + \cdots$$

EIGENVALUE OF $T_\varepsilon^*, T_\varepsilon$ FOR WEAK DIFFUSION ε WITH BOUNDARY CONDITIONS

EIGENVALUE OF $T_s^* T$ IN-COMPLEX SETTING FOR ZERO DIFFUSION

λ^{ms} : EFFECT OF DIFFUSION ON MAINSTREAM COMPONENT OF EIGENFUNCTION:

$$\lambda^{ms} = C_1 \varepsilon$$

C_1 CAN BE OBTAINED NUMERICALLY FROM EIGENFUNCTIONS.

λ^{bl} : EFFECT OF DIFFUSION IN BOUNDARY LAYERS:

$$\lambda^{bl} = C_2 \varepsilon^q \qquad q = \tfrac{1}{2}(n + \log_2 |\lambda|)$$

$q<0 \Rightarrow$ MODE EVAPORATES.

STRENGTH OF DISCONTINUITY — DEPENDS ON BOUNDARY CONDITIONS, AND EIGENFUNCTION STRUCTURE

C_2 HARD TO CALCULATE. NEED TO SOLVE BOUNDARY LAYER PROBLEM ....

OPEN QUESTIONS:

- LAMINAR FLOWS: UNDERSTAND / CLASSIFY SEPARATRIX DYNAMOS.
- FAST DYNAMOS IN MORE GENERAL HYPERBOLIC MAPS. (ALSO PASSIVE SCALARS.) — KOZLOVSKY
- FAST DYNAMOS IN "REALISTIC" FLOWS: HARD TO ANALSE; MUCH NUMERICAL EXPLORATION STILL TO BE DONE, E.G. ABC FLOWS.
- DYNAMO SATURATION:

  SUPPRESSION OF CHAOS
  LIMITING EFFECT OF HELICITY
  } SLOWING OF TIMESCALE

  Vainshtein, Cattaneo, Hughes, Brandenburg, ...

BUT:
DEPENDENCE ON Re, Rm,
MODEL DEPENDENCE (DRIVING).

MOSTLY NUMERICAL / CLOSURE.